A MODIFIED PRECONDITIONING ALGORITHM FOR SOLVING MONOTONE INCLUSION PROBLEM AND APPLICATION TO IMAGE RESTORATION PROBLEM

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In this work, we propose a modified preconditioning inertial viscosity forward-backward algorithm. We also study strong convergence behaviour of our algorithm under mild assumptions in a real Hilbert space. We use the proposed algorithm to solve the convex minimization problem. Finally, we apply our algorithm for solving the image restoration problems. We can say that the proposed algorithm exhibit to outperform the already existing algorithms in the literature.

Keywords: forward-backward splitting algorithm, monotone inclusion problem, convex minimization problem, image restoration problem.

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1. Introduction

Throughout the study, H denotes a real Hilbert space with inner product $\langle .,. \rangle$ and the induced norm $\|.\|$. One of the most significant problems in monotone operator theory is to identify a zero of the sum of two monotone operators, which is known as the monotone inclusion problem and which is defined by:

$$\text{finding } x \in H \text{ such that } 0 \in (A+B)(x) \tag{1}$$

where $A : H \to H$ is an M-cocoercive operator, where M is a linear bounded operator on underlying spaces and $B : H \to 2^H$ is a maximal monotone operator. The monotone inclusion problem occurs in many areas of applied mathematics including, signal and image processing, convex optimization, machine learning and statistical regression see e.g. : [1, 6, 7, 8, 10, 12, 13, 14, 17, 18, 21, 23, 24, 25, 26, 28].

The following forward-backward splitting algorithm, defined by Lions and Mercier [16] is the most popular technique for solving the monotone inclusion problem :

$$x_{n+1} = (I + \lambda_n A)^{-1} (I - \lambda_n B) x_n, \text{ for all } n \in \mathbb{N} \tag{2}$$

where λ_n is a step size parameter and A and B are monotone operators. If $B : H \to H$ is $1/L-$cocoercive operator for $\lambda_n \in (0, 2/L)$, then this algorithm converges weakly to a solution of the monotone inclusion problem.

After that, Moudafi and Oliny [20] presented the following algorithm for solving the monotone inclusion problem :

$$\begin{cases} y_n = x_n + \theta_n \ (x_n - x_{n-1}) \\ x_{n+1} = (I + \lambda_n A)^{-1} (y_n - \lambda_n B(x_n)), \text{ for all } n \in \mathbb{N} \end{cases}, \tag{3}$$

where θ_n is a inertial parameter on $[0, 1)$. They studied the weakly convergence of the algorithm, which satisfies the conditions $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$ and $\lambda_n < 2/L$ where L

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is the Lipschitz constant of B. The presence of the inertial parameter considerably improves the performance of algorithm.

Lately, Lorenz and Pock [18] proposed the preconditioning algorithm for solving the monotone inclusion problem in the following manner :

$$\begin{cases} y_n = x_n + \theta_n \ (x_n - x_{n-1}) \\ x_{n+1} = \left(I + \lambda_n M^{-1} A\right)^{-1} \left(I - \lambda_n M^{-1} B\right)(y_n), \text{ for all } n \in \mathbb{N}. \end{cases} \tag{4}$$

They proved the weak convergence of the algorithm. It is obvious that the Algorithm (4) is reduced to the classical forward-backward splitting algorithm (2) for $\theta_n = 0$ and $M = I$.

Next, in 2021, Dixit et al. [8] introduced the algorithm which is called accelerated preconditioning forward-backward normal S-iteration by the following way:

$$\begin{cases} y_n = x_n + \theta_n \ (x_n - x_{n-1}) \\ x_{n+1} = J_{\lambda,M}^{A,B}\left((1-\alpha_n) y_n + \alpha_n J_{\lambda,M}^{A,B}(y_n)\right), \text{for all } n \in \mathbb{N}, \end{cases} \tag{5}$$

where $J_{\lambda,M}^{A,B} = \left(I + \lambda M^{-1} A\right)^{-1} \left(I - \lambda M^{-1} B\right)$, $\alpha_n \in (0,1)$, $\lambda \in [0,1)$ and $\theta_n \in [0,1)$. They also proved weak convergence of the proposed algorithm under some assumptions in a real Hilbert space.

Recently, in 2021, Altıparmak and Karahan [3] introduced a new preconditinioning forward-backward splitting algorithm in the following manner :

$$\begin{cases} y_n = x_n + \theta_n \ (x_n - x_{n-1}) \\ z_n = J_{\lambda,M}^{A,B}\left((1-\beta_n) y_n + \beta_n J_{\lambda,M}^{A,B}(y_n)\right) \\ x_{n+1} = (1-\gamma_n) J_{\lambda,M}^{A,B}(z_n) + \gamma_n f(z_n) \end{cases} \tag{6}$$

where $\{\theta_n\} \subset [0,\theta]$ is a sequence with $\theta \in [0,1)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in (0,1)$ and f is a k-contraction mapping on H with respect to M-norm. They proved the strong convergence theorem in a real Hilbert space as well.

In this work, motivated and inspired by all these algorithms, we offer a modified preconditioning forward-backward splitting algorithm which is more effective in image restoration. We also show that the generated sequence by the proposed algorithm converges strongly to a solution of the monotone inclusion problem in real Hilbert space. On the other hand, we apply the proposed algorithm for solving the convex minimization problem. Lastly, we give an application of the proposed algorithm to the image restoration problem.

2. **Preliminaries**

In this part, we give some crucial definitions and lemmas which play a significant role in proving our main theorem.

