MIZOGUCHI-TAKAHASHI TYPE FIXED POINT THEOREM IN MODULAR FUNCTION SPACES

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This study aims to extend and refine the Mizoguchi-Takahashi's fixed point theorem in the framework of modular function spaces. Additionally, illustrative examples are provided to demonstrate the applicability of the main result.

Keywords: Reich's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem, modular function spaces.

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1. Introduction

The sequence spaces l^p, introduced in the early 20th century, provided a framework to measure and classify the behavior of sequences using the notion of p-norms. Extending this concept to function spaces, Lebesgue spaces L^p emerged, offering a powerful generalization that allowed a unifying treatment of integrable functions.

The classical Lebesgue spaces L^p were first generalized by Orlicz and Birnbaum in [1, 2, 3] in relation to orthogonal expansions. The development of Orlicz spaces was inspired by their relevance to differential and integral equations, particularly those involving kernels with growth patterns that deviate from standard power functions.

Orlicz spaces have been generalized in many directions. In 1955, Luxemburg introduced a more general class of function spaces in his Ph.D. thesis [4], which was followed by a series of papers by Luxemburg-Zaanen [5]. In 1950, Nakano [6] developed the concept of modular spaces connected to ordered spaces. This method involves substituting the specific integral representation of the nonlinear functional with a more flexible abstract functional that possesses certain properties. In 1959, this generalization was further developed by Musielak-Orlicz [7, 8] and Musielak [9]. We now present some key notions from modular theory (see Kozlowski [10]).

Definition 1.1 ([10, Preliminaries, page 88])**.** *A functional $\varrho : \mathcal{V} \to [0, \infty]$ defined on a vector space is called a* **pseudomodular** *if the following conditions hold true for arbitrary $u, v \in \mathcal{V}$:*

(A) $\varrho(0) = 0$;

(B) $\varrho(cu) = \varrho(u)$*, for every $c \in K$ ($K = \mathbb{C}$ or $K = \mathbb{R}$) such that $|c| = 1$;*

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(C) $\varrho(cu + dv) \leq \varrho(u) + \varrho(v)$ *for every* $c, d \geq 0$, $c + d = 1$.

If instead of (C) there holds

(C′) $\varrho(cu + dv) \leq c^s \varrho(u) + d^s \varrho(v)$, *for every* $c, d \geq 0$, $c^s + d^s = 1$, $s \in (0, 1]$, *then* ϱ *is called* ***s-convex***; *when* $s = 1$, *it is* ***convex***.

Replacing condition (A) with

(A′) $\varrho(0) = 0$ *and* $\varrho(\lambda u) = 0$ *for every* $\lambda > 0$ *implies* $u = 0$, *then* ϱ *is called a* ***semimodular***.

Additionally, if

(A″) $\varrho(0) = 0$ *and* $\varrho(u) = 0$ *implies* $u = 0$, *then* ϱ *is called a* ***modular***.

Definition 1.2 ([9, Definition 1.4]). *If* ϱ *is a pseudomodular in* $\mathcal{V}$, *then the corresponding* ***modular space*** *is defined as*

$$\mathcal{V}_\varrho = \{u \in \mathcal{V} : \lim_{\lambda \to 0} \varrho(\lambda u) = 0\}.$$

Remark 1.1. The modular ϱ lacks subadditivity and therefore does not exhibit the characteristics of a norm or a distance.

We will require also the following notions.

Definition 1.3 ([9, page 2]). *A functional* $\|\cdot\| : \mathcal{V} \to [0, \infty]$ *is said to define an F-pseudonorm if*

(i) $\|0\| = 0$.

(ii) *For* $c \in K$ *with* $|c| = 1$ *one has* $\|cu\| = \|u\|$, $\forall u \in \mathcal{V}$.

(iii) $\|u + v\| \leq \|u\| + \|v\|$, $\forall u, v \in \mathcal{V}$.

(iv) *If* $c_k \to c$ *and* $\|u_k - u\| \to 0$, *then* $\|c_k u_k - cu\| \to 0$.

