SCATTERING OF ELECTROMAGNETIC WAVES ON SQUARE PATCH-TYPE FREQUENCY SELECTIVE SURFACE

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Abstract *In this paper an analytical approach for calculating scattering matrix elements for the case of normal incidence of the plane electromagnetic waves on the square patch-type Frequency Selective Surface (FSS), which is placed at the interface between two dielectric media is proposed. Analytical expressions for the reflection and transmission coefficients are accurate enough for practical purposes as shown by 3D electromagnetic simulation results.*

Keywords: Boundary value problems, Meixner's condition, conformal mapping, electromagnetic wave scattering, frequency selective surface

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1. Introduction

At present, analytical and numerical methods [1–4, 6–10, 12–17] are widely used to solve problems of electromagnetic wave scattering by frequency selective surfaces (FSS).

In this paper an analytical approach for calculating scattering matrix elements for the case of normal incidence of plane electromagnetic waves on a square patch-type FSS from both sides, which is placed at an interface between two dielectric media with different dielectric permittivities, is proposed. Simple analytical expressions for the elements of the scattering matrix are derived under quasi-static assumption.

A comparison of frequency dependencies of the reflection and transmission coefficients calculated analytically by the derived formulae and computed numerically by 3D electromagnetic simulation is carried out. Approximation error estimates of these analytical expressions are given.

2. Analytical solution

2.1. Electromagnetic field

Let us consider electromagnetic oscillations near the patch-type FSS (see Figure 1) excited by two plane waves incident normally from both sides:

$$E_x^{\text{inc}} = \begin{cases} E_1^{\text{inc}} \quad \exp\bigl(ik_1 z - i\omega t\bigr), & z < 0, \\ E_2^{\text{inc}} \quad \exp\bigl(-ik_2 z - i\omega t\bigr), & z > 0, \end{cases} \tag{1a}$$

$$H_y^{\text{inc}} = \begin{cases} \dfrac{E_1^{\text{inc}}}{Z_1} \quad \exp\bigl(ik_1 z - i\omega t\bigr), & z < 0, \\[2ex] -\dfrac{E_2^{\text{inc}}}{Z_2} \quad \exp\bigl(-ik_2 z - i\omega t\bigr), & z > 0, \end{cases} \tag{1b}$$

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where $E_x^{\rm inc}, H_y^{\rm inc}$ are component of electric and magnetic fields of the **incident** waves, respectively; $k_{1,2} = \omega\sqrt{\varepsilon_0\,\varepsilon_{1,2}\,\mu_0}$ are wave numbers, $Z_{1,2} = Z_0/\sqrt{\varepsilon_{1,2}}$ are characteristic impedance, $\varepsilon_{1,2}$ are relative dielectric permittivities of medium 1 ($z<0$) and medium 2 ($z>0$), respectively, separated by the FSS plane ($z=0$); $Z_0 = \sqrt{\mu_0/\varepsilon_0}$; ε_0, μ_0 are absolute dielectric permittivity and absolute magnetic permeability of free space.

The field components of incident waves (1) are homogeneous in the plane of the FSS ($E_x^{\rm inc}, H_y^{\rm inc}$ do not depend on x, y), hence according to Floquet's principle [5] the components E_x, H_y, E_z, H_z of the excited near-field of the FSS are periodic functions with respect to x, y with the period equal to the FSS unit cell size D. Therefore restrict ourself to consideration of the unit cell (Figure 1 (b)). Here D is the unit cell size, w is the square patch width, $D-$ w is the gap between patches.