Definition 2.1. [5] *Let C be a nonempty subset of a real Hilbert space H and $x \in H$. For any $z \in H$, if there exists a unique point $y \in C$ such that*

$$\|y - x\| \leq \|z - x\|$$

then y is said to be the metric projection of x onto C and is denoted by $y = P_C x$. If $P_C x$ exists and is uniquely determined for all $x \in H$, then $P_C : H \to C$ is said to be the metric projection operator.

It is easily known that the operator P_C is nonexpansive and it is characterized by,

$$\langle x - P_C x, y - P_C x \rangle \leq 0 \text{ for all } y \in C.$$

Definition 2.2. [4] *Let $A : H \to 2^H$ be a set-valued operator. If $\langle u - v, x - y \rangle \geq 0$ for all $u \in Ax$ and $v \in Ay$, then A is said to be a monotone operator. Morever, if the graph of a monotone operator is not properly contained in the graph of any other monotone operators, then A is said to be a maximal monotone operator.*

Let $f : H \to (-\infty, +\infty]$ be a function. The subdifferential of a proper function f is defined as follows :

$$\partial f(x) = \{u \in H : \langle y - x, u\rangle \leq f(y) - f(x) \text{ for all } y \in H\}.$$

If $\partial f(x) \neq \emptyset$, then f is subdifferentiable at $x \in H$. The elements of $\partial f(x)$ are also called the subgradients of f at x.

Definition 2.3. [4] *Let $\Gamma_0(H)$ denotes the class of all proper lower semi-continuous convex functions defined from H to $(-\infty, +\infty]$. Let $g \in \Gamma_0(H)$ and $\lambda > 0$. The proximal operator of parameter λ of g at x can be defined by*

$$prox_{\lambda g}(x) = \arg\min_{y \in H} \left\{ g(y) + \frac{1}{2\lambda} \|y - x\|^2 \right\}.$$

Let $M : H \to H$ be a bounded linear operator. M is called self adjoint if $M^* = M$ where M^* is the adjoint of operator M. If $\langle M(x), x\rangle > 0$ for every $0 \neq x \in H$, then M is said to be positive definite [15]. Using the self adjoint, positive and bounded linear operator M, the M-inner product is defined by $\langle x, y\rangle_M = \langle x, M(y)\rangle$, for all $x, y \in H$. The corresponding M-norm induced from the M-inner product is also defined by $\|x\|_M^2 = \langle x, M(x)\rangle$ for all $x \in H$.

Definition 2.4. [8] *Let C be a nonempty subset of H, $T : C \to H$ be an operator and $M : H \to H$ be a positive definite operator. Then T is called:*

(i) *nonexpansive operator with respect to M-norm if*

$$\|Tx - Ty\|_M \leq \|x - y\|_M, \ \forall x, y \in H,$$

(ii) M-cocoercive operator if

$$\|Tx - Ty\|_{M^{-1}}^2 \leq \langle x - y, Tx - Ty\rangle, \forall x, y \in H.$$

T is also called $k-$contraction mapping with respect to M-norm if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|_M \leq k \|x - y\|_M, \ \forall x, y \in H \text{ [3]}.$$

Proposition 2.1. [8] *Let $B : H \to 2^H$ be a maximal monotone operator, let $A : H \to H$ be a M-cocoercive operator and $M : H \to H$ be a bounded linear self adjoint and positive definite operator. For $\lambda \in (0, 1]$, the following are satisfied:*

(1) $I - \lambda M^{-1}A$ *is nonexpansive operator with respect to M-norm,*
(2) $\left(I + \lambda M^{-1}B\right)^{-1}$ *is nonexpansive operator with respect to M-norm,*
(3) $J_{\lambda,M}^{A,B} = \left(I + \lambda M^{-1}B\right)^{-1}\left(I - \lambda M^{-1}A\right)$ *is nonexpansive operator with respect to M-norm.*

Proposition 2.2. [8] *Let $B : H \to 2^H$ be a maximal monotone operator, let $A : H \to H$ be a M-cocoercive operator and let $M : H \to H$ be a bounded linear self adjoint and positive definite operator and $\lambda \in (0, \infty)$. Then $x \in H$ is a solution of monoton inclusion problem (1) if and only if $\left(I + \lambda M^{-1}B\right)^{-1}\left(I - \lambda M^{-1}A\right)(x) = x$.*

Lemma 2.1. [9] *Let C be a nonempty closed and convex subset of a real Hilbert space H and let $T : C \to H$ be a nonexpansive operator with $F(T) \neq \emptyset$. Then the mapping $I - T$ is demiclosed at zero, that is, for any sequence $\{x_n\} \in H$ such that $x_n \rightharpoonup x \in H$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, $x \in F(T)$.*

Lemma 2.2. [4] *The following properties are satisfied:*

(i) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$,
(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$,

(iii) $\|\delta x+(1-\delta)y\|^2=\delta\|x\|^2+(1-\delta)\|y\|^2-\delta(1-\delta)\|x-y\|^2$, *for all* $x,y\in H$ *and* $\delta\in[0,1]$.

