ON AN EIGENVALUE PROBLEM INVOLVING THE VARIABLE EXPONENT AND INDEFINITE WEIGHT

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This paper is mainly concerned with the $p(x)$-Laplacian eigenvalue problem with a indefinite weight function, that is,

$$\begin{cases} -\Delta_{p(x)}u = \lambda V(x)|u|^{p(x)-2}u, \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega. \end{cases}$$

The main result of this paper establishes that any $\lambda > 0$ sufficiently small is an eigenvalue of the above nonhomogeneous quasilinear problem. The proofs will be based on the Ekeland's variational principle combined with adequate variational techniques.

Keywords: Eigenvalue problem, Variable exponent, Indefinite weight, Weak solution.

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1. Introduction

The study of eigenvalue problems involving operators with variable exponents growth conditions has captured a special attention in the last few years. For more details we refer to Mihăilescu et al. [1, 2, 3, 4, 5, 6, 7], Fan, Zhang and Zhao [8, 9], Kefi et al. [10, 11, 12] and Benouhiba [13].

Recently, Ge in [14] studied the following nonhomogeneous eigenvalue problem

$$\begin{cases} -\text{div}(a(|\nabla u|)\nabla u) = \lambda V(x)|u|^{q(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\lambda > 0$ is a real number, V is an indefinite sign-changing weight and $q : \overline{\Omega} \to (1, \infty)$ is a continuous function. In the case when $a(|\nabla u(x)|) = |\nabla u(x)|^{p(x)-2}$ with p is a continuous function on $\overline{\Omega}$, problem (P) is reduced to the following nonlinear eigenvalue problem

$$\begin{cases} -\text{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda V(x)|u|^{q(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{P_0}$$

Thus, when $1 < q(x) < \inf_{x\in\Omega} p(x) \leq \sup_{x\in\Omega} p(x) < s(x)$ for any $x \in \overline{\Omega}$, that there exists $\lambda_0 > 0$ such that any $\lambda \in (0, \lambda_0)$ is an eigenvalue for problem (P_0).

From the above cited contributions, we are interested in the existence of solutions for the following the nonhomogeneous eigenvalue problem (P_0) with $q(x) = p(x)$

$$\begin{cases} -\text{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda V(x)|u|^{p(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{P_1}$$

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On the other hand, problems like (P_0) have been largely considered in the literature in the recent years. We give in what follows a concise but complete image of the actual stage of research on this topic.

• In the case when $V(x) \equiv 1$, $\min_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x)$ and $q(x)$ has a subcritical growth Mihăilescu and Rădulescu [1] used the Ekeland's variational principle in order to prove the existence of a continuous family of eigenvalues which lies in a neighborhood of the origin.

• In the case when $V(x) \equiv 1$, $\max_{x\in\overline{\Omega}} p(x) < \min_{x\in\overline{\Omega}} q(x) \leq \max_{x\in\overline{\Omega}} q(x) < \frac{Np(x)}{N-p(x)}$, similar with those used by Fan and Zhang in the proof of Theorem 4.7 in [15], can be applied in order to show that any $\lambda > 0$ is an eigenvalue of problem (P_0).

• In the case when $V(x) \equiv 1$, $\max_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x)$ it can be proved that the energy functional associated to problem (P_0) has a nontrivial minimum for any positive λ (see Theorem 4.3 in [15]). Clearly, in this case the result in [1] can be also applied. Consequently, in this situation there exist two positive constants λ_* and λ^* such that any $\lambda \in (0, \lambda_*) \cup (\lambda^*, +\infty)$ is an eigenvalue of problem (P_0).

• The same problem, for $V(x) = 1$ and $p(x) = q(x)$ is studied by Fan, Zhang and Zhao in [9]. The authors established the existence of infinitely many eigenvalues for problem (P_0) by using an argument based on the Ljusternik- Schnirelmann critical point theory. Denoting by Λ the set of all nonnegative eigenvalues, they showed that $\sup \Lambda = +\infty$ and they pointed out that only under special conditions, which are somehow connected with a kind of monotony of the function $p(x)$, we have $\inf \Lambda > 0$ (this is in contrast with the case when $p(x)$ is a constant; then, we always have $\inf \Lambda > 0$).

