U.P.B. Sci. Bull., Series A, Vol. 85, Iss. 4, 2023 ISSN 1223-7027

CONTINUOUS FRAMES AND CAUCHY DUAL OF CLOSED RANGE OPERATORS

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Frames are basis-like systems that reconstruct elements of a vector space with different coefficients. This paper is concerned with the construction of continuous K-frames by Cauchy dual of some closed-range operators in Hilbert spaces. Also, some continuous K-frames are presented by the combinations of operators with special conditions.

Keywords: Continuous K-frames, Cauchy dual operator, EP operator, Frames.
MSC2020: 42C15, 47A15

1. Introduction

The concept of discrete frame in Hilbert spaces was first introduced by Duffin and Schaeffer in 1952 to deal with the nonharmonic Fourier series [7] even though, it was raised on the mathematical and physical scene in 1986 with the work of Daubechies, Grossmann, Meyer because of their use in wavelet analysis [4].

Nowadays, frames have been used as a powerful alternative to Hilbert bases because of their redundancy and flexibility. They are also very useful tools in the characterization of function spaces and fields of applications, such as coding and communications [22], signal processing [9], and so on.

The continuous frames were defined by Antoine and Gazeau [2] and later, independently by Kaiser [13]. These kinds of frames are the first generalization frames to family indexed by some locally compact space endowed with a Radon measure. The reader can refer to [17, 16].

Mainly in [10], L. Găvruta introduced the notion of K-frames as a generalization of discrete frames due to some potential applications in sampling theory.

The paper is organized in two sections. Section 2 contains some definitions and properties of Cauchy duals of operators and EP operators that are used in the presented results. The main goal of this paper is illustrated in Section 3. Some classes of generalized continuous frames are constructed by the Cauchy dual of closed-range operators in Hilbert spaces and EP operators.

2. Preliminaries of Special Operators

The important collections of operators in the operator theory are the Cauchy dual of closed-range operators and EP operators. Their properties help to construct classes of continuous frames in this manuscript. In this section, we introduce the Cauchy dual of closed-range operators and EP operators and review some properties.

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In this manuscript, $\mathcal{H}$ is a Hilbert space , $\mathcal{B}(\mathcal{H})$ denotes the Banach algebra of all bounded linear operators on $\mathcal{H}$ and I is the identity operator. Let $GL(\mathcal{H})$ be the set of all bounded linear operators with the bounded inverse. As usual, for $E \subset \mathcal{H}$, P_E is the orthogonal projection on E. Also, $\{\Gamma\}'$ is the set of all of the elements of $\mathcal{B}(\mathcal{H})$ which commute with every element of Γ;

$$\{\Gamma\}' = \{B \in \mathcal{B}(\mathcal{H}) : BA = AB, for\, all\, A \in \Gamma\}$$

A useful equivalency relation between the range of two operators and their norms is needed that is presented as follows.

Lemma 2.1. [6] *Let A and B be two elements of $\mathcal{B}(\mathcal{H})$. Then the following statements are equivalent.*

1. $R(A) \subseteq R(B)$;
2. $\alpha \| A^*x \|^2 \leq \| B^*x \|^2$, *for some* $\alpha > 0$.

The Moore-Penrose inverse of a closed range operator $A \in \mathcal{B}(\mathcal{H})$ is defined as the unique operator $A^\dagger \in \mathcal{B}(\mathcal{H})$ with the following property.

$$AA^\dagger x = x, \ for\ every\ x \in R(A).$$

In the following, some properties related to Moore-Penrose inverses are presented. For further information, we refer to [11].

Proposition 2.1. *For a closed range operator $A \in \mathcal{B}(\mathcal{H})$, the following statements are holding.*

1. $AA^\dagger = P_{R(A)}, \ A^\dagger A = P_{R(A^*)}$;
2. $R(A^\dagger) = R(A^*) = N(A)^\perp$;
3. $N(A^\dagger) = N(A^*) = R(A)^\perp$.
4. *For $B \in \mathcal{B}(\mathcal{H})$, if A and B commute with each other, then $A^\dagger$ and B commute too.*

In [23], S. Shimorin introduced the Cauchy dual of left-invertible operators as a powerful tool in the model theory of left-invertible operators. Recently, H. Ezzahraoui [8] introduced the notion of Cauchy dual for a closed range operator as follows; the Cauchy dual of a closed range operator A is defined by $\omega(A) := A(A^*A)^\dagger$. For example of a Cauchy dual operator, assume that $A \in \mathcal{B}(\mathcal{H})$ such that $A^2 = A$. Then $\omega(A) = P_{R(A)}P_{R(A^*)}$ because

$$A^\dagger = A^\dagger AA^\dagger = A^\dagger AAA^\dagger = P_{R(A)}P_{R(A^*)}.$$

Now, we review some properties of the Cauchy dual of a closed range operator.