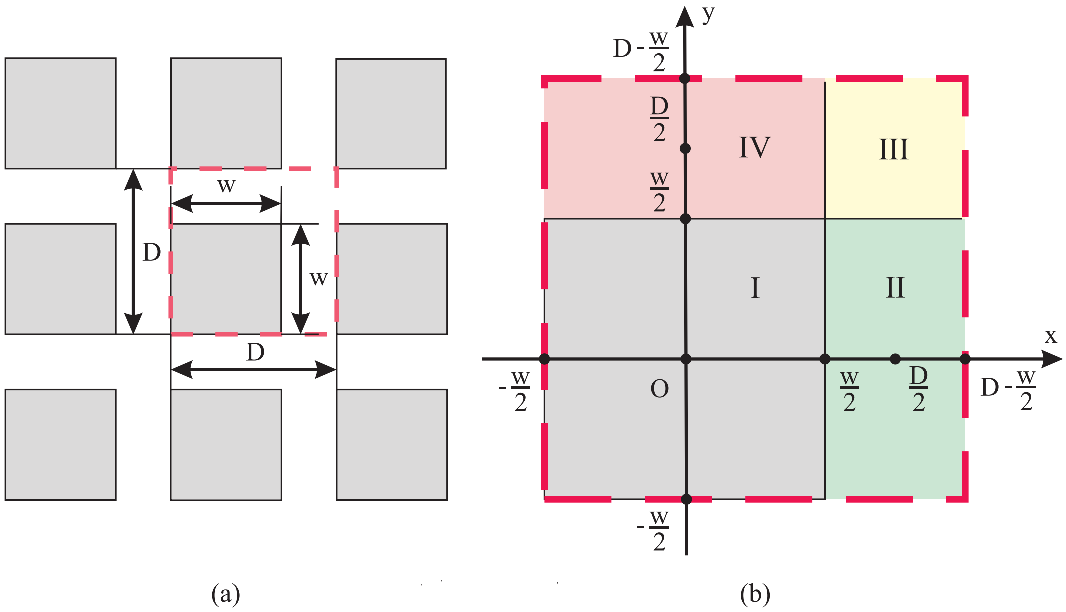

FIGURE 1. Two-dimensional view of patch-type FSS (a) and FSS unit cell (b). Region I for $z=0$ is an infinitesimally thin, perfectly conducting patch, regions II, III, IV – a dielectric medium homogeneous in the plane of the FSS ($z=0$) and in each of the half-spaces: relative dielectric permittivity equals ε_1 for $z<0$, ε_2 for $z \geq 0$.

Under the quasi-static condition ($D \ll \lambda$, where λ is a wavelength) the Helmholtz's equation is approximated by the Laplace equation:

$$\triangle E_x = 0,\ \triangle H_y = 0,\ \triangle E_z = 0,\ \triangle H_z = 0, \tag{2}$$

where $\triangle$ is the three-dimensional Laplace operator.

We find a solution of (2) as a linear combination of two particular solutions. Since any function can be written as a sum of even and odd functions, we will find the component E_x for the first particular solution as an even function with respect to z, for the second one – as an odd function. Other components H_y, E_z, H_z of these particular solutions are either even or odd functions with respect to z.

2.1.1. *The first particular solution.* For the first particular solution of (2) we will find the component E_x as an even function with respect to z. It follows from the time-harmonic

Maxwell's equations that $E_x \propto (\mathbf{rot}\,\mathbf{H})_x$, $E_z \propto (\mathbf{rot}\,\mathbf{H})_z$. Hence, H_y, E_z is odd, H_z is even with respect to z.

Asymptotic behavior of E_x, E_z near edges is defined by the following edge conditions:

$$E_x\Big|_{\substack{x=\mathrm{w}/2+\rho,\\ z=0.}} \propto E_x\Big|_{\substack{x=\mathrm{w}/2+\rho\cos\varphi,\\ z=\rho\sin\varphi,\\ z\neq 0.}} \propto \rho^{-1/2}, \qquad \rho \to +0. \tag{3a}$$

$$E_x\Big|_{\substack{y=\mathrm{w}/2+\rho,\\ z=0.}} \propto E_x\Big|_{\substack{y=\mathrm{w}/2+\rho\cos\varphi,\\ z=\rho\sin\varphi,\\ z\neq 0.}} \propto \rho^{1/2}, \qquad \rho \to +0. \tag{3b}$$

where $\rho > 0$ is the distance between the observation point (x, y, z) and the nearest edge, ρ is sufficiently small; distances to another edges are assumed to be much larger than ρ; φ is the polar angle, $-\pi < \varphi < \pi$, $\varphi \neq 0$; the sign $\propto$ means asymptotic proportionality at $\rho \to +0$ (Θ-notation).

The edge conditions (3) are the strengthening of Meixner's ones [11, 13]. Note that **exact** solution for one-dimensional planar array FSS formed by infinitesimally thin parallel perfectly conducting strips [2] satisfies (3).

It follows from (3) that in the regions I ($z \neq 0$) and II ($\forall z$) (see above 1(b)) near the edges (ρ is sufficiently small): $\left|\dfrac{\partial^2 E_x}{\partial x^2}\right| \gg \left|\dfrac{\partial^2 E_x}{\partial y^2}\right|$. Hence, in the Laplace equation (2) the second derivative with respect to y can be neglected. Therefore, we will obtain an analytical expression for E_x, in the regions I, II under the assumption of independence on y.

At a sufficiently small distance from the edge $x = \mathrm{w}/2$ (so that distances to the another edges $y = \pm\,\mathrm{w}/2$ are much larger than distance to $x = \mathrm{w}/2$) the regions I and II can be approximately regarded as **infinite along the y-axis** (I - an infinitesimally thin perfectly conducting strip, II - a dielectric medium). In this case the incident waves (1) are **E-polarized** and electro-quasi-static approximation of Maxwell equations can be used, i.e. $\mathbf{E} = -\nabla\widetilde{\varphi}$, where $\widetilde{\varphi}$ is electric scalar potential. Due to the symmetry of $\mathbf{E}$ with respect to the planes $x = 0$, $z = 0$ it is sufficient to restrict the computational domain to $\{(x, z): \quad 0 \le x \le D/2,\ z \ge 0\}$. The corresponding Laplace equation for electric potential with boundary conditions is:

$$\triangle\widetilde{\varphi} = 0, \qquad \triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \tag{4a}$$

$$\widetilde{\varphi} = \widetilde{\varphi}_1 \quad \text{for } 0 < x \le \mathrm{w}/2,\ z = 0 \text{ and } x = 0,\ z \ge 0, \tag{4b}$$

$$\widetilde{\varphi} = \widetilde{\varphi}_2 \quad \text{for } x = D/2,\ z \ge 0, \tag{4c}$$

where $\widetilde{\varphi}_1, \widetilde{\varphi}_2$ are some given constants.