Lemma 2.3. [29] *Let* $\{\varsigma_n\}$ *and* $\{\varepsilon_n\}$ *be sequences of nonnegative real numbers such that*

$$\varsigma_{n+1}\leq(1-\sigma_n)\varsigma_n+\sigma_n\delta_n+\varepsilon_n,$$

where $\{\sigma_n\}$ *is a sequence in* $[0,1]$ *and* $\{\lambda_n\}$ *is a real sequence. If the following conditions are satisfied, then* $\lim_{n\to\infty}\varsigma_n=0$:

(1) *(i)* $\sum_{n=1}^{\infty}\sigma_n=\infty$, *(ii)* $\sum_{n=1}^{\infty}\varepsilon_n<\infty$,*(iii)* $\limsup_{n\to\infty}\lambda_n\leq 0$

Lemma 2.4. [19] *Let* $\{\Theta_n\}$ *be a sequence of real numbers that does not decrease at infinity such that there exists a subsequence* $\{\Theta_{n_i}\}$ *of* $\{\Theta_n\}$ *which satisfies* $\Theta_{n_i}<\Theta_{n_{i+1}}$ *for all* $i\in\mathbb{N}$. *Let* $\{\tau(n)\}_{n\geq n_0}$ *be a sequence of integers which is defined by:*

$$\tau(n):=\max\{l\leq n:\Theta_l<\Theta_{l+1}\}.$$

Then, the following are satisfied:

(i) $\tau(n_0)\leq\tau(n_0+1)\leq\ldots$ *and* $\tau(n)\to\infty$,
(ii) $\Theta_{\tau(n)}\leq\Theta_{\tau(n)+1}$ *and* $\Theta_n\leq\Theta_{\tau(n)+1}$, *for all* $n\geq n_0$.

3. Main Results

In this part, we introduce the following modified preconditioning inertial viscosity forward-backward algorithm and then prove its strong convergence under some suitable assumptions in a real Hilbert space. We also use the obtained result to solve a convex minimization problem and so image restoration problem.

Algorithm 1 : Modified preconditining inertial viscosity forward backward algorithm

Initial : $x_0,x_1\in H$

Iterative Step. Compute ω_n,y_n,z_n and x_{n+1} using

$$\begin{cases}\omega_n=x_n+\theta_n\ (x_n-x_{n-1})\\ y_n=(1-\alpha_n)\omega_n+\alpha_nJ_{\lambda,M}^{A,B}(\omega_n)\\ z_n=J_{\lambda,M}^{A,B}\left((1-\beta_n)y_n+\beta_nJ_{\lambda,M}^{A,B}(y_n)\right)\\ x_{n+1}=(1-\gamma_n)J_{\lambda,M}^{A,B}(z_n)+\gamma_nh(z_n)\end{cases}$$

Then update $n:=n+1$ and go to Iterative Step.

Theorem 3.1. *Let* $M:H\to H$ *be a linear bounded self adjoint and positive definite operator,* $A:H\to H$ *be a* M*-cocoercive operator and* $B:H\to 2^H$ *be a maximally monotone operator such that* $\Omega=(A+B)^{-1}(0)$ *is nonempty. Let* h *be a* k*-contraction mapping on* H *with respect to* M*-norm and let* $\lambda\in(0,1]$. *Let* $\{x_n\}$ *be a sequence generated by Algorithm 1 where* $\{\theta_n\}\subset[0,\theta]$ *is a sequence with* $\theta\in[0,1)$ *and* $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}\in(0,1)$ *such that the following conditions are satisfied:*

(i) $0<a\leq\alpha_n\leq b<1$ *for some* $a,b\in\mathbb{R}$,
(ii) $0<c\leq\beta_n\leq d<1$ *for some* $c,d\in\mathbb{R}$,
(iii) $\sum_{n=1}^{\infty}\theta_n\|x_n-x_{n-1}\|_M<\infty$,
(iv) $\lim_{n\to\infty}\gamma_n=0$, $\sum_{n=1}^{\infty}\gamma_n=\infty$.

Then the sequence $\{x_n\}$*converges strongly to a point* x^* *in* Ω *where* $x^*=P_\Omega f(x^*)$.

Proof. We will analyze the proof in two parts.

Part 1 : Let $x^*\in\Omega$ such that $x^*=P_\Omega f(x^*)$. Since $J_{\lambda,M}^{A,B}=\left(I+\lambda M^{-1}B\right)^{-1}\left(I-\lambda M^{-1}A\right)$ is nonexpansive with respect to M-norm, we have the following inequalities :

$$\begin{aligned}\|\omega_n-x^*\|_M &= \|x_n+\theta_n\ (x_n-x_{n-1})-x^*\|_M\\ &\leq \|x_n-x^*\|_M+\theta_n\|\ x_n-x_{n-1}\|_M,\end{aligned}\tag{7}$$

$$\begin{aligned} \|y_n - x^*\|_M &= \left\|(1-\alpha_n)\,\omega_n + \alpha_n J_{\lambda,M}^{A,B}(\omega_n) - x^*\right\|_M \\ &\le (1-\alpha_n)\,\|\omega_n - x^*\|_M + \alpha_n \left\| J_{\lambda,M}^{A,B}(\omega_n) - x^*\right\|_M \\ &= \|\omega_n - x^*\|_M \end{aligned} \tag{8}$$

and

$$\begin{aligned} \|z_n - x^*\|_M &= \left\|J_{\lambda,M}^{A,B}\left((1-\beta_n)\,y_n + \beta_n J_{\lambda,M}^{A,B}(y_n)\right) - x^*\right\|_M \\ &\le \left\|(1-\beta_n)\,y_n + \beta_n J_{\lambda,M}^{A,B}(y_n) - x^*\right\|_M \\ &\le (1-\beta_n)\,\|y_n - x^*\|_M + \beta_n \left\|J_{\lambda,M}^{A,B}(y_n) - x^*\right\|_M \\ &\le \|y_n - x^*\|_M\,. \end{aligned} \tag{9}$$