Motivated by all results mentioned above, it is very natural for us to pose an interesting question, that is,

Question: In the case that $V(x)$ is allowed to be sign-changing and $p(x) = q(x)$. Can we obtain the same results as described in [1, 14] by replacing them with some suitable assumptions?

Few papers have treated the existence of nontrivial solutions for problem (P_1). Can we achieve the result? In the present paper, we restrict our attention to the existence of a continuous family of eigenvalues for problem (P_1) and are most interested in seeking definite answers to **Question**. To be precise, we make the following hypotheses on p, r, V.

(h_1) $p, r \in C_+(\overline{\Omega})$, $1 < p(x) \leq N$, $r(x) > \frac{Np(x)}{p(x)-1}$, $\forall x \in \overline{\Omega}$, $V \in L^{r(x)}(\Omega) \cap C(\Omega)$ and $V > 0$ in $\Omega_0 \subset \Omega$, where $|\Omega_0| > 0$.

(h_2) There exists an open subset $U \subset \Omega_0$ and a point $x_0 \in U$ such that $p(x_0) < p(x)$ for all $x \in \partial U$.

Thus, the case considered here is different from all the cases studied before. In this new situation we will show the existence of a continuous family of eigenvalues for problem (P_1) in a neighborhood of the origin. More precisely, we show that there exists $\lambda_0 > 0$ such that any $\lambda \in (0, \lambda_0)$ is an eigenvalue for problem (P_1).

We start with some preliminary basic results on the theory of variable exponent Sobolev space $W_0^{1,p(x)}(\Omega)$. For more details we refer to the book by Diening-Harjulehto-Hästö-Ružička [16] and the papers by Fan et al. [17, 18], Kováčík-Rákosník [19], and Edmunds-Rákosník [20, 21]. Throughout this article, we assume that $p \in C(\overline{\Omega})$ and $p(x) > 1$, for all $x \in \overline{\Omega}$.

Set $C_+(\overline{\Omega}) = \{h \mid h \in C(\overline{\Omega}), h(x) > 1 \text{ for all } x \in \overline{\Omega}\}$. For any $h \in C_+(\overline{\Omega})$ we define $h^+ = \sup_{x\in\Omega} h(x)$ and $h^- = \inf_{x\in\Omega} h(x)$. For any $p \in C_+(\overline{\Omega})$, we define the generalized Lebesgue space

$$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} \mid u \text{ is measurable and } \int_\Omega |u|^{p(x)} dx < +\infty\}.$$

This Luxemburg type norm $|u|_{p(x)} = \inf\left\{\mu > 0 \mid \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}$ makes $L^{p(x)}(\Omega)$ a Banach space. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$, $v \in L^{p'(x)}(\Omega)$ the Hölder-type inequality $\left|\int_\Omega uvdx\right| \le 2|u|_{p(x)}|v|_{p'(x)}$ holds true.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by $\rho(u) = \int_\Omega |u|^{p(x)}dx$. If $u_n, u \in L^{p(x)}(\Omega)$, then the following relations hold:

$$\begin{aligned} &|u|_{p(x)} > 1 \Rightarrow |u|^{p^-}_{p(x)} \le \rho(u) \le |u|^{p^+}_{p(x)}, \\ &|u|_{p(x)} < 1 \Rightarrow |u|^{p^+}_{p(x)} \le \rho(u) \le |u|^{p^-}_{p(x)}, \\ &|u_n - u|_{p(x)} \to 0 \Leftrightarrow \rho(u_n - u) \to 0. \end{aligned} \tag{1}$$