The exact solution of (4) obtained using the conformal mapping method by analogy with [2], up to a constant multiplier, is: $\widetilde{\varphi}(x, z) = \Im V_0(z + ix)$, where $V_0(z + ix) = \operatorname{arccosh}\left(\dfrac{\cosh\left(\pi(z + ix)/D\right)}{\cos\alpha}\right)$, $\alpha = \dfrac{\pi \mathrm{w}}{2D}$. Hence,

$$\begin{aligned} E_x^{\mathrm{I,\,II}}(x, z) &= -\frac{\partial}{\partial x}\Im V_0(z + ix) = \frac{\partial}{\partial x}\Im V(x + iz), \qquad z \ge 0, \\ E_z^{\mathrm{I,\,II}}(x, z) &= -\frac{\partial}{\partial z}\Im V_0(z + ix) = \frac{\partial}{\partial z}\Im V(x + iz), \qquad z \ge 0, \end{aligned} \tag{5}$$

since $V_0(z + ix) = \overline{V(x + iz)}$, where

$$V(x + iz) = \operatorname{arccosh}\left(\frac{\cos\left(\pi(x + iz)/D\right)}{\cos\alpha}\right), \qquad \alpha = \frac{\pi \mathrm{w}}{2D}. \tag{6}$$

Since in the first particular solution E_x must be even function with respect to z, we construct an even extension of E_x to the half-space $z < 0$:

$$E_x^{\mathrm{I,\,II}}(x,z) = \mp \frac{\partial}{\partial x}\Im V(x+iz), \qquad \forall z, \tag{7}$$

where the upper sign ("-") corresponds to the half-space $z \le 0$, the lower sign ("+") corresponds to the half-space $z \ge 0$; on the plane $z = 0$ both expressions coincide. Here $\mp$ is due to the oddness of the function $\frac{\partial}{\partial x}\Im V(x+iz)$ with respect to z.

Thus, $E_x^{\mathrm{I,\,II}}$ satisfies the edge condition (3a) and approximately satisfies (due to neglecting the second derivative with respect to y) the Laplace equation in three dimensions (2).

In analogy with (7), we derive from (5) that:

$$E_z^{\mathrm{I,\,II}}(x,z) = \mp \frac{\partial}{\partial z}\Im V(x+iz), \qquad \forall z. \tag{8}$$

The z-component of the time-harmonic Maxwell's equations: $(\mathbf{rot\,H})_z = -i\,\omega\,\varepsilon_0\,\varepsilon_{1,2}\,E_z$, $\forall z$, therefore

$$\frac{\partial H_y^{\mathrm{I,\,II}}}{\partial x} = \pm\, i\,\omega\,\varepsilon_0\,\varepsilon_{1,2}\frac{\partial}{\partial z}\Im V(x+iz).$$

The function $V(x+iz)$ defined by (6) is an analytic function in the regions under consideration ($-\mathrm{w}/2 < x < D - \mathrm{w}/2$, $z < 0$ and $-\mathrm{w}/2 < x < D - \mathrm{w}/2, z > 0$) (in each half-spaces $z < 0$, $z > 0$ separately). Hence, its real and imaginary parts, considered as functions of two real variables, satisfy the Cauchy–Riemann equations in these regions, so that

$$\frac{\partial H_y^{\mathrm{I,\,II}}}{\partial x} = \pm\, i\,\omega\,\varepsilon_0\,\varepsilon_{1,2}\frac{\partial}{\partial x}\Re V(x+iz). \tag{9}$$

Integrating (9) with respect to x yields

$$H_y^{\mathrm{I,\,II}}(x,z) = \pm\, i\,\omega\,\varepsilon_0\,\frac{\varepsilon_1+\varepsilon_2}{2}\Re V(x+iz). \tag{10}$$

Here an additive **function of integration** is defined so that H_y is odd function with respect to z, as the first particular solution requires.

It follows from boundary conditions for the electromagnetic field at the planar interface between two dielectric media that H_y in the regions II, III, IV is continuous across the interface ($z = 0$): $H_y\Big|_{z=-0} = H_y\Big|_{z=+0}$. Continuity and oddness of H_y with respect to z results in

$$H_y^{\mathrm{II,\,III,\,IV}}\Big|_{z=\mp 0} = 0. \tag{11}$$

Note that H_y^{II} defined by (10) satisfies the interface condition (11).