Combining (7), (9) and (8) and since h is k-contractive mapping with respect to M-norm we also have the following inequality, for all $n \ge 1$,

$$\begin{aligned} \|x_{n+1} - x^*\|_M &= \left\|(1-\gamma_n)\,J_{\lambda,M}^{A,B}(z_n) + \gamma_n h(y_n) - x^*\right\|_M \\ &\le \gamma_n \|h(y_n) - h(x^*)\|_M + \gamma_n \|h(x^*) - x^*\|_M \\ &\quad + (1-\gamma_n)\left\|J_{\lambda,M}^{A,B}(z_n) - x^*\right\|_M \\ &\le \gamma_n k \|y_n - x^*\|_M + \gamma_n \|h(x^*) - x^*\|_M + (1-\gamma_n)\|z_n - x^*\|_M \\ &\le \gamma_n k \|y_n - x^*\|_M + \gamma_n \|h(x^*) - x^*\|_M + (1-\gamma_n)\|y_n - x^*\|_M \\ &\le (1-\gamma_n(1-k))\,\|x_n - x^*\|_M \\ &\quad + \theta_n \|x_n - x_{n-1}\|_M + \gamma_n \|h(x^*) - x^*\|_M \\ &\le (1-\gamma_n(1-k))\,\|x_n - x^*\|_M \\ &\quad + \gamma_n . \frac{\theta_n}{\gamma_n}\|x_n - x_{n-1}\|_M + \gamma_n \|h(x^*) - x^*\|_M\,. \end{aligned} \tag{10}$$

It follows from the assumptions (ii) and (iii) that $\lim_{n\to\infty} \frac{\theta_n}{\gamma_n}\|x_n - x_{n-1}\|_M = 0$. Thus, there exists a positive constant $N_1 > 0$ such that $\frac{\theta_n}{\gamma_n}\|x_n - x_{n-1}\|_M \le N_1$. By using (10) we find

$$\begin{aligned} \|x_{n+1} - x^*\|_M &\le (1-\gamma_n(1-k))\,\|x_n - x^*\|_M + \gamma_n\left(N_1 + \|f(x^*) - x^*\|_M\right) \\ &\le \max\left\{\|x_n - x^*\|_M\,, \frac{N_1 + \|h(x^*) - x^*\|_M}{(1-k)}\right\} \\ &\;\;\vdots \\ &\le \max\left\{\|x_1 - x^*\|_M\,, \frac{N_1 + \|h(x^*) - x^*\|_M}{(1-k)}\right\} \end{aligned}$$

for all $n \ge 1$. It means that $\{x_n\}$ is bounded so $\{y_n\}, \{z_n\}$ and $\{\omega_n\}$ are also bounded.

Part 2 : We show that $x_n \to x^* = P_\Omega f(x^*)$. By using Lemma 2.2 we observe that,

$$\|\omega_n - x^*\|_M^2 \le \|x_n - x^*\|_M^2 + 2\theta_n \|x_n - x^*\|_M \|x_n - x_{n-1}\|_M + \theta_n^2 \|x_n - x_{n-1}\|_M^2\,, \tag{11}$$

$$\begin{aligned} \|y_n - x^*\|_M^2 &= \alpha_n \left\|J_{\lambda,M}^{A,B}(\omega_n) - x^*\right\|_M^2 + (1-\alpha_n)\|\omega_n - x^*\|_M^2 \\ &\quad - \alpha_n(1-\alpha_n)\left\|J_{\lambda,M}^{A,B}(\omega_n) - \omega_n\right\|_M^2 \\ &\le \|\omega_n - x^*\|_M^2 \end{aligned} \tag{12}$$

and

$$\begin{aligned}
\|z_n - x^*\|_M^2 &= \left\| J_{\lambda,M}^{A,B}\left((1-\beta_n) y_n + \beta_n J_{\lambda,M}^{A,B}(y_n)\right) - x^* \right\|_M^2 \\
&\leq \beta_n \left\| J_{\lambda,M}^{A,B}(y_n) - x^* \right\|_M^2 + (1-\beta_n)\|y_n - x^*\|_M^2 \\
&\quad -\beta_n (1-\beta_n) \left\| J_{\lambda,M}^{A,B}(y_n) - y_n \right\|_M^2 \\
&\leq \|y_n - x^*\|_M^2, \text{ for all } n \geq 1. \qquad (13)
\end{aligned}$$

It follows from (11), (12), (13) and Lemma 2.2 that

$$\begin{aligned}
\|x_{n+1} - x^*\|_M^2 &\leq \left\| (1-\gamma_n)\left(J_{\lambda,M}^{A,B}(z_n) - x^*\right) + \gamma_n (h(y_n) - h(x^*)) \right\|_M^2 \\
&\quad +2\gamma_n \langle h(x^*) - x^*, x_{n+1} - x^* \rangle_M \\
&\leq (1-\gamma_n) \left\| J_{\lambda,M}^{A,B}(z_n) - x^* \right\|_M^2 + \gamma_n \|h(y_n) - h(x^*)\|_M^2 \\
&\quad +2\gamma_n \langle h(x^*) - x^*, x_{n+1} - x^* \rangle_M \\
&\leq \gamma_n k^2 \|y_n - x^*\|_M^2 + (1-\gamma_n)\|z_n - x^*\|_M^2 + 2\gamma_n \langle h(x^*) - x^*, x_{n+1} - x^* \rangle_M \\
&\leq \left(1 - \gamma_n\left(1-k^2\right)\right) \|y_n - x^*\|_M^2 + 2\gamma_n \langle h(x^*) - x^*, x_{n+1} - x^* \rangle_M \\
&\leq \left(1 - \gamma_n\left(1-k^2\right)\right) \Big[\|x_n - x^*\|_M^2 + 2\theta_n \|x_n - x^*\|_M \| x_n - x_{n-1}\|_M \\
&\quad +\theta_n^2 \| x_n - x_{n-1}\|_M^2 \Big] + 2\gamma_n \langle h(x^*) - x^*, x_{n+1} - x^* \rangle_M \\
&\leq \left(1 - \gamma_n\left(1-k^2\right)\right) \|x_n - x^*\|_M^2 + \theta_n \| x_n - x_{n-1}\|_M [2\|x_n - x^*\|_M \\
&\quad +\theta_n \| x_n - x_{n-1}\|_M] + 2\gamma_n \langle h(x^*) - x^*, x_{n+1} - x^* \rangle_M, \text{ for all } n \geq 1. \quad (14)
\end{aligned}$$

