# THE MAIN EQUATION OF INVERSE PROBLEM FOR DIRAC OPERATORS

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*In this paper, the main equation or Gelfand-Levitan-Marchenko type equation of inverse problem for Dirac operators with piecewise continuous coefficient is derived. The uniqueness theorem for inverse spectral problem according to the sequences of eigenvalues and normalized numbers is proved.*

**Keywords:** Dirac operator, main equation, inverse problem.
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## 1. Introduction

The theory of inverse problems for differential operators plays an important role in the development of the spectral theory of linear operators. The inverse spectral problem is the reconstruction of a linear operator from some of its spectral characteristics, such as spectral data, spectral function, spectra (for different boundary conditions), scattering data, Weyl function, etc. According to the spectral characteristic, different inverse problems can be considered. The most comprehensive information on the theory of inverse problems can be found in the books [4, 8, 13].

The direct and inverse problems for Dirac operators have attracted considerable attention in both mathematics and physics. Especially, since Dirac equation is related to nonlinear wave equation (this was discovered in [1, 3]), there has been many investigations based on Dirac equation and the investigations have been continuing to be developed in many directions.

In this paper, our aim is to prove the uniqueness theorem of inverse problems for Dirac operators with discontinuous coefficient according to the sequences of eigenvalues and normalized numbers and give an algorithm to construct the potential function. Then, we consider the following boundary value problem

$$Bu' + Q(x)u = \lambda r(x)u, \quad 0 < x < \pi, \tag{1}$$

$$u_1(0) = (\lambda + h_1)\, u_1(\pi) + h_2 u_2(\pi) = 0, \tag{2}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} q_1(x) & q_2(x) \\ q_2(x) & -q_1(x) \end{pmatrix}, \quad u = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix},$$

the functions $q_1(x) \in L_2(0,\pi)$ and $q_2(x) \in L_2(0,\pi)$ are real valued, $\lambda$ is a spectral parameter,

$$r(x) = \begin{cases} 1, & 0 \le x \le a, \\ \alpha, & a < x \le \pi, \end{cases}$$

$0 < \alpha \neq 1$, $h_1$ and $h_2 > 0$ are real numbers.

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In the finite interval, in the case of $r(x) \equiv 1$ in the equation (**1**.1) and the potential function $Q(x)$ is continuous, the solvability of inverse problem according to two spectra was examined in [5] and according to one spectrum and normalized numbers was given in [2]. The inverse problem contained spectral parameter in boundary condition by spectral function was studied in [9]. Inverse spectral problems for Dirac operator with summable potential were worked in [14, 17]. An algorithm for reconstructing the Dirac operator was given in [10, 12, 15, 20, 21]. The uniqueness theorem of the inverse problem for Dirac operators with spectral parameter in boundary conditions by Weyl function was proved in [11]. Moreover, the works [16, 18, 19] can be examined for the physical applications of Dirac equation.

As different from other works, the problem (**1**.1), (**1**.2) has piecewise continuous coefficient, so the integral representation (not operator transformation) for the solution of equation (**1**.1) obtained in [6] is used. This paper is organized as follows: In section 2, we give an operator formulation of the problem (**1**.1), (**1**.2) and the asymptotic behaviour of eigenvalues, eigenfunctions and normalized numbers of problem (**1**.1), (**1**.2) obtained by using the integral representation. In section 3, Gelfand-Levitan-Marchenko type equation with respect to the kernel of this integral representation is derived and it is obtained that this equation has a unique solution. Then, we prove the uniqueness theorem for the solution of inverse problem by its eigenvalues and normalized numbers. Finally, we give an algorithm to construct the potential function $Q(x)$.

## 2. **Operator Formulation and Some Spectral Properties**

We denote the inner product in Hilbert space $H_r = L_{2,r}(0,\pi;\mathbb{C}^2) \oplus \mathbb{C}$ by

$$\langle U,V\rangle := \int_0^\pi \left[u_1(x)\overline{v}_1(x) + u_2(x)\overline{v}_2(x)\right] r(x)dx + \frac{1}{h_2}u_3\overline{v}_3,$$

where

$$U = (u_1(x),\ u_2(x),\ u_3)^T \in H_r, \qquad V = (v_1(x),\ v_2(x),\ v_3)^T \in H_r.$$

Let us define the operator $L$ by

$$L(U) := \begin{pmatrix} l(u) \\ -h_1u_1(\pi) - h_2u_2(\pi) \end{pmatrix}$$

with the domain

$$D(U) = \left\{ \begin{array}{c} U \mid U = (u_1(x), u_2(x), u_3)^T \in H_r,\ u_1(x),\ u_2(x) \in AC[0,\pi], \\ u_3 = u_1(\pi),\ l(u) \in L_{2,r}(0,\pi;\mathbb{C}^2),\ u_1(0) = 0 \end{array} \right\}$$

where

$$l(u) = \frac{1}{r(x)}\left(Bu' + Q(x)u\right).$$

Thus, the considered problem (**1**.1), (**1**.2) is equivalent the equation $LU = \lambda U$.