It follows from (11), (10), (6) that

$$\begin{aligned}\langle H_y\rangle &\stackrel{\mathrm{def}}{=} \frac{1}{D^2}\int\limits_{-\mathrm{w}/2}^{D-\mathrm{w}/2}\int\limits_{-\mathrm{w}/2}^{D-\mathrm{w}/2} H_y\Big|_{z=\mp 0}\,dx\,dy = \frac{\mathrm{w}}{D^2}\int\limits_{-\mathrm{w}/2}^{\mathrm{w}/2} H_y^{\mathrm{I}}\Big|_{z=\mp 0}\,dx = \\ &= \mp\, i\,\omega\,\varepsilon_0\,\frac{\varepsilon_1+\varepsilon_2}{2}\,\mathrm{w}/D\ln\cos\alpha,\end{aligned} \tag{12}$$

since $\int\limits_{-\mathrm{w}/2}^{\mathrm{w}/2}\Re V(x+iz)\Big|_{z=\mp 0}dx = \Re\int\limits_{-\mathrm{w}/2}^{\mathrm{w}/2}\operatorname{arccosh}\left(\frac{\cos(\pi x/D)}{\cos\alpha}\right)dx = -D\ln\cos\alpha.$

Let us obtain an analytical expression for E_x in the regions I ($\forall z$), IV ($\forall z$). For this purpose, we first derive analytical expressions for H_y.

Asymptotic behavior of H_y near the edge $y = \mathrm{w}/2$ is defined by:

$$H_y\Big|_{\substack{y=\mathrm{w}/2-\rho,\\ z=0.}} \propto H_y\Big|_{\substack{y=\mathrm{w}/2+\rho\cos\varphi,\\ z=\rho\sin\varphi,\\ z\neq 0.}} \propto \rho^{-1/2}, \qquad \rho \to +0, \tag{13a}$$

and near the edge $x = \mathrm{w}/2$ by:

$$H_y\Big|_{\substack{x=\mathrm{w}/2-\rho,\\ z=0.}} \propto H_y\Big|_{\substack{x=\mathrm{w}/2+\rho\cos\varphi,\\ z=\rho\sin\varphi,\\ z\neq 0.}} \propto \rho^{1/2}, \qquad \rho \to +0. \tag{13b}$$

The edge conditions (13) are the strengthening of Meixner's ones [11, 13]. Note that **exact** solution for one-dimensional planar array FSS formed by infinitesimally thin parallel perfectly conducting strips [2] satisfies (13).

It follows from (13) that in the Laplace equation (2) in the regions I ($\forall z$) and IV ($z \neq 0$) near the edges (ρ is sufficiently small): $\left|\dfrac{\partial^2 H_y}{\partial y^2}\right| \gg \left|\dfrac{\partial^2 H_y}{\partial x^2}\right|$. Hence, dependence of H_y on x is weak (compared with dependence on y) in the regions I, IV ($\forall z$) (here, at $z = 0$, consistently with (11), $H_y^{\mathrm{IV}} = 0$).

At a sufficiently small distance from the edge $y = \mathrm{w}/2$ (so that distances to the another edges $x = \pm\mathrm{w}/2$ are much larger than distance to $y = \mathrm{w}/2$) the regions I and IV can be approximately regarded as **infinite along the x-axis** (I - an infinitesimally thin perfectly conducting strip, II - a dielectric medium).

In this case the incident waves (1) are **H-polarized** and magneto-quasi-static approximation of Maxwell equations can be used, i.e. $\mathbf{H} = -\nabla\psi$, where ψ is magnetic scalar potential. Due to the symmetry of $\mathbf{H}$ with respect to the planes $y = 0$, $z = 0$ it is sufficient to restrict the computational domain to $\{(y, z): \quad 0 \le y \le D/2,\ z \ge 0\}$. The corresponding Laplace equation for magnetic potential with boundary conditions is:

$$\triangle\psi = 0, \qquad \triangle = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \tag{14a}$$

$$\psi = \psi_1 \quad \text{for } y = 0,\ z \ge 0, \tag{14b}$$

$$\psi = \psi_2 \quad \text{for } \mathrm{w}/2 \le y < D/2,\ z = 0 \text{ and } y = D/2,\ z \ge 0, \tag{14c}$$

where ψ_1, ψ_2 are some given constants.