If, moreover

(i') $\|u\| = 0$ *implies* $u = 0$, *then* $\|\cdot\|$ *is called an* F*-norm.*

If $\|\cdot\|$ *satisfies the above conditions* $(i) - (iii)$ *and the condition*

(iv') $\|\alpha u\| = |\alpha|^s \|u\|$, *for* $0 < s \leq 1$, *then* $\|\cdot\|$ *is called an s-pseudonorm and adding* (i') *we obtain an s-norm in* $\mathcal{V}$ *which is denoted* $\|\cdot\|^s$. *If* $s = 1$, *then* $\|\cdot\|$ *is called a norm.*

Remark 1.2 ([10, Preliminaries, page 88]). If ϱ is modular a (pseudomodular) on $\mathcal{V}$, then

$$\|u\|_\varrho = \inf\{\alpha > 0 : \varrho(u/\alpha) \leq \alpha\}$$

is an F-norm (F-pseudonorm) on $\mathcal{V}_\varrho$.

If ϱ is an s-convex modular (s-convex pseudomodular) on $\mathcal{V}$, then the functional

$$\|u\|_\varrho^s = \inf\{\alpha > 0 : \varrho(u/\alpha^{1/s}) \leq 1\}$$

defines an s-norm (s-pseudonorm) on $\mathcal{V}_\varrho$(a norm (pseudonorm) for $s = 1$, for which one denotes $\|u\|_\varrho^1 = \|u\|_\varrho$).

In 1959, modular theory was further generalized to concepts in connection to Musielak-Orlicz spaces in [7], i.e., Orlicz spaces with a function ϕ depending on a parameter. This led to further generalizations of Orlicz spaces.

Numerous challenges linked to metric fixed point theory can be reexamined and tackled using the concept of modular spaces, even without a conventional metric framework. A landmark contribution in this direction was made by Khamsi [15], who introduced various results on fixed points for single-valued mappings within modular function spaces.

On the other side, in 1969 Nadler [11] extended the Banach contraction principle to multivalued mappings in metric spaces. Building on this foundational result, Reich [12] extended the contractiveness condition. Specifically, he replaced the constant contraction coefficient with a function $\theta : (0,\infty) \to [0,1)$, requiring that the inequality $H(T\xi, T\eta) \leq \theta(d(\xi,\eta))d(\xi,\eta)$ holds for each ξ, $\eta \in \mathcal{X}$. Additionally, θ was subject to the condition $\limsup_{j\to z^+} \theta(j) < 1$ for each $z \in (0,\infty)$, which allowed a broader class of mappings to satisfy the fixed point criterion.

Furthermore, Reich [12] questioned whether the range of the mapping T could be extended from $K(\mathcal{X})$ to $CB(\mathcal{X})$ or $CL(\mathcal{X})$. This conjecture was addressed affirmatively by Mizoguchi-Takahashi [13], who proved the validity of the result, including also the limit case $z = 0$. Their work revealed that if $T : \mathcal{X} \to CB(\mathcal{X})$ satisfies the condition $H(T\xi, T\eta) \leq \theta(d(\xi,\eta))d(\xi,\eta)$ with a function θ ensuring $\limsup_{j\to z^+} \theta(j) < 1$ for any $z \in [0,\infty)$, then T possesses a fixed point.

Kamran [14] broadened the findings of Mizoguchi-Takahashi [13], applying the concept of T-orbitally lower semi-continuous mappings to closed multi-valued operators. Kamran [14] showed that, if for a mapping T from $\mathcal{X}$ to $CL(\mathcal{X})$ the inequality $d(\eta, T\eta) \leq \theta(d(\xi,\eta))d(\xi,\eta)$ holds true for all $\xi \in \mathcal{X}$ and $\eta \in T\xi$, where $\theta : (0,\infty) \to [0,1)$ is a function that satisfies $\limsup_{j\to z^+} \theta(j) < 1$ for each $z \in [0,\infty)$, then certain structural properties of the mapping guarantee the existence of an orbit $\{\xi_n\}$ that converges to a fixed point $\zeta \in \mathcal{X}$. Moreover, if the fixed point ζ exists, then the function $\varphi(\xi) = d(\xi, T\xi)$ is T-orbitally lower semi-continuous at ζ, and conversely.

Our primary aim is to extend these fixed point results within the framework of modular function spaces. Modular function spaces, which generalize normed spaces, provide a more flexible framework for studying fixed point theory by replacing norms with modulars. This generalized framework allows us to address a broader class of problems, including those where traditional methods may not apply. Here are some notions from modular function spaces. For further information, see also [15, 16, 17, 18, 19].