Moreover, if $s(x) \in L^\infty(\Omega)$ with $1 \le p(x)s(x) \le +\infty$ for all $x \in \overline{\Omega}$, then for any $u \in L^{s(x)}(\Omega)$ with $u \ne 0$, we have

$$\begin{aligned} &|u|_{p(x)s(s)} > 1 \Rightarrow |u|^{p^-}_{p(x)s(s)} \le \left||u|^{p(x)}\right|_{s(x)} \le |u|^{p^+}_{p(x)s(s)}, \\ &|u|_{p(x)s(s)} < 1 \Rightarrow |u|^{p^+}_{p(x)s(s)} \le \left||u|^{p(x)}\right|_{s(x)} \le |u|^{p^-}_{p(x)s(s)}. \end{aligned} \tag{2}$$

We also define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm $\|u\| = |\nabla u|_{p(x)}$. Thus, the space $W_0^{1,p(x)}(\Omega)$ is a separable and reflexive Banach space. Next, we recall some embedding results regarding variable exponent Lebesgue-Sobolev spaces. We note that if $\alpha \in C_+(\overline{\Omega})$ and $\alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$ is compact and continuous, where $p^*(x)$ denotes the corresponding critical Sobolev exponent, that is $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ or $p^*(x) = +\infty$ if $p(x) \ge N$. We refer to [19] for more properties of Lebesgue and Sobolev spaces with variable exponent.

For applications of Sobolev spaces with variable exponent we refer to Acerbi and Mingione [22], Chen, Levine, Rao [23], Ruzicka [24], and Zhikov [25].

2. The main results and proof of the theorem

We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (P_1) if there exists $u \in W_0^{1,p(x)}(\Omega)\backslash\{0\}$ such that

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx = \lambda \int_\Omega V(x)|u|^{p(x)-2}uvdx,$$

for all $v \in W_0^{1,p(x)}(\Omega)$. We point out that if λ is an eigenvalue of problem (P_1), then the corresponding $u \in W_0^{1,p(x)}(\Omega)\backslash\{0\}$ is a weak solution of (P_1).

Our main result is given by the following theorem.

Theorem 2.1. *Assume that conditions (h_1) and (h_2) are fulfilled. Then there exists $\lambda_0 > 0$ such that any $\lambda \in (0, \lambda_0)$ is an eigenvalue for problem (P_1).*

Proof. Let E denote the generalized Sobolev space $W_0^{1,p(x)}(\Omega)$. Define the functionals $J, I : E \to \mathbb{R}$ by

$$J(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx \text{ and } I(u) = \int_\Omega \frac{V(x)}{p(x)}|u|^{p(x)}dx.$$

Standard arguments imply that $J, I \in C^1(E, \mathbb{R})$ with

$$\langle J'(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx \text{ and } \langle I'(u), v\rangle = \int_\Omega V(x)|u|^{p(x)-2}uvdx,$$

for any $u, v \in E$. Next, for any $\lambda > 0$, we define the functional associated with problem (P), $\varphi_\lambda : E \to \mathbb{R}$ by

$$\varphi_\lambda(u) = J(u) - \lambda I(u), \quad \forall u \in E.$$

We divide the proof of Theorem 2.1 into three steps.

• **Step 1.** There exists $\lambda_0 > 0$ such that for any $0 < \lambda < \lambda_0$ there exist $\rho, \nu > 0$ such that $\varphi_\lambda(u) \geq \nu$ for any $u \in E$ with $\|u\| = \rho$.

Let $\frac{1}{r(x)} + \frac{1}{r'(x)} = 1$. Assumption (h_1) implies that $r(x) > N$, $\forall x \in \overline{\Omega}$, furthermore, $p(x)r(x) > N$, $\forall x \in \overline{\Omega}$. Thus, $p(x)r'(x) < p^*(x)$, $\forall x \in \overline{\Omega}$ it follows that the embeddings $E \hookrightarrow L^{p(x)r'(x)}(\Omega)$ is compact and continuous. So, there exists a positive constant $c_1 > 0$ such that