Denote by $\phi(x,\lambda)$ and $\vartheta(x,\lambda)$ the solutions of the equation (**1**.1) under the initial conditions

$$\begin{array}{cc} \phi_1(0,\lambda) = 0, & \phi_2(0,\lambda) = -1, \\ \vartheta_1(\pi,\lambda) = h_2, & \vartheta_2(\pi,\lambda) = -\lambda - h_1. \end{array} \tag{3}$$

The integral representation of the solution $\phi(x,\lambda)$ has the form

$$\phi(x,\lambda) = \phi_0(x,\lambda) + \int_0^{\mu(x)} G(x,y)\begin{pmatrix} \sin\lambda y \\ -\cos\lambda y \end{pmatrix} dy, \tag{4}$$

where

$$\phi_0(x,\lambda) = \begin{pmatrix} \sin\lambda\mu(x) \\ -\cos\lambda\mu(x) \end{pmatrix}, \qquad \mu(x) = \begin{cases} x, & 0 \le x \le a, \\ \alpha x - \alpha a + a, & a < x \le \pi, \end{cases}$$

$G_{ij}(x,.) \in L_2(0,\pi),\ i,j=1,2$ for fixed $x \in [0,\pi]$ (see [6, 10]). Moreover, the kernel $G(x,y)$ satisfies the following problem

$$\begin{aligned} BG'_x(x,y)+r(x)G'_y(x,y)B &= -Q(x)G(x,y),\\ Q(x) &= r(x)[G(x,\mu(x))B - BG(x,\mu(x))],\\ G_{11}(x,0) &= G_{21}(x,0)=0. \end{aligned} \tag{5}$$

Here, we specify that the relation (**2**.3) expresses the connection between the kernel $G(x,t)$ and the potential function $Q(x)$ of the equation (**1**.1) and this relation is used to prove the uniqueness theorem for inverse problem.

Define the characteristic function $\omega(\lambda)$ of $L$ by

$$\omega(\lambda) := \phi_2(x,\lambda)\vartheta_1(x,\lambda) - \phi_1(x,\lambda)\vartheta_2(x,\lambda). \tag{6}$$

Then, it follows from (**2**.4) that

$$\omega(\lambda) = -\vartheta_1(0,\lambda) = (\lambda+h_1)\,\phi_1(\pi,\lambda) + h_2\phi_2(\pi,\lambda).$$

**Lemma 2.1.** [11] *The zeros $\lambda_n$ of the characteristic function $\omega(\lambda)$ coincide with the eigenvalues of the problem $L$. The functions $\phi(x,\lambda_n)$ and $\vartheta(x,\lambda_n)$ are eigenfunctions and there exists the sequence $\kappa_n$ such that*

$$\vartheta(x,\lambda_n) = \kappa_n\phi(x,\lambda_n), \quad \kappa_n \neq 0. \tag{7}$$

The normalized numbers are defined by

$$\gamma_n := \int_0^\pi \left(|\phi_1(x,\lambda_n)|^2 + |\phi_2(x,\lambda_n)|^2\right) r(x)dx + \frac{1}{h_2}\,|\phi_1(\pi,\lambda_n)|^2 .$$

**Lemma 2.2.** [11] *The relation*

$$\dot{\omega}(\lambda_n) = \kappa_n\gamma_n, \tag{8}$$

*is valid, where $\dot{\omega}(\lambda) = \frac{d}{d\lambda}\omega(\lambda)$.*

**Remark 2.1.** *The following estimates are obtained by using the integral representation (**2**.2) as $|\lambda| \to \infty$ uniformly in $x \in [0,\pi]$*

$$\begin{aligned} \phi_1(x,\lambda) &= \sin\lambda\mu(x) + O\left(\tfrac{1}{|\lambda|}e^{|Im\lambda|\mu(x)}\right),\\ \phi_2(x,\lambda) &= -\cos\lambda\mu(x) + O\left(\tfrac{1}{|\lambda|}e^{|Im\lambda|\mu(x)}\right). \end{aligned} \tag{9}$$

Let us substitute the estimates (**2**.7) in the characteristic function $\omega(\lambda)$. Then, we find

$$\omega(\lambda) = \lambda\sin\lambda\mu(\pi) + O\left(e^{|Im\lambda|\mu(\pi)}\right), \quad |\lambda| \to \infty. \tag{10}$$