Since $\lim_{n\to\infty} \theta_n \|x_n - x_{n-1}\|_M = 0$, there exists a positive constant $N_2 > 0$ such that $\theta_n \|x_n - x_{n-1}\|_M \leq N_2$. From the inequality (14) we can observe,

$$\begin{aligned}
\|x_{n+1} - x^*\|_M^2 &\leq \left(1 - \gamma_n\left(1-k^2\right)\right) \|x_n - x^*\|_M^2 + 3N_3\theta_n \| x_n - x_{n-1}\|_M \\
&\quad +\gamma_n\left(1-k^2\right) \frac{2}{(1-k^2)} \langle h(x^*) - x^*, x_{n+1} - x^* \rangle_M, \qquad (15)
\end{aligned}$$

where $N_3 = \sup_{n\geq 1}\{\|x_n - x^*\|_M, N_2\}$. In inequality (15), if we get $\sigma_n = \gamma_n\left(1-k^2\right)$, $\varsigma_n = \|x_n - x^*\|_M^2$, $\delta_n = \frac{2}{(1-k^2)} \langle h(x^*) - x^*, x_{n+1} - x^* \rangle_M$ and $\varepsilon_n = 3N_3\theta_n \| x_n - x_{n-1}\|_M$ then we have $\varsigma_{n+1} \leq (1-\sigma_n)\varsigma_n + \sigma_n\delta_n + \varepsilon_n$ for all $n \geq 1$.

We next show that $\limsup_{n\to\infty} \langle h(x^*) - x^*, x_{n+1} - x^* \rangle_M \leq 0$. We observe the following two cases.

In the first case, suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - x^*\|_M\}_{n\geq n_0}$ is a nonincreasing sequence. The sequence $\{\|x_n - x^*\|_M\}$ is convergent since it is bounded from below by 0. Using the assumption (iv), we have $\sum_{n=1}^{\infty} \sigma_n = \infty$.

Assert that $\limsup_{n\to\infty} \langle h(x^*) - x^*, x_{n+1} - x^* \rangle_M \le 0$. Together (11) and (12) with Lemma 2.2 we can get, for all $n \ge 1$,

$$\begin{aligned}
\|x_{n+1} - x^*\|_M^2 &= \gamma_n \|h(y_n) - x^*\|_M^2 + (1-\gamma_n)\left\|J_{\lambda,M}^{A,B}(z_n) - x^*\right\|_M^2 \\
&\quad -\gamma_n(1-\gamma_n)\left\|h(y_n) - J_{\lambda,M}^{A,B}(z_n)\right\|_M^2 \\
&\le \gamma_n \|h(y_n) - x^*\|_M^2 + (1-\gamma_n)\|z_n - x^*\|_M^2 \\
&\le \gamma_n \|h(y_n) - x^*\|_M^2 + (1-\gamma_n)\|y_n - x^*\|_M^2 \\
&\le \gamma_n \|h(y_n) - x^*\|_M^2 + (1-\gamma_n)\left[\|\omega_n - x^*\|_M^2 - \alpha_n(1-\alpha_n)\left\|J_{\lambda,M}^{A,B}(\omega_n) - \omega_n\right\|_M^2\right] \\
&\le \gamma_n \|h(y_n) - x^*\|_M^2 + (1-\gamma_n)\Big[\|x_n - x^*\|_M^2 + 2\theta_n \|x_n - x^*\|_M \|x_n - x_{n-1}\|_M \\
&\quad +\theta_n^2 \| x_n - x_{n-1}\|_M^2 - \alpha_n(1-\alpha_n)\left\|J_{\lambda,M}^{A,B}(\omega_n) - \omega_n\right\|_M^2\Big] \\
&= \gamma_n \|h(y_n) - x^*\|_M^2 + (1-\gamma_n)\|x_n - x^*\|_M^2 \\
&\quad +2(1-\gamma_n)\theta_n \|x_n - x^*\|_M \| x_n - x_{n-1}\|_M \\
&\quad (1-\gamma_n)\theta_n^2 \| x_n - x_{n-1}\|_M^2 - \alpha_n(1-\alpha_n)(1-\gamma_n)\left\|J_{\lambda,M}^{A,B}(\omega_n) - \omega_n\right\|_M^2
\end{aligned}$$

This implies that

$$\begin{aligned}
\alpha_n(1-\alpha_n)(1-\gamma_n)\left\|J_{\lambda,M}^{A,B}(\omega_n) - \omega_n\right\|_M^2 &\le \gamma_n\left(\|h(y_n) - x^*\|_M^2 - \|x_n - x^*\|_M^2\right) \\
&\quad - \|x_{n+1} - x^*\|_M^2 + \|x_n - x^*\|_M^2 \\
&\quad + (1-\gamma_n)\theta_n \| x_n - x_{n-1}\|_M \\
&\quad \times \left(2\|x_n - x^*\|_M^2 + \theta_n \| x_n - x_{n-1}\|_M\right).
\end{aligned}$$

Due to the assunptions (iii), (iv) and the convergence of the sequence $\{\|x_n - x^*\|_M\}$, we can derive that

$$\lim_{n\to\infty} \left\|J_{\lambda,M}^{A,B}(\omega_n) - \omega_n\right\|_M^2 = 0. \tag{16}$$