2. Preliminaries

Consider a set $\Omega \neq \emptyset$, a nontrivial σ-algebra Σ and a nontrivial σ-ring $\mathscr{P}$ of subsets of Ω. Let $A \cap B \in \mathscr{P}$ for any $B \in \Sigma$ and $A \in \mathscr{P}$. Assume that there exists an increasing sequence of sets $H_n \in \mathscr{P}$ such that $\Omega = \bigcup H_n$. Define $\mathscr{E}$ as the linear space of all simple functions with supports from $\mathscr{P}$. Let $\mathscr{M}$ denote the space of all measurable functions, i.e., all functions $p : \Omega \to \mathbb{R}$ such that there exists a sequence $\{q_n\} \subset \mathscr{E}$, $|q_n| \leq |p|$, and $q_n(w) \to p(w)$ for all $w \in \Omega$. The characteristic function of A is denoted by 1_A.

Definition 2.1 ([16, Definition 3]). *If a functional $\varrho\colon \mathscr{E} \times \Sigma \to [0,\infty]$ satisfies:*

(P_1) *for any $A \in \Sigma$, $\varrho(0, A) = 0$,*

(P_2) *$\varrho(p, A) \leq \varrho(q, A)$ whenever $|p(w)| \leq |q(w)|$ for each $w \in \Omega$, $A \in \Sigma$ and $p, q \in \mathscr{E}$,*

(P_3) *for any $p \in \mathscr{E}$, $\varrho(p, .) : \Sigma \to [0,\infty]$ is a σ-subadditive measure,*

(P_4) *$\varrho(\gamma, B) \to 0$ as $\gamma \downarrow 0$ for every $B \in \mathscr{P}$, where $\varrho(\gamma, B) = \varrho(\gamma 1_B, B)$,*

(P_5) *if there exits $\gamma > 0$ such that $\varrho(\gamma, B) = 0$, then $\varrho(\eta, B) = 0$ for each $\eta > 0$,*

(P_6) *for any $\gamma > 0$, $\varrho(\gamma, .)$ is order continuous on $\mathscr{P}$, that is $\varrho(\gamma, B_n) \to 0$ for every sequence $\{B_n\} \subset \mathscr{P}$ such that $B_n \downarrow \emptyset$,* *then ϱ is called* ***function*** **modular**.

To extend the function modular ϱ to the space of measurable functions $\mathscr{M}$, we approximate each p in $\mathscr{M}$ using simple functions. Since every measurable function can be expressed as the pointwise limit of an increasing sequence of simple functions [20, page 62], the definition of ϱ is then extended to measurable p in $\mathscr{M}$ as

$$\varrho(p, A) = \sup\{\varrho(q, A) : q \in \mathscr{E},\ |q(w)| \leq |p(w)|\ \ \forall\ w \in \Omega\}.$$

This ensures that the modular preserves the fundamental properties established for simple functions in $\mathscr{E}$ and extends naturally to the larger space $\mathscr{M}$. This allows us to define $\varrho(\gamma, A)$ for sets A that are not necessarily in $\mathscr{P}$. For convenience, we denote it simply as $\varrho(p)$ instead of $\varrho(p, \Omega)$.

Definition 2.2 ([18, Definition 2.1.3]). *If for all $\gamma > 0$, $\varrho(\gamma, A) = 0$ then the set A is called ϱ-null. A property $\mathfrak{b}(w)$ holds ϱ-almost everywhere if the exceptional set of elements in Ω such that $\mathfrak{b}(w)$ does not hold is ϱ-null.*

Theorem 2.1 ([18, Theorem 2.1.4]). *The functional ϱ from $\mathscr{M}$ to $[0, \infty]$ is a modular.*

The modular function space induced by a function modular ϱ is given by

$$L_\varrho = \{p \in \mathscr{M} : \varrho(\alpha p) \to 0 \,\text{as}\, \alpha \to 0\}.$$

Definition 2.3 ([19, Definition 2.3]). *A function modular ϱ is said to satisfy the Δ_2-condition if, whenever $\{p_n\} \subset \mathscr{M}$, $E_i \in \Sigma$, $E_i \downarrow \emptyset$ and $\sup_{n\geq 1} \varrho(p_n, E_i) \to 0$ as $i \to \infty$, one has*

$$\sup_{n\geq 1} \varrho(2p_n, E_i) \to 0.$$

Definition 2.4 ([19, Definition 2.4]). *A function modular ϱ satisfies the Δ_2-type condition if there exists $M > 0$ such that*

$$\varrho(2p) \leq M\varrho(p), \text{ for each } p \in L_\varrho.$$

The Δ_2-type condition guarantees the Δ_2-condition. However, the reverse implication does not necessarily hold.