**Theorem 2.1.** *The asymptotic formulas for eigenvalues $\lambda_n$ for $n \in \mathbb{Z}$, eigenfunctions and normalized numbers of boundary value problem (**1**.1), (**1**.2) are as follows, respectively:*

$$\lambda_n = \lambda_n^0 + \epsilon_n, \quad \{\epsilon_n\} \in l_2, \tag{11}$$

$$\phi(x,\lambda_n) = \begin{pmatrix} \sin\frac{n\pi\mu(x)}{\mu(\pi)} \\ -\cos\frac{n\pi\mu(x)}{\mu(\pi)} \end{pmatrix} + \begin{pmatrix} \zeta_n^{(1)}(x) \\ \zeta_n^{(2)}(x) \end{pmatrix}, \tag{12}$$

$$\gamma_n = \mu(\pi) + \tau_n, \quad \{\tau_n\} \in l_2, \tag{13}$$

*where $\lambda_n^0 = \frac{n\pi}{\mu(\pi)}$, $\left\{\zeta_n^{(1)}(x)\right\} \in l_2$ and $\left\{\zeta_n^{(2)}(x)\right\} \in l_2$ for all $x \in [0,\pi]$.*

*Proof.* The proof of this theorem is similarly obtained in [11]. □

Moreover, in consideration of (**2**.8), since the function $\omega(\lambda)$ is entire function, it is obtained from Hadamard's theorem (see [7]) that

$$\omega(\lambda) = -\mu(\pi)(\lambda_0^2 - \lambda^2) \prod_{n=1}^{\infty} \frac{(\lambda_n^2 - \lambda^2)}{(\lambda_n^0)^2}. \tag{14}$$

**Lemma 2.3.** [11] *The eigenfunction expansion formula*

$$g(x) = \sum_{n=-\infty}^{\infty} \beta_n \phi(x, \lambda_n), \qquad \beta_n = \frac{1}{\gamma_n} \langle g(x), \phi(x, \lambda_n) \rangle \tag{15}$$

*holds for the absolutely continuous function $g(x)$, $x \in [0, \pi]$ and the series converges uniformly in $x \in [0, \pi]$.*

## 3. The Uniqueness Theorem for Inverse Problem

In this section, the uniqueness of the solution of inverse problem will be proved by using the Gelfand-Levitan-Marchenko method. In this method, the transformation operator is used and the main role is played by linear integral equation with respect to the kernel of the transformation operator. On the other hand, it should be pointed out that since the equation (**1**.1) has $r(x)$ piecewise continuous coefficient, the solution of this problem forms the integral representation not operator transformation and we use this integral representation for the solution of inverse problem of considered problem (**1**.1), (**1**.2). First of all, we derive the linear integral equation by the kernel of the integral representation and then we show that this equation has a unique solution. Finally, we prove the uniqueness theorem of inverse problem.

Now, we will refer to the sequences $\{\lambda_n\}$ and $\{\gamma_n\}$, $(n \in \mathbb{Z})$ as the spectral data of the boundary value problem (**1**.1), (**1**.2). Consider the functions

$$F_0(x, y) = \sum_{n=-\infty}^{\infty} \left[ \frac{1}{\gamma_n} \begin{pmatrix} \sin \lambda_n x \\ -\cos \lambda_n x \end{pmatrix} \phi_0^T(y, \lambda_n) - \frac{1}{\mu(\pi)} \begin{pmatrix} \sin \lambda_n^0 x \\ -\cos \lambda_n^0 x \end{pmatrix} \phi_0^T(y, \lambda_n^0) \right] \tag{16}$$

and

$$F(x, y) = F_0(\mu(x), y). \tag{17}$$

Then, it is obtained from (**3**.1) and (**3**.2) that

$$F(x, y) = \sum_{n=-\infty}^{\infty} \left[ \frac{1}{\gamma_n} \phi_0(x, \lambda_n) \phi_0^T(y, \lambda_n) - \frac{1}{\mu(\pi)} \phi_0(x, \lambda_n^0) \phi_0^T(y, \lambda_n^0) \right]. \tag{18}$$

**Theorem 3.1.** *The following linear integral equation named by Gelfand-Levitan-Marchenko type equation is satisfied for each fixed $x \in (0, \pi]$ by the kernel $G(x, y)$ of the integral representation (**2**.2):*

$$G(x, \mu(y)) + F(x, y) + \int_0^{\mu(x)} G(x, s) F_0(s, y) ds = 0, \qquad 0 < y < x. \tag{19}$$

*Proof.* It can be written from (**2**.2) that

$$\phi_0(x, \lambda) = \phi(x, \lambda) - \int_0^{\mu(x)} G(x, y) \begin{pmatrix} \sin \lambda y \\ -\cos \lambda y \end{pmatrix} dy. \tag{20}$$