The exact solution of (14) obtained using the conformal mapping method by analogy with [2], up to a constant multiplier, is:

$$\psi(y, z) = \Im\Psi_0(z + iy),$$

where $\Psi_0(z + iy) = \operatorname{arcsinh}\left(\dfrac{\sinh(\pi(z + iy)/D)}{\sin\alpha}\right), \quad \alpha = \dfrac{\pi\mathrm{w}}{2D}$. Hence,

$$H_y^{\mathrm{I,IV}}(y, z) = -\frac{\partial}{\partial y}\Im\Psi_0(z + iy) = -\frac{\partial}{\partial y}\Re\Psi(y + iz), \qquad z \ge 0, \tag{15}$$

since $\Psi_0(z + iy) = i\,\overline{\Psi(y + iz)}$, where

$$\Psi(y + iz) = \arcsin\left(\frac{\sin(\pi(y + iz)/D)}{\sin\alpha}\right), \qquad \alpha = \frac{\pi\mathrm{w}}{2D}, \tag{16}$$

Thus, $H_y^{\mathrm{I,IV}}(y, z)$ defined by (15) satisfies only the edge condition (13a) and **exactly** satisfies the corresponding **two**-dimensional Laplace equation ((2) without the second derivative with respect to x).

As we have remarked, dependence of H_y on x is weak in the regions I ($\forall z$), IV ($\forall z$), hence H_y can be constructed by combining (15) and (10) in a manner similar to [3].

$$H_y(x,y,z) = C\Re V(x)\frac{\partial}{\partial y}\Re\Psi(y+iz), \qquad z \geq 0,$$

where C is an undetermined constant multiplier. Since in the first particular solution H_y must be odd function with respect to z, we construct an odd extension of H_y to the half-space $z < 0$:

$$H_y(x,y,z) = \pm C\Re V(x)\frac{\partial}{\partial y}\Re\Psi(y+iz), \qquad \forall z. \tag{17}$$

Here $\pm$ is due to the evenness of the function $\frac{\partial}{\partial y}\Re\Psi(y+iz)$ with respect to z. Furthermore, $H_y(x,y,z)$ satisfies the condition (11).

Thus, $H_y(x,y,z)$ defined by (17) in the regions I ($\forall z$), IV ($\forall z$) is valid also in II ($z=0$), III ($z=0$) consistently with (11). Here, $H_y(x,y,z)$, in contrast to $H_y^{\mathrm{I,IV}}(y,z)$, satisfies both (13a) and (13b) edge conditions and approximately satisfies (due to the smallness of the second derivative with respect to x) the Laplace equation in **three** dimensions (2).

To determine the constant C, calculate $\langle H_y\rangle$ again:

$$\langle H_y\rangle = \pm C\frac{1}{D^2}\Re\Psi(y\mp i\,0)\Bigg|_{-\mathrm{w}/2}^{\mathrm{w}/2} \cdot \int\limits_{-\mathrm{w}/2}^{\mathrm{w}/2}\Re V(x)\,dx = \mp C\frac{\pi}{D}\ln\cos\alpha, \tag{18}$$

since $\int\limits_{-\mathrm{w}/2}^{\mathrm{w}/2}\Re V(x)\,dx = -D\ln\cos\alpha,\quad \Re\Psi(y\mp i\,0)\Bigg|_{-\mathrm{w}/2}^{\mathrm{w}/2} = \Re\big[\arcsin(1\mp i\,0)-$
$-\arcsin(-1\mp i\,0)\big] = \pi$. Comparing (18) with (12), we obtain

$C = i\,\omega\,\varepsilon_0\,\dfrac{\varepsilon_1+\varepsilon_2}{2}\dfrac{\mathrm{w}}{\pi}$, then (17) takes the form:

$$H_y(x,y,z) = \pm i\,\omega\,\varepsilon_0\,\frac{\varepsilon_1+\varepsilon_2}{2}\frac{\mathrm{w}}{\pi}\Re V(x)\frac{\partial}{\partial y}\Re\Psi(y+iz). \tag{19}$$

The y-component of the time-harmonic Maxwell's equations: $(\mathbf{rot\,E})_y = = i\,\omega\mu_0 H_y$, $\forall z$, therefore

$$\frac{\partial E_x}{\partial z} = \mp\omega^2\,\varepsilon_0\,\frac{\varepsilon_1+\varepsilon_2}{2}\mu_0\frac{\mathrm{w}}{\pi}\Re V(x)\frac{\partial}{\partial y}\Re\Psi(y+iz).$$

The function $\Psi(y+iz)$ defined by (16) is an analytic function in the regions under consideration. Hence, its real and imaginary parts, considered as functions of two real variables, satisfy the Cauchy–Riemann equations, so that

$$\frac{\partial E_x}{\partial z} = \mp\omega^2\,\varepsilon_0\,\frac{\varepsilon_1+\varepsilon_2}{2}\mu_0\frac{\mathrm{w}}{\pi}\Re V(x)\frac{\partial}{\partial z}\Im\Psi(y+iz). \tag{20}$$

Integrating (20) with respect to z yields in the regions I ($\forall z$), IV ($\forall z$)

$$E_x(x,y,z) = \mp\omega^2\,\varepsilon_0\,\frac{\varepsilon_1+\varepsilon_2}{2}\mu_0\frac{\mathrm{w}}{\pi}\Re V(x)\Im\Psi(y+iz). \tag{21}$$

Here an additive function of integration is defined from interface conditions for the electromagnetic field.