Proposition 2.2. [8] *Let $A \in \mathcal{B}(\mathcal{H})$ be a closed range operator. Then the following assertations valid.*

1. $\omega(A) = A^{*\dagger}$;
2. $\omega(\omega(A)) = A$;
3. $\omega(A)A^* = P_{R(A)}$;
4. *If $B \in \mathcal{B}(\mathcal{H})$ is unitary, then $\omega(BAB^*) = B\omega(A)B^*$.*

Furthermore, the concept of EP matrix, as an extension of normal matrices, was introduced by Schwerdtfeger [19] and was extended by Campbell and Meyer [21] to closed range operators on a Hilbert space. The operator $A \in \mathcal{B}(\mathcal{H})$ is called to be an EP operator if $R(A)$ is closed and $R(A) = R(A^*)$. (See [14]) Equivalently, A is an EP operator if and only if $AA^\dagger = A^\dagger A$. For seeing an example for an EP operator, considering $A \in \mathcal{B}(l^2(\mathbb{C}))$ be defined by:

$$A\left((x_j)_{j\geq 1}\right) = \left((y_j)_{j\geq 1}\right),$$

where

$$y_j = \begin{cases} x_2 + x_3 & \text{if } j = 3 \\ x_j & \text{if } j \neq 3. \end{cases}$$

Now, some families of EP operators that will be needed in this study, are introduced.

Theorem 2.1. [20] *Let $A \in \mathcal{B}(\mathcal{H})$ be a closed range operator. Then the following statements are satisfying.*

1. *If A is normal, then A is an EP operator.* [12]
2. *For a closed range operator $\mathcal{B}(\mathcal{H})$ that $AB = BA$ and $R(A^*) \cap R(B^*) = R(AB)$, the operator AB is an EP operator.*

3. Constructing Some Continuous K-frames

Cauchy duals obtain some new classes of continuous K-frames. In this section, the new continuous K-frames are constructed in two parts by Cauchy duals. Firstly, some closed range operators $K \in \mathcal{B}(\mathcal{H})$ are considered that construct a continuous K-frame. And, in the second part, we characterize some properties for operator A that generates a continuous AK-frame.

Throughout this section, the operator $K \in \mathcal{B}(\mathcal{H})$ is a closed range operator. In 2006, Rahimi and at. introduced Continuous frames for a Hilbert space $\mathcal{H}$ and a measure space (Ω, μ) as a mapping $F : \Omega \longrightarrow \mathcal{H}$ with the following properties.

1. F is weakly-measurable, i.e., for all $x \in \mathcal{H}$, the mapping : $t \longrightarrow \langle x, F(t)\rangle$ is a measurable function on Ω.
2. There exist positive constants α and β such that

$$\alpha \parallel x \parallel^2 \leq \int_\Omega \mid \langle x, F(t)\rangle \mid^2 d\mu(t) \leq \beta \parallel x \parallel^2 .$$

After, in [10], this concept is generalized to K-frames as a generalization of discrete frames due to some potential applications in sampling theory.

A mapping $F : \Omega \longrightarrow \mathcal{H}$ is called a continuous K- frame with respect to a measure space (Ω, μ), if

1. F is weakly-measurable, i.e., for all $x \in \mathcal{H}$, the mapping : $t \longrightarrow \langle x, F(t)\rangle$ is a measurable function on Ω.
2. There exist positive constants α and β such that

$$\alpha \parallel K^* x \parallel^2 \leq \int_\Omega \mid \langle x, F(t)\rangle \mid^2 d\mu(t) \leq \beta \parallel x \parallel^2, \quad \forall x \in \mathcal{H}.$$

In the following, we see an example of continuous K- frames.

Example 3.1. *Let $K \in \mathcal{B}\left(\mathbb{C}^2\right)$ be defined as follows.*

$$K = \begin{bmatrix} 0 & 0 \\ 1 & \sqrt{3}i \end{bmatrix}.$$

And, assume

$$\begin{array}{cccc} F & : & \mathbb{R} & \longrightarrow & \mathbb{C}^2 \\ & & t & \longmapsto & \left(0; e^{-(2+i)t^2}\right). \end{array}$$

For $x = (x_1, x_2) \in \mathbb{C}^2$, we have

$$\int_{\mathbb{R}} \mid \langle x, F(t)\rangle \mid^2 d\mu(\omega) = \frac{\sqrt{\pi}}{2} \mid x_2 \mid^2,$$

and

$$\parallel K^* x \parallel^2 = 4 \mid x_2 \mid^2 .$$

Consequently, for $x \in \mathbb{C}^3$, we obtain the follwing inequalities.

$$\frac{\sqrt{\pi}}{9} \parallel K^* x \parallel^2 \leq \int_{\mathbb{R}} \mid \langle x, F_\omega \rangle \mid^2 d\mu(\omega) \leq \parallel x \parallel^2 .$$

Therefore, F is a continuous K-frames for $\mathbb{C}^3$.