It is obtained from (**2**.2) and (**3**.5) that

$$\sum_{n=-N}^{N} \frac{1}{\gamma_n} \phi(x, \lambda_n) \phi_0^T(y, \lambda_n) = \sum_{n=-N}^{N} \frac{1}{\gamma_n} \phi_0(x, \lambda_n) \phi_0^T(y, \lambda_n) +$$

$$+\int_0^{\mu(x)} G(x,s)\left(\sum_{n=-N}^{N}\frac{1}{\gamma_n}\begin{pmatrix}\sin\lambda_n s\\ -\cos\lambda_n s\end{pmatrix}\phi_0^T(y,\lambda_n)\right)ds,$$

$$\sum_{n=-N}^{N}\frac{1}{\gamma_n}\phi(x,\lambda_n)\phi_0^T(y,\lambda_n)=\sum_{n=-N}^{N}\frac{1}{\gamma_n}\phi(x,\lambda_n)\phi^T(y,\lambda_n)-$$

$$-\sum_{n=-N}^{N}\frac{1}{\gamma_n}\phi(x,\lambda_n)\int_0^{\mu(y)}(\sin\lambda_n s,\ -\cos\lambda_n s)\,G^T(y,s)ds.$$

Then, according to these equalities, we can write

$$\Psi_N(x,y)=\Phi_N(x,y)+\Phi_N'(x,y)+\Phi_N''(x,y)+\Phi_N'''(x,y), \tag{21}$$

where

$$\Psi_N(x,y)=\sum_{n=-N}^{N}\left[\frac{1}{\gamma_n}\phi(x,\lambda_n)\phi^T(y,\lambda_n)-\frac{1}{\mu(\pi)}\phi(x,\lambda_n^0)\phi^T(y,\lambda_n^0)\right],$$

$$\Phi_N(x,y)=\sum_{n=-N}^{N}\left[\frac{1}{\gamma_n}\phi_0(x,\lambda_n)\phi_0^T(y,\lambda_n)-\frac{1}{\mu(\pi)}\phi_0(x,\lambda_n^0)\phi_0^T(y,\lambda_n^0)\right],$$

$$\Phi_N'(x,y)=\int_0^{\mu(x)}G(x,s)\sum_{n=-N}^{N}\left[\frac{1}{\mu(\pi)}\begin{pmatrix}\sin\lambda_n^0 s\\ -\cos\lambda_n^0 s\end{pmatrix}\phi_0^T(y,\lambda_n^0)\right]ds,$$

$$\begin{aligned}\Phi_N''(x,y)\quad=\quad&\int_0^{\mu(x)}G(x,s)\sum_{n=-N}^{N}\left[\frac{1}{\gamma_n}\begin{pmatrix}\sin\lambda_n s\\ -\cos\lambda_n s\end{pmatrix}\phi_0^T(y,\lambda_n)-\right.\\ &\left.-\frac{1}{\mu(\pi)}\begin{pmatrix}\sin\lambda_n^0 s\\ -\cos\lambda_n^0 s\end{pmatrix}\phi_0^T(y,\lambda_n^0)\right]ds,\end{aligned}$$

$$\Phi_N'''(x,y)=\sum_{n=-N}^{N}\frac{1}{\gamma_n}\phi(x,\lambda_n)\int_0^{\mu(y)}(\sin\lambda_n s,\ -\cos\lambda_n s)\,G^T(y,s)ds.$$

Now, let us examine these expressions respectively. Assume that $g(x)$, $x\in[0,\pi]$ is absolutely continuous function. Then, from Lemma **2**.3, it is calculated uniformly with respect to $x\in[0,\pi]$ that

$$\lim_{N\to\infty}\int_0^{\pi}\Psi_N(x,y)g(y)r(y)dy=\sum_{n=-\infty}^{\infty}\beta_n\phi(x,\lambda_n)-\sum_{n=-\infty}^{\infty}\beta_n^0\phi(x,\lambda_n^0)=0. \tag{22}$$

It follows from (**3**.3) that

$$\begin{aligned}&\lim_{N\to\infty}\int_0^{\pi}\Phi_N(x,y)g(y)r(y)dy=\\ &=\lim_{N\to\infty}\int_0^{\pi}\sum_{n=-N}^{N}\left[\frac{1}{\gamma_n}\phi_0(x,\lambda_n)\phi_0^T(y,\lambda_n)-\frac{1}{\mu(\pi)}\phi_0(x,\lambda_n^0)\phi_0^T(y,\lambda_n^0)\right]g(y)r(y)dy\\ &=\int_0^{\pi}F(x,y)g(y)r(y)dy.\end{aligned} \tag{23}$$