Definition 2.5 ([15, Definition 3.4]).

(*i*) *A sequence $\{p_n\} \subset L_\varrho$ is said to be ϱ-convergent to $p \in L_\varrho$ if $\varrho(p_n - p) \to 0$; we write shortly $p_n \to p$ (ϱ).*

(*ii*) *$\{p_n\} \subset L_\varrho$ is called ϱ-Cauchy if $\varrho(p_n - p_m) \to 0$ as $n, m \to \infty$.*

(*iii*) *A set $B \subset L_\varrho$ is called ϱ-closed if for any sequence $\{p_n\}$ in B, the convergence $p_n \to p$ (ϱ) implies that $p \in B$.*

(*iv*) *A set $B \subset L_\varrho$ is called ϱ-bounded if*

$$\delta_\varrho(B) = \sup\{\varrho(p - q) : p \in B, q \in B\} < \infty.$$

(*v*) *ϱ has the Fatou property if $\varrho(p - q) \leq \varrho(p_n - q_n)$ whenever $p_n \to p$ (ϱ) and $q_n \to q$ (ϱ).*

Theorem 2.2 ([18, Theorem 2.3.7]). *$(L_\varrho, \|\cdot\|_\varrho)$ is a complete metric space.*

Proposition 2.1 ([18, Proposition 3.1.6]). *If ϱ satisfies the Δ_2-condition, then convergence in norm and convergence in modular are equivalent.*

This equivalence extends to cases where the Δ_2-type condition is satisfied. In the following discussion, we assume that ϱ is a convex function modular with the Δ_2-type condition.

Definition 2.6 ([15, Definition 3.7]). *Define a growth function ψ as follows*

$$\psi(z) = \sup\left\{\frac{\varrho(zp)}{\varrho(p)} : p \in L_\varrho, 0 < \varrho(p) < \infty\right\}, \quad \textit{for all } 0 \leq z < \infty.$$

Lemma 2.1 ([15, Lemma 3.1]). *The following are the properties of the growth function ψ:*

(*i*) *for all $z \in [0, \infty)$, $\psi(z) < \infty$.*

(*ii*) *ψ from $[0, \infty)$ to $[0, \infty)$ is both convex and strictly increasing. Therefore, ψ is continuous.*

(*iii*) *For all $\gamma, \eta \in [0, \infty)$, $\psi(\gamma\eta) \leq \psi(\gamma)\psi(\eta)$.*

(*iv*) *For all $\gamma, \eta \in [0, \infty)$, $\psi^{-1}(\gamma)\psi^{-1}(\eta) \leq \psi^{-1}(\gamma\eta)$, where the function ψ^{-1} is the inverse of ψ.*

The subsequent lemma demonstrates that ψ can provide an upper bound.

Lemma 2.2 ([15, Lemma 3.1]). *Let ϱ be as above. Then*

$$\|p\|_\varrho \leq \frac{1}{\psi^{-1}\left(\frac{1}{\varrho(p)}\right)}, \quad \textit{whenever } p \in L_\varrho \backslash \{0\}.$$

Let $C \subseteq L_\varrho$ and let $\mathcal{C}_\varrho(C)$ denote the collection of all nonempty ϱ-closed subsets of C. The map $H : \mathcal{C}_\varrho(L_\varrho) \times \mathcal{C}_\varrho(L_\varrho) \to \mathbb{R}^+$ defined as

$$H_\varrho(A, B) = \max\{\sup_{p \in A} \operatorname{dist}_\varrho(p, B), \sup_{q \in B} \operatorname{dist}_\varrho(q, A)\}, \quad \mathrm{A, B} \in \mathcal{C}_\varrho(L_\varrho)$$

is the generalized Hausdorff distance over $\mathcal{C}_\varrho(L_\varrho)$, where

$$\operatorname{dist}_\varrho(p, B) = \inf\{\varrho(p - q) : q \in B\}.$$

A fixed point of a multivalued mapping $T \colon C \to \mathcal{C}_\varrho(C)$ is a point satisfying $p \in Tp$. In the following theorems we will assume that ϱ is a convex function modular satisfying the Δ_2-type condition, and C is a nonempty ϱ-bounded ϱ-closed subset of L_ϱ.