On the other hand, the following expressions are obtained

$$\lim_{n\to\infty} \|\omega_n - x_n\|_M = \lim_{n\to\infty} \theta_n \| x_n - x_{n-1}\|_M = 0 \tag{17}$$

and

$$\begin{aligned}
\|y_n - \omega_n\|_M &= \left\|(1-\alpha_n)\omega_n + \alpha_n J_{\lambda,M}^{A,B}(\omega_n) - \omega_n\right\|_M \\
&= \alpha_n \left\|J_{\lambda,M}^{A,B}(\omega_n) - \omega_n\right\|_M
\end{aligned}$$

which implies

$$\lim_{n\to\infty} \|y_n - \omega_n\|_M = \lim_{n\to\infty} \left\|J_{\lambda,M}^{A,B}(\omega_n) - \omega_n\right\|_M = 0. \tag{18}$$

Note that

$$\begin{aligned}
\|z_n - \omega_n\|_M &\le \left\|z_n - J_{\lambda,M}^{A,B}(\omega_n)\right\|_M + \left\|J_{\lambda,M}^{A,B}(\omega_n) - \omega_n\right\|_M \\
&\le \left\|(1-\beta_n)y_n + \beta_n J_{\lambda,M}^{A,B}(y_n) - \omega_n\right\|_M + \left\|J_{\lambda,M}^{A,B}(\omega_n) - \omega_n\right\|_M \\
&\le \|y_n - \omega_n\|_M + \beta_n \left\|J_{\lambda,M}^{A,B}(\omega_n) - \omega_n\right\|_M.
\end{aligned}$$

The inequalities (16) and (18) imply that $\lim_{n\to\infty} \|z_n - \omega_n\|_M = 0$. Using (16), (17), (18) and the assumption (iv), we consider

$$\begin{aligned} \|x_{n+1} - \omega_n\|_M &\leq \left\|x_{n+1} - J_{\lambda,M}^{A,B}(\omega_n)\right\|_M + \left\|J_{\lambda,M}^{A,B}(\omega_n) - \omega_n\right\|_M \\ &= \left\|(1-\gamma_n) J_{\lambda,M}^{A,B}(z_n) + \gamma_n h(z_n) - J_{\lambda,M}^{A,B}(\omega_n)\right\|_M + \left\|J_{\lambda,M}^{A,B}(\omega_n) - \omega_n\right\|_M \\ &\leq \gamma_n \left\|h(z_n) - J_{\lambda,M}^{A,B}(z_n)\right\|_M + \left\|J_{\lambda,M}^{A,B}(z_n) - J_{\lambda,M}^{A,B}(\omega_n)\right\|_M + \left\|J_{\lambda,M}^{A,B}(\omega_n) - \omega_n\right\|_M \\ &\leq \gamma_n \left\|h(z_n) - J_{\lambda,M}^{A,B}(z_n)\right\|_M + \|z_n - \omega_n\|_M + \left\|J_{\lambda,M}^{A,B}(\omega_n) - \omega_n\right\|_M \end{aligned}$$

which implies

$$\lim_{n\to\infty} \|x_{n+1} - \omega_n\|_M = 0. \tag{19}$$

Thus, from the inequalities of (17) and (19) we obtain that $\lim_{n\to\infty} \|x_{n+1} - x_n\|_M = 0$. We next set $\limsup_{n\to\infty} \langle h(x^*) - x^*, x_{n+1} - x^*\rangle_M = u$. Since the sequence $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup t$ and $\lim_{i\to\infty} \langle h(x^*) - x^*, x_{n_i+1} - x^*\rangle_M = u$. By using (16) and (17) we can find

$$\begin{aligned} \left\|J_{\lambda,M}^{A,B}(x_n) - x_n\right\|_M &= \left\|J_{\lambda,M}^{A,B}(x_n) - x_n + \omega_n - \omega_n + J_{\lambda,M}^{A,B}(\omega_n) - J_{\lambda,M}^{A,B}(\omega_n)\right\|_M \\ &\leq 2\|\omega_n - x_n\|_M + \left\|J_{\lambda,M}^{A,B}(\omega_n) - \omega_n\right\|_M. \end{aligned}$$

Also, this implies that $\lim_{n\to\infty} \left\|J_{\lambda,M}^{A,B}(x_n) - x_n\right\|_M = 0$. Then, it is easly seen from Lemma 2.1 that $t \in F\left(J_{\lambda,M}^{A,B}\right)$. On the other hand, since $\|x_{n+1} - x_n\|_M \to 0$ as $n \to \infty$ and $x_{n_i} \rightharpoonup t$ we have $x_{n_{i+1}} \to t$. Furthermore, by combining $x^* = P_\Omega f(x^*)$ and property of the metric projection operators we can write

$$\lim_{i\to\infty} \langle f(x^*) - x^*, x_{n_i+1} - x^*\rangle_M = \langle f(x^*) - x^*, t - x^*\rangle_M \leq 0.$$

In this case, this implies that

$$\limsup_{n\to\infty} \langle h(x^*) - x^*, x_{n+1} - x^*\rangle_M \leq 0. \tag{20}$$

It follows from (20) that $\limsup_{n\to\infty} \delta_n \leq 0$. Consequently, we obtain that $x_n \to x^*$.