It follows from (7), (21) that

$$\langle E_x\rangle \overset{\text{def}}{=} \frac{1}{D^2}\int\limits_{-\mathrm{w}/2}^{D-\mathrm{w}/2}\int\limits_{-\mathrm{w}/2}^{D-\mathrm{w}/2} E_x\Big|_{z=\mp 0} dx\,dy = \frac{\mathrm{w}}{D^2}\int\limits_{\mathrm{w}/2}^{D-\mathrm{w}/2} E_x^{\mathrm{II}}(x,z)\Big|_{z=\mp 0} dx + \tag{22}$$

$$+\frac{1}{D^2}\int\limits_{\mathrm{w}/2}^{D-\mathrm{w}/2}\int\limits_{-\mathrm{w}/2}^{\mathrm{w}/2} E_x(x,y,z)\Big|_{z=\mp 0} dx\,dy = -\frac{\pi \mathrm{w}}{D^2} + \omega^2\varepsilon_0\frac{\varepsilon_1+\varepsilon_2}{2}\mu_0\frac{\mathrm{w}}{\pi}\ln\cos\alpha\cdot\ln\sin\alpha,$$

since $\displaystyle\int\limits_{\mathrm{w}/2}^{D-\mathrm{w}/2} E_x^{\mathrm{II}}(x,z)\Big|_{z=\mp 0} dx = \mp\left[\Im V(D-\mathrm{w}/2 \mp i\,0) - \Im V(\mathrm{w}/2 \mp i\,0)\right] =$

$= \mp\,\Im\left[\operatorname{arccosh}(-1 \pm i\,0) - \operatorname{arccosh}(1 \pm i\,0)\right] = -\pi,$

$$\int\limits_{\mathrm{w}/2}^{D-\mathrm{w}/2} \Im\Psi(y+iz)\Big|_{z=\mp 0} dy = \mp\int\limits_{\mathrm{w}/2}^{D-\mathrm{w}/2} \operatorname{arccosh}\left(\frac{\sin(\pi y/D)}{\sin\alpha}\right) dy = \pm D\ln\sin\alpha,$$

2.1.2. *The second particular solution.* Components of the second particular solution of (2) are supplementary to ones of the first particular solution, i.e. E_x, H_z are odd functions, H_y, E_z are even functions with respect to z. The simplest nontrivial solution of (2) satisfying these requirements is

$$E_x = 0,\quad H_y = H_0,\quad E_z = 0,\quad H_z = 0, \tag{23}$$

where $H_0 \neq 0$ is some constant.

Thus, electric and magnetic field components $\widetilde{E}_x, \widetilde{H}_y$, which are a linear combination of the first and second particular solutions with the coefficients c_1, c_2, have the following average values:

$$\langle\widetilde{E}_x\rangle = c_1\langle E_x\rangle,\qquad \langle\widetilde{H}_y\rangle = c_1\langle H_y\rangle + c_2 H_0, \tag{24}$$

respectively, where $\langle E_x\rangle, \langle H_y\rangle$ are defined by (22), (12).

2.2. Scattering matrix

The considered patch-type FSS can be modelled as a two-port network: some domains to the left ($z < 0$) and right ($z > 0$) of the FSS plane ($z = 0$) are viewed as port 1 and 2, respectively. The relationship between the incident and scattered waves is described by scattering matrix $\mathbf{S}$:

$$\mathbf{b} = \mathbf{S}\,\mathbf{a}, \tag{25}$$

where $\mathbf{a} = (a_1, a_2)^\intercal$, $\mathbf{b} = (b_1, b_2)^\intercal$.

Here a_1, a_2 are the normalized complex amplitudes of the incident waves and b_1, b_2 are ones of the scattered waves at port 1 and 2, respectively [6]:

$$a_{1,2} = E_{1,2}^{\mathrm{inc}}/\sqrt{Z_{1,2}}\,,\qquad b_{1,2} = E_{1,2}^{\mathrm{sct}}/\sqrt{Z_{1,2}}\,, \tag{26a}$$

where $E_{1,2}^{\mathrm{inc}}$ are the complex amplitudes of the electric fields of the incident waves (1a), $E_{1,2}^{\mathrm{sct}}$ are the complex amplitudes of the electric fields of the scattered waves at port 1 and 2, respectively.

It follows from (26a) that

$$H_{1,2}^{\mathrm{inc}} = \pm\, a_{1,2}/\sqrt{Z_{1,2}}\,,\qquad H_{1,2}^{\mathrm{sct}} = \mp\, b_{1,2}/\sqrt{Z_{1,2}}, \tag{26b}$$

where the upper and lower signs correspond to the port 1 and 2, respectively; $H^{\rm inc}_{1,2}$, $H^{\rm sct}_{1,2}$ are the complex amplitudes of the magnetic fields of the incident (1b) and scattered waves, respectively.