Definition 2.7. *Let C be a nonempty subset of L_ϱ and $T : C \to \mathcal{C}_\varrho(C)$. If, for a given point $p_0 \in C$, there exists a sequence $\{p_n\} \subset C$ such that $p_n \in Tp_{n-1}$, then $\mathcal{O}(p_0) = \{p_0, p_1, p_2, \dots\}$ is called an orbit of p_0 through operator T.*

Definition 2.8. *A mapping $\lambda \colon C \to \mathbb{R}$ is T-orbitally ϱ-lower semi-continuous if, for any sequence $\{p_n\}$ in $\mathcal{O}(p_0)$ such that $p_0 \in C$ and $p_n \to p$ (ϱ),*

$$\lambda(p) \leq \lim_{n \to \infty} \inf \lambda(p_n).$$

Dhompongsa [19] generalized the contraction theorem in modular function spaces in the following manner (also see [21]).

Theorem 2.3 ([19, Theorem 3.1]). *Let $T\colon C \to \mathcal{C}_\varrho(C)$ be a ϱ-contraction mapping, i.e., there exists $k \in [0,1)$ such that*

$$H_\varrho(Tp, Tq) \le k\varrho(p-q),\ \forall p, q \in C.$$

Then T has a fixed point.

Alfuraidan [22] extended Theorem **2**.3 in the following way.

Theorem 2.4 ([22, Theorem 3.1]). *Let G be a reflexive directed graph which is defined on C and $T\colon C \to \mathcal{C}_\varrho(C)$ be a Reich (G,ϱ)-contraction mapping and $C_T = \{p \in C; (p,q) \in E(G)$ for some q in $Tp\}$. If C has Property (1), then T has a fixed point provided that $C_T \neq \emptyset$.*

Remark 2.1 ([22, Remark 3.1]). If we assume that G satisfies $E(G) = C \times C$, then G is connected, and Theorem **2**.4 yields the Mizoguchi-Takahashi's theorem [13] on modular function spaces (and consequently Nadler's theorem [11] when $\theta(j)$ is assumed to be constant). Moreover, if T is single-valued, then we obtain Reich's extension of the Banach contraction principle [12].

3. Main results

We now present the principal outcome of our work. We assume that ϱ is a convex function modular satisfying the Δ_2-type condition, and C is a nonempty ϱ-bounded ϱ-closed subset of L_ϱ.

Theorem 3.1. *Let ϱ be as above and assume additionally that it satisfies the Fatou property. Let $T\colon C \to \mathcal{C}_\varrho(C)$ be a multi-valued mapping satisfying*

$$\operatorname{dist}_\varrho(q, Tq) \le \theta(\varrho(p-q))\varrho(p-q) \quad \text{for all } p \in C \text{ and } q \in Tp, \tag{3.1}$$

where $\theta : (0,\infty) \to [0,1)$ is such that

$$\limsup_{j\to z^+} \theta(j) < 1 \quad \text{for all } z \in [0,\infty). \tag{3.2}$$

Then,

1. *for every $p_0 \in C$, there exists an orbit $\{p_n\}$ of p_0 through operator T and $\zeta \in C$ such that $\lim_n p_n = \zeta$;*
2. *if ζ is a fixed point of T, then the function $\lambda(p) = \operatorname{dist}_\varrho(p, Tp)$ is T-orbitally ϱ-lower semi-continuous at ζ, and conversely.*

The proof of Theorem **3**.1 relies on an argument presented in [23, Theorem 1.2.1]. We resume it by stating the next key result.

Lemma 3.1. *Let ϱ be a convex function modular, C be nonempty ϱ-bounded ϱ-closed subset of L_ϱ, and let $B \in \mathcal{C}_\varrho(C)$. Then, for any $p \in C$ and $\ell > 1$, there exists an element $q \in B$ satisfying*

$$\varrho(p-q) \le \ell \operatorname{dist}_\varrho(p, B). \tag{3.3}$$

Proof. Assume first that $\text{dist}_\varrho(p, B) = 0$. It follows that $p \in \overline{B} = B$, as B is a closed subset of C. Further, choosing $p = q$ clearly makes inequality (**3**.3) valid.