It can be written from (**2**.2) that

$$\begin{pmatrix}\sin\lambda s\\ -\cos\lambda s\end{pmatrix}=\begin{cases}\phi_0(s,\lambda), & s<a,\\ \phi_0\left(\frac{s}{\alpha}+a-\frac{a}{\alpha},\lambda\right), & s>a.\end{cases} \tag{24}$$

Using (**2**.13) and (**3**.9), we have

$$\lim_{N\to\infty}\int_0^\pi \Phi_N'(x,y)g(y)r(y)dy=$$

$$=\lim_{N\to\infty}\int_0^\pi\left[\int_0^{\mu(x)}G(x,s)\sum_{n=-N}^{N}\frac{1}{\mu(\pi)}\begin{pmatrix}\sin\lambda_n^0 s\\ -\cos\lambda_n^0 s\end{pmatrix}\phi_0^T(y,\lambda_n^0)ds\right]g(y)r(y)dy$$

$$=\int_0^\pi\left[\int_0^a G(x,s)\sum_{n=-\infty}^{\infty}\frac{1}{\mu(\pi)}\phi_0(s,\lambda_n^0)\phi_0^T(x,\lambda_n^0)ds\right]g(y)r(y)dy$$

$$+\int_0^\pi\left[\int_a^{\alpha x-\alpha a+a}G(x,s)\sum_{n=-\infty}^{\infty}\frac{1}{\mu(\pi)}\phi_0\left(\frac{s}{\alpha}+a-\frac{a}{\alpha},\lambda_n^0\right)\right.$$

$$\left.\times\phi_0^T(x,\lambda_n^0)ds\right]g(y)r(y)dy$$

$$=\int_0^a G(x,s)g(s)ds+\int_a^{\alpha x-\alpha a+a}G(x,s)g\left(\frac{s}{\alpha}+a-\frac{a}{\alpha}\right)ds.$$

Substituting $\frac{s}{\alpha}+a-\frac{a}{\alpha}\to\eta$ and then changing the denotation for integration variables, we get

$$\lim_{N\to\infty}\int_0^\pi \Phi_N'(x,y)g(y)r(y)dy=$$

$$=\int_0^a G(x,s)g(s)ds+\alpha\int_a^x G(x,\alpha\eta-\alpha a+a)g(\eta)d\eta$$

$$=\int_0^a G(x,y)g(y)dy+\alpha\int_a^x G(x,\alpha y-\alpha a+a)g(y)dy$$

$$=\int_0^x G(x,\mu(y))g(y)r(y)dy. \tag{25}$$

It is found that

$$\lim_{N\to\infty}\int_0^\pi \Phi_N''(x,y)g(y)r(y)dy=$$

$$=\lim_{N\to\infty}\int_0^\pi\int_0^{\mu(x)}G(x,s)\sum_{n=-N}^{N}\left[\frac{1}{\gamma_n}\begin{pmatrix}\sin\lambda_n s\\ -\cos\lambda_n s\end{pmatrix}\phi_0^T(y,\lambda_n)-\right.$$

$$\left.-\frac{1}{\mu(\pi)}\begin{pmatrix}\sin\lambda_n^0 s\\ -\cos\lambda_n^0 s\end{pmatrix}\phi_0^T(y,\lambda_n^0)\right]g(y)r(y)dsdy$$

$$=\int_0^\pi\left[\int_0^{\mu(x)}G(x,s)F_0(s,y)ds\right]g(y)r(y)dy. \tag{26}$$

According to the expressions (**2**.5), (**2**.6) and residue theorem, we have

$$\lim_{N\to\infty}\int_0^\pi \Phi_N'''(x,y)g(y)r(y)dy=$$

$$=\lim_{N\to\infty}\int_0^\pi\left[\sum_{n=-N}^{N}\frac{1}{\gamma_n}\phi(x,\lambda_n)\int_0^{\mu(y)}(\sin\lambda_n s,-\cos\lambda_n s)\,G^T(y,s)ds\right]g(y)r(y)dy$$

$$=\lim_{N\to\infty}\int_0^\pi\left[\sum_{n=-N}^{N}\frac{\vartheta(x,\lambda_n)}{\dot\omega(\lambda_n)}\int_0^{\mu(y)}(\sin\lambda_n s,-\cos\lambda_n s)\,G^T(y,s)ds\right]g(y)r(y)dy$$

$$= \lim_{N\to\infty}\int_0^{\pi}\left[\sum_{n=-N}^{N}\underset{\lambda=\lambda_n}{Res}\frac{\vartheta(x,\lambda)}{\omega(\lambda)}\int_0^{\mu(y)}(\sin\lambda s, -\cos\lambda s)\,G^T(y,s)ds\right]g(y)r(y)dy$$