Boundary conditions for the electromagnetic field at ports 1 and 2 are of the form

$$\begin{aligned} E^{\rm inc}_{1,2} + E^{\rm sct}_{1,2} &= \langle \widetilde{E}_x \rangle, \\ H^{\rm inc}_{1,2} + H^{\rm sct}_{1,2} &= \langle \widetilde{H}_y \rangle, \end{aligned} \tag{27}$$

where $\langle \widetilde{E}_x \rangle, \langle \widetilde{H}_y \rangle$ are defined by (24).

Substituting (24), (26) into (27) gives

$$\begin{aligned} a_{1,2}\sqrt{Z_{1,2}} \quad + b_{1,2}\sqrt{Z_{1,2}} \quad &= c_1 \langle E_x \rangle \\ \pm a_{1,2}/\sqrt{Z_{1,2}} \mp b_{1,2}/\sqrt{Z_{1,2}} &= c_1 \langle H_y \rangle + c_2 H_0, \end{aligned} \tag{28}$$

where $\langle E_x \rangle, \langle H_y \rangle$ are defined by (22), (12).

Let us rewrite the system of linear algebraic equations (28) in the form (25). Denote $Z = -1/2 \ \langle E_x \rangle / \langle H_y \rangle$, where $\langle E_x \rangle$, $\langle H_y \rangle$ are evaluated, for the sake of definiteness, at port 1. Then

$$\begin{aligned} b_1 &= \frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2} - Z_0/Z}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2} + Z_0/Z}\, a_1 + \frac{2\sqrt[4]{\varepsilon_1 \varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2} + Z_0/Z}\, a_2, \\ b_2 &= \frac{2\sqrt[4]{\varepsilon_1 \varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2} + Z_0/Z}\, a_1 + \frac{\sqrt{\varepsilon_2} - \sqrt{\varepsilon_1} - Z_0/Z}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2} + Z_0/Z}\, a_2. \end{aligned} \tag{29}$$

Substituting $Z = i\,X$ into (29) and using (22), (12), we obtain

$$\mathbf{S} = \frac{1}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2} - iZ_0/X} \begin{pmatrix} \sqrt{\varepsilon_1} - \sqrt{\varepsilon_2} + iZ_0/X & 2\sqrt[4]{\varepsilon_1 \varepsilon_2} \\ 2\sqrt[4]{\varepsilon_1 \varepsilon_2} & \sqrt{\varepsilon_2} - \sqrt{\varepsilon_1} + iZ_0/X \end{pmatrix}, \tag{30a}$$

$$X = \omega \mu_0 \frac{D}{2\pi} \ln \sin\alpha - \frac{\pi}{\omega \varepsilon_0 (\varepsilon_1 + \varepsilon_2) D} \frac{1}{\ln \cos\alpha}, \tag{30b}$$

where $\alpha = \dfrac{\pi \mathrm{w}}{2D}$, $Z_0 = \sqrt{\mu_0/\varepsilon_0}$.

Electromagnetic wave propagation can be described using a transmission line equivalent circuit model. Then Z is the normalized electrical impedance of the unit cell of the patch-type FSS, X is the normalized electrical reactance, the first term in (30b) is inductive reactance, the second one is capacitive reactance.

3. Numerical solution and comparison of the results

Let us estimate the approximation error of the formulae derived in this paper. For this purpose, the absolute values of the reflection coefficient S_{11} and the transmission coefficient S_{21} calculated by the analytical expression (30), have been compared with ones computed numerically with high accuracy by 3D electromagnetic simulation with CST MWS. Note that $S_{21} = S_{12}$ and $|S_{11}| = |S_{22}|$. Figure 2 shows frequency dependencies of $|S_{11}|, |S_{21}|$ of the electromagnetic waves incident normally on the patch-type FSS with the fixed period D and the variable relative width of the patch w/D.

The reflection coefficients increase and the transmission coefficients decrease with increasing the relative width of the patch, as displayed in Figure 2.

Figure 3 shows frequency dependencies of $|S_{11}|$ of the electromagnetic waves incident normally on the patch-type FSS with the fixed gap between the patches $D - \mathrm{w}$ and the variable period D.