Now, let us assume that $\text{dist}_\varrho(p, B) > 0$ and select

$$\epsilon = (\ell - 1)\text{dist}_\varrho(p, B). \tag{3.4}$$

Since $\text{dist}_\varrho(p, B) = \inf \{\varrho(p - q_1) : q_1 \in B\}$, then there is $q \in B$ such that

$$\begin{aligned} \varrho(p - q) &\leq \text{dist}_\varrho(p, B) + \epsilon \\ &\leq \ell\, \text{dist}_\varrho(p, B) \quad (\text{using } (\mathbf{3}.4)). \end{aligned}$$

□

Proof of Theorem 3.1. Let $p_0 \in C$. Since $Tp_0 \neq \emptyset$, there exists $p_1 \in C$ such that $p_1 \in Tp_0$. We may assume that $p_1 \neq p_0$, otherwise the conclusion comes easily. This implies that $\varrho(p_0 - p_1) > 0$.

Since $p_1 \in C$ and $Tp_1 \in \mathcal{C}_\varrho(C)$, by taking $\ell = \frac{1}{\sqrt{\theta(\varrho(p_0 - p_1))}} > 1$ there exists $p_2 \in Tp_1$ such that

$$\varrho(p_1 - p_2) \leq \frac{1}{\sqrt{\theta(\varrho(p_0 - p_1))}} \text{dist}_\varrho(p_1, Tp_1) \quad [\text{by Lemma } \mathbf{3}.1].$$

By applying the same reasoning for $p_2 \in C$, $Tp_2 \in \mathcal{C}_\varrho(C)$, $\ell = \frac{1}{\sqrt{\theta(\varrho(p_1 - p_2))}} > 1$, there exists $p_3 \in Tp_2$ such that $\varrho(p_2 - p_3) \leq \dfrac{1}{\sqrt{\theta(\varrho(p_1 - p_2))}}\text{dist}_\varrho(p_2, Tp_2)$.

Proceeding similarly, we obtain a sequence $\{p_n\} \subset C$ such that $\varrho(p_n - p_{n+1}) \leq \dfrac{1}{\sqrt{\theta(\varrho(p_{n-1} - p_n))}}\text{dist}_\varrho(p_n, Tp_n),\ p_n \in Tp_{n-1}, n = 1, 2, 3, \cdots$.

As assumed $p_n \neq p_{n-1}$, otherwise p_{n-1} is fixed point of T. Then, it follows from (**3**.1) that

$$\begin{aligned} \varrho(p_n - p_{n+1}) &\leq \sqrt{\theta(\varrho(p_{n-1} - p_n))}\varrho(p_{n-1} - p_n) \\ &\leq \varrho(p_{n-1} - p_n). \end{aligned} \tag{3.5}$$

Hence, $\{\varrho(p_n - p_{n+1})\}$ is a decreasing sequence, so it converges. Say that b is the limit of this decreasing sequence. By taking limit in (**3**.5) we get $b = 0$. Select $\epsilon > 0$ and $0 < a < 1$ such that $\theta(z) < a^2$, for $z \in (0, \epsilon)$ $\quad$ (*using* (**3**.2)).

Let M be such that $\varrho(p_{n-1} - p_n) < \epsilon$, for $n \geq M$.

Now, from (**3**.5) we get $\dfrac{1}{a^{n-M+1}(\varrho(p_0 - p_1))} < \dfrac{1}{\varrho(p_n - p_{n+1})}$. Using properties of $\psi(t)$, we find $\psi^{-1}\left(\dfrac{1}{a^{n-M+1}(\varrho(p_0 - p_1))}\right) < \psi^{-1}\left(\dfrac{1}{\varrho(p_n - p_{n+1})}\right)$. So, $\dfrac{1}{\psi^{-1}\left(\dfrac{1}{\varrho(p_n - p_{n+1})}\right)} <$

$$\frac{1}{\psi^{-1}\left(\dfrac{1}{a}\right)^{n-M+1} \psi^{-1}\left(\dfrac{1}{\varrho(p_0 - p_1)}\right)}.$$