$$|u|_{p(x)r'(x)} \leq c_1 \|u\|. \tag{3}$$

We fix $\rho \in (0, 1)$ such that $\rho < \frac{1}{c_1}$. Then relation (3) implies $|u|_{p(x)r'(x)} < 1$, $\forall\ u \in E$ with $\|u\| = \rho$. Taking into account relations (2) and (3) we deduce that for any $u \in E$ with $\|u\| = \rho$ the following inequalities hold true:

$$\begin{aligned}
\varphi_\lambda(u) \geq & \frac{1}{p^+} \int_\Omega |\nabla u|^{p(x)} dx - \frac{\lambda}{p^-} \int_\Omega |V(x)| |u|^{p(x)} dx \\
\geq & \frac{1}{p^+} \|u\|^{p^+} - \frac{\lambda}{p^-} |V|_{r(x)} \big| |u|^{p(x)} \big|_{q'(x)} \\
\geq & \frac{1}{p^+} \|u\|^{p^+} - \frac{\lambda}{p^-} |V|_{r(x)} |u|^{p^-}_{p(x)r'(x)} \\
\geq & \frac{1}{p^+} \|u\|^{p^+} - \frac{\lambda}{p^-} |V|_{r(x)} c_1^{p^-} \|u\|^{p^-} \\
= & \frac{1}{p^+} \rho^{p^+} - \frac{\lambda}{p^-} |V|_{r(x)} c_1^{p^-} \rho^{p^-} \\
= & \rho^{p^-} \Big(\frac{1}{p^+} \rho^{p^+ - p^-} - \frac{\lambda}{p^-} |V|_{r(x)} c_1^{p^-} \Big).
\end{aligned}$$

By the above inequality we remark that if we define $\lambda_0 = \frac{p^- \rho^{p^+ - p^-}}{2 c_1^{p^-} p^+ |V|_{r(x)}}$, then for any $\lambda \in (0, \lambda_0)$ any $u \in E$ with $\|u\| = \rho$ there exists $\nu = \frac{\rho^{p^+}}{2p^+} > 0$ such that $\varphi_\lambda(u) \geq \nu > 0$. The Step 1 is completed.

• **Step 2.** There exists $\eta \in E$ such that $\eta \geq 0$, $\eta \neq 0$ and $\varphi_\lambda(\eta) < 0$, for $t > 0$ small enough.

From (h_2), we may assume that $\overline{U} \subset \Omega_0$, then there is $\varepsilon_1 > 0$ such that $p(x_0) < p(x) - 4\varepsilon_1$ for any $x \in \partial U$, and there is $\varepsilon_2 > 0$ such that

$$p(x_0) < p(x) - 2\varepsilon_1,\ \forall x \in B_{\varepsilon_2}(\partial U), \tag{4}$$

where $B_{\varepsilon_2}(\partial U) = \{x : \exists y \in \partial U \text{ s.t. } |x - y| < \varepsilon_2\}$, and there is $\varepsilon_3 > 0$ such that $B_{\varepsilon_3}(x_0) \subset U \setminus B_{\varepsilon_2}(\partial U)$ and

$$|p(x_0) - p(x)| < \varepsilon_1,\ \forall x \in B_{\varepsilon_3}(x_0). \tag{5}$$

From (4) and (5) it follows that

$$p(x) > p(y) + \varepsilon_1,\ \forall x \in B_{\varepsilon_2}(\partial U), \forall y \in B_{\varepsilon_3}(x_0). \tag{6}$$

Let $\eta \in C_0^\infty(\Omega_0)$ such that $|\nabla \eta(x)| \leq c$, $0 \leq \eta(x) \leq 1$ for any $x \in \Omega_0$, and

$$\eta(x) = \begin{cases} 0,\ x \notin U \cup B_{\varepsilon_2}(\partial U), \\ 1,\ x \in U \setminus B_{\varepsilon_2}(\partial U). \end{cases}$$