$$= \lim_{N\to\infty}\int_0^{\pi}\left[\frac{1}{2\pi i}\int_{I_N}\frac{\vartheta(x,\lambda)}{\omega(\lambda)}\int_0^{\mu(y)}(\sin\lambda s,\ -\cos\lambda s)\,G^T(y,s)dsd\lambda\right]g(y)r(y)dy$$

$$= \lim_{N\to\infty}\int_0^{\pi}\left[\frac{1}{2\pi i}\int_{I_N}\frac{\vartheta(x,\lambda)}{\omega(\lambda)}e^{|Im\lambda|\mu(y)}\times\right.$$

$$\left.\times e^{-|Im\lambda|\mu(y)}\int_0^{\mu(y)}(\sin\lambda s,\ -\cos\lambda s)\,G^T(y,s)dsd\lambda\right]g(y)r(y)dy, \qquad (27)$$

where $I_N=\left\{\lambda:|\lambda|=\lambda_N^0+\frac{\pi}{2\mu(\pi)}\right\}$, $N$ is sufficiently large number. Since as $|\lambda|\to\infty$, the estimates

$$\vartheta_1(x,\lambda)=h_2\cos\lambda\alpha(\pi-x)-(\lambda+h_1)\sin\lambda\alpha(\pi-x)+O\left(e^{|Im\lambda|\alpha(\pi-x)}\right),$$

$$\vartheta_2(x,\lambda)=-h_2\sin\lambda\alpha(\pi-x)-(\lambda+h_1)\cos\lambda\alpha(\pi-x)+O\left(e^{|Im\lambda|\alpha(\pi-x)}\right),$$

and the following expressions ([13], Lemma 1.3.1)

$$\lim_{|\lambda|\to\infty}\max_{0\le y\le\pi}e^{-|Im\lambda|\mu(y)}\left|\int_0^{\mu(y)}G_{i,1}(y,s)\sin\lambda s ds\right|=0,$$

$$\lim_{|\lambda|\to\infty}\max_{0\le y\le\pi}e^{-|Im\lambda|\mu(y)}\left|\int_0^{\mu(y)}G_{i,2}(y,s)\cos\lambda s ds\right|=0,\quad i=1,2,$$

are valid, it is obtained from (**2**.9) and (**3**.12) that

$$\lim_{N\to\infty}\int_0^{\pi}\Phi_N'''(x,y)g(y)r(y)dy=0. \qquad (28)$$

In this way, it is calculated by using (**3**.6), (**3**.7), (**3**.8), (**3**.10) (**3**.11) and (**3**.13) that

$$\int_0^x G(x,\mu(y))g(y)r(y)dy+\int_0^{\pi}F(x,y)g(y)r(y)dy+$$

$$+\int_0^{\pi}\int_0^{\mu(x)}G(x,s)F_0(s,y)g(y)r(y)dsdy=0.$$

Then, in view of the arbitrariness of $g(x)$, this yields

$$G(x,\mu(y))+F(x,y)+\int_0^{\mu(x)}G(x,s)F_0(s,y)ds=0,\quad 0<y<x.$$

□

**Theorem 3.2.** *The equation (**3**.4) has a unique solution $G(x,.)\in L_2(0,\mu(x))$ for each fixed $x\in(0,\pi]$.*

*Proof.* In the case of $x<a$, due to $r(x)\equiv 1$, this theorem is proved in [12]. Now, assume that $a<x$. Then, the equation (**3**.4) can be rewritten as

$$T_xG(x,.)+K_xG(x,.)=-F(x,.),$$

where

$$(T_xg)(y)=\begin{cases} g(y), & y\le a<x,\\ g(\alpha y-\alpha a+a), & a<y\le x,\end{cases} \qquad (29)$$

$$(K_xg)=\int_0^{\alpha x-\alpha a+a}g(s)F_0(s,y)ds,\quad 0<y<x.$$

Let us obtain that $T_x$ has a bounded inverse in $L_2(0,\pi)$. Suppose that $(T_x g)(y) = \varphi(y)$, $\varphi(y) \in L_2(0,\pi)$ and $\varphi(y) = 0$ for $y > x$. Using this and (**3**.14), we can write

$$g(y) = \left(T_x^{-1}\varphi\right)(y) = \begin{cases} \varphi(y), & y \le a, \\ \varphi\left(\frac{y}{\alpha} + a - \frac{a}{\alpha}\right), & a < y. \end{cases}$$