Then, by Lemma **2**.2, one has

$$\|p_n - p_{n+1}\|_\varrho < \left(\frac{1}{\psi^{-1}(\frac{1}{a})}\right)^{n-M+1} \left(\frac{1}{\psi^{-1}\left(\frac{1}{\varrho(p_0 - p_1)}\right)}\right).$$

Since $\psi(1) = 1$ and $a < 1$, then $\frac{1}{\psi^{-1}\left(\frac{1}{a}\right)} < 1$. This implies that sequence $\{p_n\}$ is a Cauchy in $(L_\varrho, \|\cdot\|_\varrho)$. By Theorem **2**.2, there exists $\zeta \in L_\varrho$ such that $\{p_n\}$ is $\|\cdot\|_\varrho$-convergent to ζ. Because the Δ_2-type condition holds, then $\{p_n\} \to \zeta$ (ϱ) and $\zeta \in C$ since C is ϱ-closed. Next, we must show that ζ is fixed point of mapping T. Since $p_n \in Tp_{n-1}$, using (**3**.1) and (**3**.2) we have $\text{dist}_\varrho(p_n, Tp_n) < \theta(\varrho(p_{n-1}, p_n))\varrho(p_{n-1}, p_n), < \varrho(p_{n-1}, p_n)$. Letting n $\to \infty$, it follows that $\lim_{n\to\infty} \text{dist}_\varrho(p_n, Tp_n) = 0$. $(\Leftarrow)$ $\text{dist}_\varrho(\zeta, T\zeta) = \lambda(\zeta) \leq \liminf_n \lambda(p_n) = \liminf_n \text{dist}_\varrho(p_n, Tp_n) = 0$. Since $T\zeta$ is closed, it follows that $\zeta \in T\zeta$.
$(\Rightarrow)$ if ζ is fixed point of T then $\lambda(\zeta) = 0 \leq \liminf_n \lambda(p_n)$. □

As $\text{dist}_\varrho(q, Tq) \leq H_\varrho(Tp, Tq)$ for q $\in Tp$, we may state the following subsequent result.

Corollary 3.1. *Let $T : C \to \mathcal{C}_\varrho(C)$ satisfying $H_\varrho(Tp, Tq) \leq \theta(\varrho(p-q))\varrho(p-q)$, for any $p \in C$ and $q \in Tp$,*

where $\theta : (0, \infty) \to [0, 1)$ is such that $\limsup_{j\to z^+} \theta(j) < 1$ for any $z \in [0, \infty)$. Then,

1. *for any $p_0 \in C$, there is an orbit $\{p_n\}$ of p_0 and a point $\zeta \in C$ such that $\lim_n p_n = \zeta$;*
2. *if ζ is fixed point of T, then the function $\lambda(p) = \text{dist}_\varrho(p, Tp)$ is T-orbitally ϱ-lower semi-continuous at ζ and conversely.*

Remark 3.1.

- *Corollary* **3**.1 extends the result of the Mizoguchi-Takahashi's theorem [13].
- To show that a fixed point exists, it suffices to ensure that the $\text{dist}_\varrho(p, Tp)$ is T-orbitally continuous at ζ, whereas the condition in the extension of the Mizoguchi-Takahashi's theorem [13] specifies that T is continuous from C into $\mathcal{C}_\varrho(C)$.

Let us give a simple example to validate Theorem **3**.1.

Example 3.1. The real number system $\mathbb{R}$ is a space modulared by $\varrho(p) = |p|$. Let $C = \left[0, \frac{3}{5}\right] = \{p \in L_\varrho : 0 \leq p \leq \frac{3}{5}\}$. Obviously C is nonempty ϱ-closed and ϱ-bounded subset of $\mathbb{R}$. Define the mapping $T\colon C \to \mathcal{C}_\varrho(C),\ Tp = \left[0, p^2\right], \quad p \geq 0$.

When $p \geq 0$ and $q \in$ Tp $= \left[0, p^2\right]$ we get

$$\text{dist}_\varrho(q, Tq) \leq \text{dist}_\varrho(Tq, Tp) = |p^2 - q^2| = |p+q||p-q| \leq \frac{24}{25}\varrho(p-q).$$

We may consider $\theta(z)=k$, where $\frac{24}{25} \leq k < 1$. See the 2D and 3D graphical behaviors in Figures 1 and 2. Note that, since all the requirements of Theorem **3**.1 and Corollary **3**.1 are fulfilled, then T has a fixed point.