In the second case, suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - x^*\|_M\}_{n\geq n_0}$ is a monotonically decrasing sequence. Let us denote $\Theta = \|x_n - x^*\|_M^2$ for all $n \geq 1$. For this reason, there exists a subsequence $\{\Theta_j\}$ of $\{\Theta_n\}$ such that $\Theta_{n_j} < \Theta_{n_{j+1}}$ for all $n \geq n_0$. Define $\tau : \{n : n \geq n_0\}$ by $\tau(n) = \max\{l \in \mathbb{N} : l \leq n, \Theta_l \leq \Theta_{l+1}\}$. It is clear that the sequence τ is nondecreasing. By Lemma 2.4 we say that $\Theta_{\tau(n)} \leq \Theta_{\tau(n)+1}$ for all $n \geq n_0$. So, we have $\left\|\Theta_{\tau(n)} - x^*\right\|_M \leq \left\|\Theta_{\tau(n)+1} - x^*\right\|_M$.

In the first case, by taking $\tau(n)$ instead of n, we can here obtain similar results. Namely, we get $\limsup_{n\to\infty} \left\|\Theta_{\tau(n)} - x^*\right\|_M^2 \leq 0$. We also have

$$\left\|\Theta_{\tau(n)} - x^*\right\|_M^2 \to 0 \text{ and } \left\|\Theta_{\tau(n)+1} - x^*\right\|_M \to 0 \text{ as } n \to \infty. \tag{21}$$

Thus, by using (21) and Lemma 2.4, we deduce that $\|\Theta_n - x^*\|_M \leq \left\|\Theta_{\tau(n)+1} - x^*\right\|_M \to 0$ as $n \to \infty$. In conclusion, we conclude that $x_n \to x^*$, and then the proof is finished. □

Remark 3.1. *In Algorithm 1,*

(1) *If we take $\alpha_n = 0$, then we derive Algorithm (6),*
(2) *If we take $\alpha_n = 0$, $\gamma_n = 1$ and $h = I$, then we derive Algorithm (5).*
(3) *If we take $\alpha_n = \beta_n = 0$, $\gamma_n = 1$ and $h = I$, then we derive Algorithm (4).*
(4) *If we take $\alpha_n = \beta_n = \theta_n = 0$, $\gamma_n = 1$ and $h = M = I$, then we derive Algorithm (2).*

4. **Applications to Convex Minimization Problem**

Consider the following convex minimization problem :

$$\text{finding } x^* \in H \text{ such that } f(x^*) + g(x^*) = \min_{x \in H} \{f(x) + g(x)\}. \tag{22}$$

Let $g : H \to \mathbb{R}$ be a proper convex and lower semi-continuous function and let $f : H \to \mathbb{R}$ be convex and differentiable with L_f-Lipschitz gradient which is Lipschitz constant of ∇f . Then Baillon-Hadded Theorem states that ∇F is cocoercive with respect to L_f^{-1}. Morever, the subdifferential of g is maximal monotone see, for detail [4]. It is well known that a point x^* is a solution of convex minimization problem (22) if and only if $0 \in \nabla f(x^*) + \partial g(x^*)$. In Theorem 3.1, set $A = \nabla f$, $B = \partial g$ and $M(x) = L_f(x)$. So, the next Algorithm 2 and theorem can be derived.

Algorithm 2

Initial : $x_0, x_1 \in H$

Iterative Step. Compute ω_n, y_n, z_n and x_{n+1} using

$$\begin{cases} \omega_n = x_n + \theta_n\ (x_n - x_{n-1}) \\ y_n = (1-\alpha_n)\,\omega_n + \alpha_n J_{\lambda,L_f}^{\nabla f,\partial g}(\omega_n) \\ z_n = J_{\lambda,L_f}^{\nabla f,\partial g}\left((1-\beta_n)\,y_n + \beta_n J_{\lambda,L_f}^{\nabla f,\partial g}(y_n)\right) \\ x_{n+1} = (1-\gamma_n)\,J_{\lambda,L_f}^{\nabla f,\partial g}(z_n) + \gamma_n h(z_n) \end{cases}$$

Then update $n := n + 1$ and go to Iterative Step.

Theorem 4.1. *Let $f : H \to \mathbb{R}$ be a differentiable and convex function with L_f-Lipschitz gradient and $g : H \to \mathbb{R}$ be a proper convex and lower semi continuous function. Assume that the solution set of convex minimization problem (22) is nonempty. The parameters $\{\theta_n\} \subset [0, \theta]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in (0, 1)$ satisfy the same condition as in Theorem 3.1. Let $\{x_n\}$ be a sequence generated by Algorithm 2. Then $\{x_n\}$ converges strongly to a x^*solution of convex minimization problem.*

5. **Applications to Image Restoration Problem**

The goal of this part is to demonstrate that Algorithm 2 can be applied to solve the image restoration problem. We also conduct a comparison of Algorithm 2 with Algorithm (4), Algorithm (5) and Algorithm (6) in image restoration. All codes are implemented in MATLAB R2020a running on a Dell with Intel (R) Core (TM) i5 CPU and 8 GB of RAM. The image restoration problem can be formulated by the inversion of the following manner:

$$z = Ax + y \tag{23}$$

where $x \in \mathbb{R}^d$ is the original image, $A : \mathbb{R}^d \to \mathbb{R}^m$ is the blur function, $z \in \mathbb{R}^m$ is the observed image and y is the additive noise. The aim of the image restoration problem is to minimize the additive noise y by using the degenerated image z. Thus, the image restoration problem (23) reduces to l_1- regularization problem which can be formulated by

$$\min_{x \in \mathbb{R}^d} \left\{ \frac{1}{2} \|Ax - z\|^2 + \tau \|x\|_1 \right\}, \tag{24}$$