Then, we calculate

$$\begin{aligned} \|g\|_{L_2} &= \int_0^\pi \left(|g_1(y)|^2 + |g_2(y)|^2\right) dy = \int_0^a \left(|\varphi_1(y)|^2 + |\varphi_2(y)|^2\right) dy + \\ &+ \int_a^\pi \left(\left|\varphi_1\left(\frac{y}{\alpha} + a - \frac{a}{\alpha}\right)\right|^2 + \left|\varphi_2\left(\frac{y}{\alpha} + a - \frac{a}{\alpha}\right)\right|^2\right) dy \\ &= \int_0^a \left(|\varphi_1(y)|^2 + |\varphi_2(y)|^2\right) dy + \alpha \int_a^{\frac{\pi+\alpha a - a}{\alpha}} \left(|\varphi_1(y)|^2 + |\varphi_2(y)|^2\right) dy \\ &\le c \int_0^\pi \left(|\varphi_1(y)|^2 + |\varphi_2(y)|^2\right) dy = c\,\|\varphi\|_{L_2}. \end{aligned}$$

Thus, we have

$$\|g\|_{L_2} = \left\|T_x^{-1}\varphi\right\|_{L_2} \le c\,\|\varphi\|_{L_2}.$$

Therefore, the operator $T_x$ is invertible in $L_2(0,\pi)$ and the main equation (**3**.4) can be expressed as follows

$$G(x,.) + T_x^{-1} K_x G(x,.) = -T_x^{-1} F(x,.),$$

where $T_x^{-1} K_x$ is completely continuous operator in $L_2(0,\pi)$. In that case, it suffices to show that the homogeneous equation

$$t(\mu(y)) + \int_0^{\mu(x)} t(s) F_0(s,y) ds = 0 \tag{30}$$

has only trivial solution $t(y) = 0$. Let $t(y)$ be a non-trivial solution of (**3**.15) and $t(y) = 0$ for $y \in (x,\pi)$. Then, from (**3**.1) and (**3**.15), we have

$$\begin{aligned} &\int_0^x \left(t_1^2(\mu(y)) + t_2^2(\mu(y))\right) r(y) dy + \\ &+ \int_0^x \int_0^{\mu(x)} t(s) \sum_{n=-\infty}^{\infty} \left[\frac{1}{\gamma_n} \begin{pmatrix} \sin\lambda_n s \\ -\cos\lambda_n s \end{pmatrix} \phi_0^T(y,\lambda_n) - \right. \\ &\left. - \frac{1}{\mu(\pi)} \begin{pmatrix} \sin\lambda_n^0 s \\ -\cos\lambda_n^0 s \end{pmatrix} \phi_0^T(y,\lambda_n^0)\right] t^T(\mu(y)) r(y) ds dy = 0. \end{aligned}$$

In this equality, using (**3**.9) we get

$$\begin{aligned} &\int_0^x \left(t_1^2(\mu(y)) + t_2^2(\mu(y))\right) r(y) dy + \\ &+ \int_0^x \int_0^a t(s) \sum_{n=-\infty}^{\infty} \frac{1}{\gamma_n} \phi_0(s,\lambda_n) \phi_0^T(y,\lambda_n) t^T(\mu(y)) r(y) ds dy \\ &- \int_0^x \int_0^a t(s) \sum_{n=-\infty}^{\infty} \frac{1}{\mu(\pi)} \phi_0(s,\lambda_n^0) \phi_0^T(y,\lambda_n^0) t^T(\mu(y)) r(y) ds dy \\ &+ \int_0^x \int_0^{\alpha x - \alpha a + a} t(s) \sum_{n=-\infty}^{\infty} \frac{1}{\gamma_n} \phi_0\left(\frac{s}{\alpha} + a - \frac{a}{\alpha}, \lambda_n\right) \phi_0^T(y,\lambda_n) t^T(\mu(y)) r(y) ds dy \\ &- \int_0^x \int_0^{\alpha x - \alpha a + a} t(s) \sum_{n=-\infty}^{\infty} \frac{1}{\mu(\pi)} \phi_0\left(\frac{s}{\alpha} + a - \frac{a}{\alpha}, \lambda_n^0\right) \times \\ &\times \phi_0^T(y,\lambda_n^0) t^T(\mu(y)) r(y) ds dy = 0. \end{aligned}$$

Substituting $\frac{s}{\alpha}+a-\frac{a}{\alpha}\to s$, we find

$$\int_0^x \left(t_1^2(\mu(y))+t_2^2(\mu(y))\right)r(y)dy+$$

$$+\int_0^x\int_0^a t(s)\sum_{n=-\infty}^{\infty}\frac{1}{\gamma_n}\phi_0(s,\lambda_n)\phi_0^T(y,\lambda_n)t^T(\mu(y))r(y)dsdy$$

$$-\int_0^x\int_0^a t(s)\sum_{n=-\infty}^{\infty}\frac{1}{\mu(\pi)}\phi_0(s,\lambda_n^0)\phi_0^T(y,\lambda_n^0)t^T(\mu(y))r(y)dsdy$$