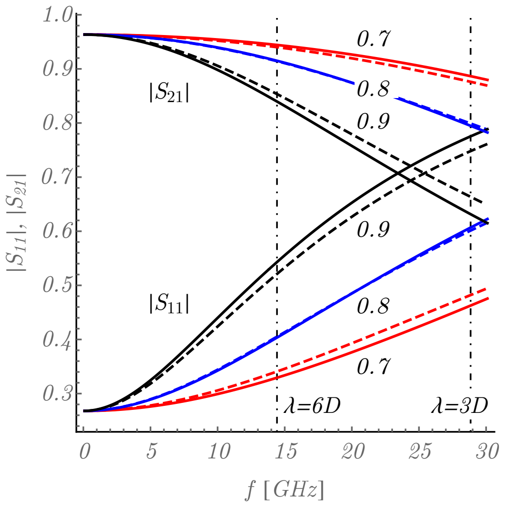

FIGURE 2. Frequency dependencies of the coefficients $|S_{11}|, |S_{21}|$ for D=2 mm, $\mathrm{w}/D = 0.7, 0.8, 0.9$ and $\varepsilon_1 = 1$, $\varepsilon_2 = 3$. Dashed lines correspond to the analytical solution, solid lines represent a numerical solution.

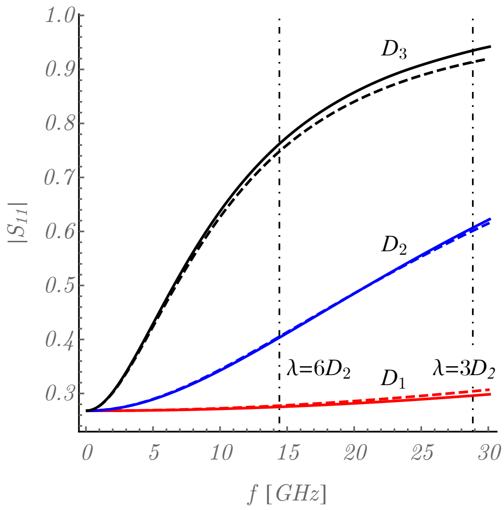

FIGURE 3. Frequency dependencies of the coefficient $|S_{11}|$ for $D - \mathrm{w} = 0.4$, $D_1 = 1$ mm, $D_2 = 2$ mm, $D_3 = 4$ mm and $\varepsilon_1 = 1$, $\varepsilon_2 = 3$. Dashed lines correspond to the analytical solution, solid lines represent a numerical solution.

The reflection coefficients increase with increasing the patch-type FSS period for the fixed gap between the patches, as displayed in Figure 3.

Recall that the approximate analytical solution has been obtained under $\lambda \gg D$, therefore, as expected, the approximation error for $\lambda = 6D$ is smaller than for $\lambda = 3D$, as shown in Figures 2, 3. Here λ is the wavelength in the second medium ($\varepsilon_2 = 3$).

TABLE 1. Frequency and wavelength dependencies of the relative deviations δS_{11}, δS_{21} for $D = 2$ mm, $\mathrm{w} = 0.8D$, $\varepsilon_1 = 1$, $\varepsilon_2 = 3$; here λ is for $\varepsilon_2 = 3$.

Frequency [GHz]	Wavelength [mm]	δS_{11}	δS_{21}
2	86.5	4.1×10^{-4}	3.2×10^{-5}
4	43.3	1.4×10^{-3}	1.2×10^{-4}
6	28.8	2.7×10^{-3}	2.6×10^{-4}
8	21.6	3.8×10^{-3}	4.3×10^{-4}
10	17.3	4.5×10^{-3}	6.0×10^{-4}
12	14.4	4.7×10^{-3}	7.4×10^{-4}
14	12.4	4.3×10^{-3}	8.0×10^{-4}
16	10.8	3.4×10^{-3}	7.5×10^{-4}

Table 1 presents the estimation of the relative approximation error of the analytical expressions for different frequencies. The relative deviations δS_{11}, δS_{21} of the reflection and transmission coefficients S_{11}, S_{21} calculated by the analytical expression (30) from S_{11}^{CST}, S_{21}^{CST} obtained with CST MWS. The relative deviations δS_{11} and δS_{21} do not exceed 0.5 % and 0.08 %, respectively, in the frequency range up to $\widehat{f} = 16$ GHz, i.e. in the wavelength range down to 10.8 mm ($\lambda > 6D$), as shown in Table 1.

4. **Conclusions**

Thus, in this paper the simple, but quite accurate analytical expressions for the elements of the scattering matrix have been derived under the quasi-static assumption for the case of normal incidence of the plane electromagnetic waves on the square patch-type FSS from both sides, which is placed at the interface between two dielectric media with the different dielectric permittivities.

The comparison of frequency dependencies of the reflection and transmission coefficients calculated analytically by the derived formulae and computed numerically with high accuracy by 3D electromagnetic simulation with CST MWS has shown good agreement between both approaches. Numerical results have demonstrated that the formulae obtained in this paper are accurate enough for practical purposes in their applicability domain.

The derived analytical expressions can be used in design of multi-layer patch-type FSS structures. They can help to **analytically** optimize the FSS structure parameters and hence avoid extensive numerical simulations, and therefore reduce computational costs. Desired reflective properties of such structures can be achieved by varying both the relative width of the patch w/D and the FSS period D.

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