where $\tau > 0$ is a regularization parameter. By taking $f(x) = \frac{1}{2}\|Ax - z\|^2$, $g(x) = \tau \|x\|_1$ the convex minimization problem reduces to as l_1- regularization problem. The gradient of f is the following form $\nabla f(x) = A^T(Ax - z)$, where A^T is a transpose of operator. Image restoration problem (23) is solved by adapting Algorithm 2 and also this algorithm is compared with Algorithm (4), Algorithm (5) and Algorithm (6). In numerical comparisons, we consider the blur functions Gaussian and Motion in MATLAB and add random noise. Cameraman, Goldhill, Mountain and Barbara are used as test images. The signal to noise ratio (SNR) is used to determine the quality of the restored images, and it is defined as follows:

$$SNR = 20 \log \frac{\|x\|_2}{\|x - x_n\|_2}$$

where x and x_n are the original image and the estimated image at iteration n, respectively. In all numerical comparisons, we choose parameters as $\alpha_n = \frac{1}{2}$, $\theta_n = \frac{1}{10}$, $\beta_n = \frac{1}{2n}$, $\gamma_n = \frac{1}{10n}$, $\lambda = 0.99$, $f(x) = 0.99x$ and $\tau = 0.0001$.

We compare Algorithm 2 to algorithms (4), (5) and (6), and use cameraman and barbara images and motion and gaussian blur functions. Numerical results corresponding to the above selections are given in the following figures.

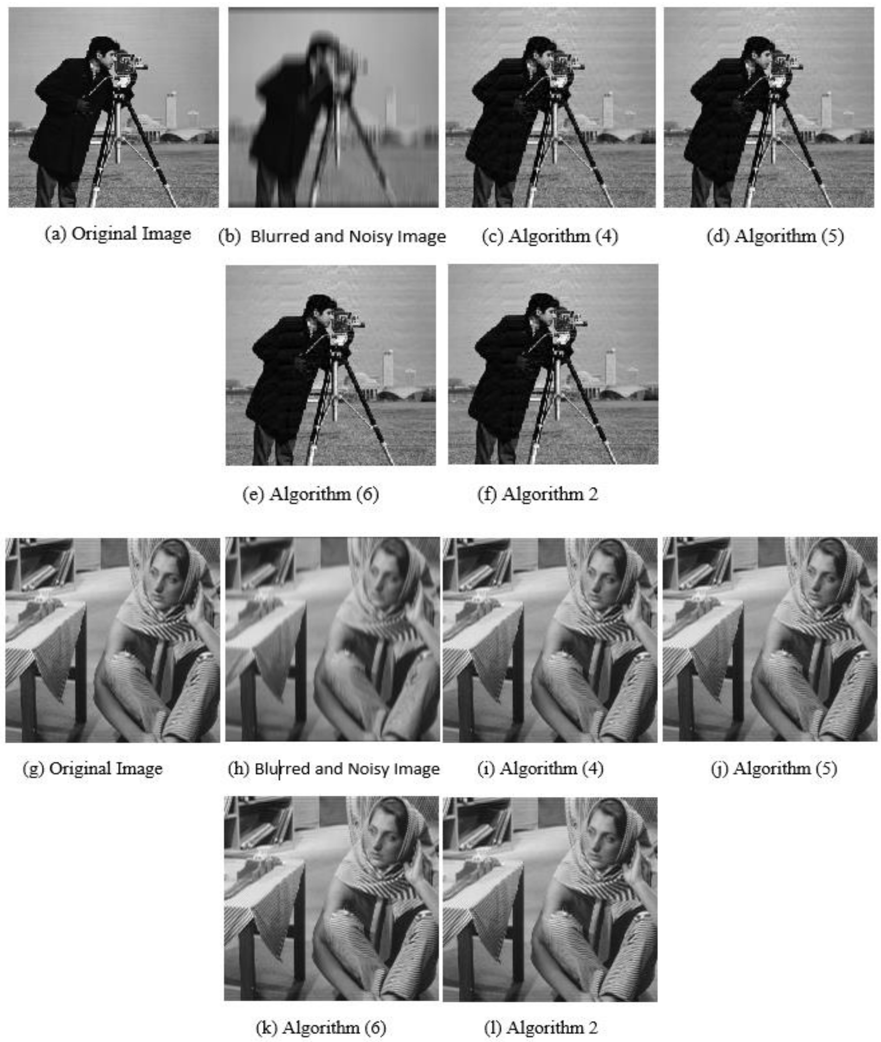
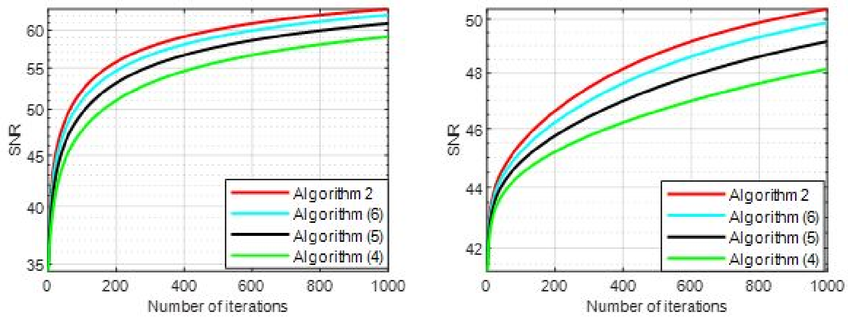


Figure 1. Comparisons of algorithms for Cameraman and Barbara images.



(a) SNR values for Cameraman image (b) SNR values for Barbara image

FIGURE 2. Comparisons of SNR values according to the algorithms

Experimental results show that our algorithm has a better image restoration than other algorithms.

6. Conclusion

We presented a modified preconditioning forward-backward technique to handle the image restoration problem effectively in this work.We also showed that the Algorithm 1 has strong convergence under certain conditions. Experimental findings demonstrate that the Algorithm 2 provides superior image restoration with higher SNR than Algorithm (4), Algorithm (5) and Algorithm (6).

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