Thus, for all $t \in (0, \delta)$ with $\delta = \min\{1, \frac{1}{c}\}$, we have

$$\varphi_\lambda(t\eta) = \int_\Omega \frac{1}{p(x)} |\nabla t\eta|^{p(x)} dx - \lambda \int_\Omega \frac{1}{p(x)} V(x) |t\eta|^{p(x)} dx$$
$$= \int_{B_{\varepsilon_2}(\partial U)} \frac{1}{p(x)} |\nabla t\eta|^{p(x)} dx - \lambda \int_{B_{\varepsilon_3}(x_0)} \frac{1}{p(x)} V(x) |t\eta|^{p(x)} dx$$
$$\leq \frac{1}{p^-} \int_{B_{\varepsilon_2}(\partial U)} |ct|^{p(x)} dx - \frac{\lambda V_0}{p^+} \int_{B_{\varepsilon_3}(x_0)} |t|^{p(x)} dx \leq \frac{1}{p^-} |B_{\varepsilon_2}(\partial U)| (ct)^{p(\xi_1)} - \frac{\lambda V_0}{p^+} |B_{\varepsilon_3}(x_0)| t^{p(\xi_2)},$$

where $\xi_1 \in B_{\varepsilon_2}(\partial U)$, $\xi_2 \in B_{\varepsilon_3}(x_0)$ and $V_0 = \min_{x \in \overline{B}_{\varepsilon_3}(x_0)} V(x) > 0$.

On the other hand, using (6) it follows that $p(\xi_1) > p(\xi_2) + \varepsilon_1 > p(\xi_2)$. Therefore $\varphi_\lambda(t\eta) < 0$ $t > 0$ small enough. The proof of Step 2 is completed.

By Step 1, we have

$$\inf_{v \in \partial B_\rho(0)} \varphi_\lambda(v) > 0. \tag{7}$$

On the other hand, by Step 2, there exists $\eta \in E$ such again $\varphi_\lambda(t\eta) < 0$ for $t > 0$ small enough. Using (2) and (3), we have $\varphi_\lambda(u) \geq \frac{1}{p^+} \|u\|^{p^+} - \lambda c_1^{p^-} |V|_{r(x)} \|u\|^{p^-}$, $\forall u \in B_\rho(0)$. Thus, $-\infty < c_\lambda := \inf_{v \in \overline{B_\rho(0)}} \varphi_\lambda(v) < 0$. Now let ε be such that $0 < \varepsilon < \inf_{v \in \partial B_\rho(0)} \varphi_\lambda(v) - \inf_{v \in B_\rho(0)} \varphi_\lambda(v)$. Then, by applying Ekeland's variational principle to the functional $\varphi_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, there exist $u_\varepsilon \in \overline{B_\rho(0)}$ such that $\varphi_\lambda(u_\varepsilon) \leq \inf_{v \in \overline{B_\rho(0)}} \varphi_\lambda(v) + \varepsilon$, and $\varphi_\lambda(u_\varepsilon) < \varphi_\lambda(u) + \varepsilon \|u - u_\varepsilon\|$, $u \neq u_\varepsilon$. Since $\varphi_\lambda(u_\varepsilon) \leq \inf_{v \in \overline{B_\rho(0)}} \varphi_\lambda(v) + \varepsilon \leq \inf_{v \in B_\rho(0)} \varphi_\lambda(v) + \varepsilon < \inf_{v \in \partial B_\rho(0)} \varphi_\lambda(v)$, we deduce that $u_\varepsilon \in B_\rho(0)$.