$$+\alpha\int_0^x\int_a^x t(\alpha s-\alpha a+a)\sum_{n=-\infty}^{\infty}\frac{1}{\gamma_n}\phi_0(s,\lambda_n)\phi_0^T(y,\lambda_n)t^T(\mu(y))r(y)dsdy$$

$$-\alpha\int_0^x\int_a^x t(\alpha s-\alpha a+a)\sum_{n=-\infty}^{\infty}\frac{1}{\mu(\pi)}\phi_0(s,\lambda_n^0)\phi_0^T(y,\lambda_n^0)t^T(\mu(y))r(y)dsdy$$

$$=\int_0^x \left(t_1^2(\mu(y))+t_2^2(\mu(y))\right)r(y)dy$$

$$+\int_0^x\int_0^x t(\mu(s))\sum_{n=-\infty}^{\infty}\frac{1}{\gamma_n}\phi_0(s,\lambda_n)\phi_0^T(y,\lambda_n)t^T(\mu(y))r(y)r(s)dsdy$$

$$-\int_0^x\int_0^x t(\mu(s))\sum_{n=-\infty}^{\infty}\frac{1}{\mu(\pi)}\phi_0(s,\lambda_n^0)\phi_0^T(y,\lambda_n^0)t^T(\mu(y))r(y)r(s)dsdy=0. \quad (31)$$

In the expression (**3**.16), using Parseval equality

$$\|t(\mu(y))\|^2=\sum_{n=-\infty}^{\infty}\frac{1}{\mu(\pi)}\left(\int_0^x t(\mu(y))\phi_0(y,\lambda_n^0)r(y)dy\right)^2$$

we obtain

$$\sum_{n=-\infty}^{\infty}\frac{1}{\gamma_n}\left(\int_0^x t(\mu(y))\phi_0(y,\lambda_n)r(y)dy\right)^2.$$

The system $\{\varphi_0(y,\lambda_n)\}$, $(n\in\mathbb{Z})$ is complete in $L_{2,r}(0,\pi)$ (see [11]), therefore $t(\mu(y))\equiv 0$, namely $(T_xt)(y)=0$. Since $T_x$ has a bounded inverse in $L_2(0,\pi)$, we have $G(x,.)=0$. □

Consequently, the following theorem is proved by using Theorem **3**.1 and Theorem **3**.2:

Assume that $L(Q(x),h_1,h_2)$ and $\tilde{L}(\tilde{Q}(x),\tilde{h}_1,\tilde{h}_2)$ be two boundary value problems and also if a certain symbol $\theta$ denotes an object related to $L$, then the symbol $\tilde{\theta}$ denotes the corresponding object to related $\tilde{L}$.

**Theorem 3.3.** *If $\lambda_n=\tilde{\lambda}_n$, $\gamma_n=\tilde{\gamma}_n$, $(n\in\mathbb{Z})$, then $Q(x)=\tilde{Q}(x)$ a.e. on $(0,\pi)$ and $h_1=\tilde{h}_1$, $h_2=\tilde{h}_2$. Namely, the spectral data $\{\lambda_n,\gamma_n\}$, $(n\in\mathbb{Z})$ uniquely determines the boundary value problem (**1**.1), (**1**.2).*

*Proof.* Considering (**3**.1) and (**3**.2), we have $F_0(x,y)=\tilde{F}_0(x,y)$ and $F(x,y)=\tilde{F}(x,y)$. It is obtained from the main equation (**3**.4) that $G(x,y)=\tilde{G}(x,y)$. The expression (**2**.3) implies that $Q(x)=\tilde{Q}(x)$ a.e. on $(0,\pi)$. Now, according to order of precedence, respectively in the consideration of the equalities (**2**.2), (**2**.12) and (**2**.6), we calculate $\phi(x,\lambda_n)=\tilde{\phi}(x,\lambda_n)$, $\dot{\omega}(\lambda_n)=\dot{\tilde{\omega}}(\lambda_n)$ and $\kappa_n=\tilde{\kappa}_n$. Finally, $h_1=\tilde{h}_1$ and $h_2=\tilde{h}_2$ are obtained by using (**2**.1) and (**2**.5). □

**Algorithm 3.1.** *According to spectral data $\{\lambda_n, \gamma_n\}$, $(n \in \mathbb{Z})$, the construction of the potential function $Q(x)$ is as follows:*

- *From the given numbers $\{\lambda_n, \gamma_n\}$, $(n \in \mathbb{Z})$ construct the functions $F_0(x, y)$ and $F(x, y)$ respectively by the formulas (**3**.1) and (**3**.2),*
- *Find the function $G(x, y)$ by solving the main equation (**3**.4),*
- *Calculate $Q(x)$ by the formula (**2**.3).*

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