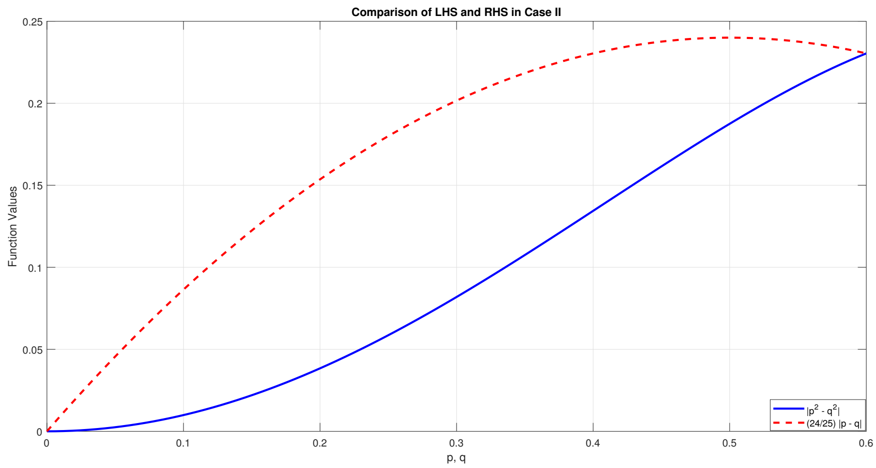

FIGURE 1. Depicts the 2D behavior of left hand and right hand side inequality in Example **3**.1

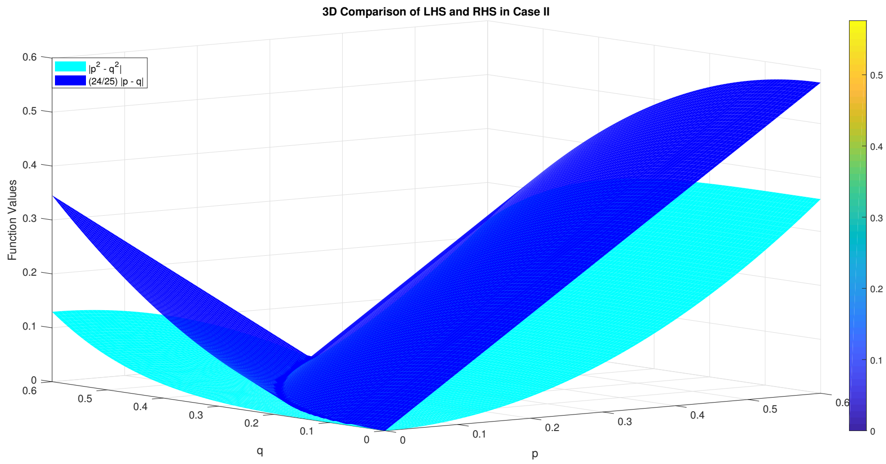

FIGURE 2. Depicts the 3D behavior of left hand and right hand side inequality in Example **3**.1

Example 3.2. The real number system $\mathbb{R}$ is a space modulared by $\varrho(p) = |p|$. Take $C = \left[-\frac{3}{5}, \frac{3}{5}\right]$ and define the mapping $T\colon C \to \mathcal{C}_\varrho(C)$, $Tp = \begin{cases} \left[-\frac{3}{5}, \frac{p}{2}\right] & \text{if} \quad p \leq 0, \\ \left[0, p^2\right] & \text{if} \quad p \in \left(0, \frac{3}{5}\right]. \end{cases}$

It is no difficult to see that all conditions of Theorem **3**.1 are satisfied and T has a fixed point. Note that [13, Theorem 5] and Theorem **2**.3 are not applicable here, since at $p = 0$ and $q = \frac{3}{5}$, we have $\frac{H_\varrho(Tp, Tq)}{\varrho(p-q)} = 1$.

4. Conclusion

We extended and refined the Mizoguchi-Takahashi's fixed point theorem within the framework of modular function spaces, broadening its applicability under more general conditions. We provided an illustrative example to demonstrate the practical relevance and effectiveness of the derived result. Additionally, we presented a graphical representation of the example through MATLAB, using both 2D and 3D plots to visually validate the theoretical findings. These contributions not only enhanced the existing field of knowledge but also opened new directions for further research in fixed point theory and its applications in modular function spaces.

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