Now, we define $T_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $T_\lambda(u) = \varphi_\lambda(u) + \varepsilon \|u - u_\varepsilon\|$. It is clear that u_ε is an minimum of T_λ. Therefore, for small $t > 0$ and $v \in B_1(0)$, we have $\frac{T_\lambda(u_\varepsilon + tv) - T_\lambda(u_\varepsilon)}{t} \geq 0$, which implies that $\frac{\varphi_\lambda(u_\varepsilon + tv) - \varphi_\lambda(u_\varepsilon)}{t} + \varepsilon \|v\| \geq 0$. As $t \to 0$, we have $\langle d\varphi_\lambda(u_\varepsilon) + \varepsilon \|v\| \geq 0$, $\forall v \in B_1(0)$. Hence, $\|\varphi'_\lambda(u_\varepsilon)\|_{E^*} \leq \varepsilon$. We deduce that there exists a sequence $\{u_n\}_n^\infty \subset B_\rho(0)$ such that

$$\varphi_\lambda(u_n) \to c_\lambda \quad \text{and} \quad \varphi'_\lambda(u_n) \to 0. \tag{8}$$

It is clear that $\{u_n\}_n^\infty$ is bounded in E. Thus, there exists $u \in E$ such that, up to a subsequence, $u_n \rightharpoonup u$ in E.

• **Step 3.** We will show that $u_n \to u$ in E.

Let $\alpha(x) = \frac{r(x)p(x)}{r(x) - p(x)}$. Assumption (h_1) implies that $r(x) > N$, $\forall x \in \overline{\Omega}$. Thus, $\alpha(x) < p^*(x)$, $\forall x \in \overline{\Omega}$. Using again the fact that $p(x) < p^*(x)$, $\forall x \in \overline{\Omega}$, we deduce that the embeddings $E \hookrightarrow L^{\alpha(x)}(\Omega)$ and $E \hookrightarrow L^{p(x)}(\Omega)$ are compact and continuous. So, there exists a positive constant $c_2 > 0$ such that $|u|_{p(x)} \leq c_2 \|u\|$, $\forall u \in E$. Thus

$$|u_n|_{p(x)} \leq c_2 \|u_n\| \text{ and } u_n \to u \text{ in } L^{\alpha(x)}(\Omega). \tag{9}$$

The Hölder's type inequality and relation (9) imply

$$\begin{aligned} \Big| \int_\Omega V(x) |u_n|^{p(x)-2} u_n (u_n - u) dx \Big| &\leq |V|_{r(x)} \Big| |u_n|^{p(x)-1} \Big|_{p'(x)} |u_n - u|_{\alpha(x)} \\ \leq& |V|_{r(x)} \big(1 + |u_n|_{p(x)}^{p^+ - 1}\big) |u_n - u|_{\alpha(x)} \leq |V|_{r(x)} \big(1 + c_2^{p^+ - 1} \|u_n\|^{p^+ - 1}\big) |u_n - u|_{\alpha(x)} \\ \leq& |V|_{r(x)} \big(1 + c_2^{p^+ - 1} \rho^{p^+ - 1}\big) |u_n - u|_{\alpha(x)} \to 0, \text{ as } n \to +\infty. \end{aligned} \tag{10}$$

Moreover, since $d\varphi_\lambda(u_n) \to 0$ and $\{u_n\}_n^\infty$ is bounded in E, we have

$$|\langle d\varphi_\lambda(u_n), u_n - u \rangle| \leq |\langle d\varphi_\lambda(u_n), u_n \rangle| + |\langle d\varphi_\lambda(u_n), u \rangle| \leq \|d\varphi_\lambda(u_n)\|_{E^*} \|u_n\| + \|d\varphi_\lambda(u_n)\|_{E^*} \|u\|,$$

that is, $\lim_{n\to+\infty}\langle d\varphi_\lambda(u_n), u_n - u\rangle = 0$. Using (10) and the last relation we deduce that

$$\lim_{n\to+\infty}\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla(u_n - u)dx = 0. \tag{11}$$

From (11) and the fact that $u_n \rightharpoonup u$ in E it follows that $\lim_{n\to\to+\infty}\langle J'(u_n), u_n - u\rangle = 0$, and by Theorem 3.1 in Fan and Zhang [15] we deduce that $u_n \to u$ in E. Thus, in view of (8), we obtain$\varphi_\lambda(u) = c_\lambda < 0$ and $\varphi'_\lambda(u) = 0$. The proof of Theorem 2.1 is completed. □

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