To continue, we need to know about some properties of measurable functions that the following lemma illustrates.

Lemma 3.1. [24] *Let (Ω, μ) be a measure space, $g \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator and $f : \Omega \longrightarrow \mathcal{H}$ be a measurable function. Then gf is a measurable function and the following equality holds.*

$$g \int_\Omega f d\mu = \int_\Omega g f d\mu.$$

At first, we see that the classes of continuous K-frames and continuous $\omega(K)$-frames are the same.

Theorem 3.1. *Let K be a closed range operator and let F be a continuous K-frame for $\mathcal{H}$. Then the following conditions are equivalent.*

1. *F is a continuous K-frame for $\mathcal{H}$.*
2. *F is a continuous $\omega(K)$-frame for $\mathcal{H}$.*

Before, the relation between the ranges of a closed range operator and its Cauchy duals is considered.

Lemma 3.2. *Let $K \in \mathcal{B}(\mathcal{H})$ be closed range. Then*

$$R(\omega(K)) = R(K).$$

Proof. By the properties of the Moore-Penrose inverse, the following equalities hold.

$$R(\omega(K)) = R\left(K^{\dagger *}\right) = N\left(K^\dagger\right)^\perp = N(K^*)^\perp = R(K).$$

This complete the proof. □

Proof. **of Theorem 3.1** $1 \Rightarrow 2$: By Lemma 3.2, we have

$$R(\omega(K)) = R(K).$$

Using Lemma 2.1, there exists $\alpha > 0$ such that for $x \in \mathcal{H}$,

$$\alpha \parallel \omega(K)^* x \parallel^2 \leq \parallel K^* x \parallel^2 .$$

Since F is a continuous K-frame for $\mathcal{H}$, there exist positive constants α' and β such that for $x \in \mathcal{H}$,

$$\alpha' \parallel K^* x \parallel^2 \leq \int_\Omega \mid \langle x, F(t) \rangle \mid^2 d\mu(t) \leq \beta \parallel x \parallel^2,$$

and this implies

$$\alpha\alpha' \parallel \omega(K)^* x \parallel^2 \leq \int_\Omega \mid \langle x, F(t) \rangle \mid^2 d\mu(t) \leq \beta \parallel x \parallel^2, \quad \forall x \in \mathcal{H}.$$

Therefore, F is an $\omega(K)$-continuous frames for $\mathcal{H}$.
$2 \Rightarrow 1$: The converse is valid by $\omega(\omega(K)) = K$ in Proposition 2.2. □

Corollary 3.1. *Let F be a continuous K-frame for $\mathcal{H}$. Then F is a continuous $P_{R(K)}$-frame for $\mathcal{H}$.*

Proof. For $x \in \mathcal{H}$, we have

$$\| (\omega (K) K^*)^* x \|^2 \leq \| K \|^2 \| \omega (K)^* x \|^2 .$$

Since F is a continuous K-frame for $\mathcal{H}$, for some positive constants α and β, one obtaines

$$\alpha \| \omega (K)^* x \|^2 \leq \int_{\Omega} | \langle x, F(t) \rangle |^2 \, d\mu (t) \leq \beta \| x \|^2, \quad \forall x \in \mathcal{H}.$$

Hence

$$\alpha \| K \|^{-2} \| (\omega (K) K^*)^* x \|^2 \leq \int_{\Omega} | \langle x, F(t) \rangle |^2 \, d\mu (t) \leq \beta \| x \|^2, \; \forall x \in \mathcal{H}.$$

By Proposition 2.2, we have $K^* \omega (K) = P_{R(K)}$, and this completes the proof. □

Now, we study EP operators for constructing continuous K-frames. The following theorem presents some equivalent conditions for continuous K-frames, continuous K^*-frames, and continuous $K^\dagger$-frames.

Theorem 3.2. *Let K be an EP operator. Then the following statements are equivalent.*

1. *F is a continuous K-frame for $\mathcal{H}$;*
2. *F is a continuous K^*-frame for $\mathcal{H}$;*
3. *F is a continuous $K^\dagger$-frame for $\mathcal{H}$.*

Proof. $1 \Rightarrow 2$. Assume that K is an EP operator. Then

$$R(K) = R(K^*) .$$

By Lemma 2.1, there are positive constants α and β such that

$$\alpha \| Kx \|^2 \leq \| K^* x \|^2 \leq \beta \| Kx \|^2, \quad \forall x \in \mathcal{H}.$$

Since F is a continuous K-frame for $\mathcal{H}$, the following inequalities are valid for positive constants α' and β'.

$$\alpha' \| K^* x \|^2 \leq \int_{\Omega} | \langle x, F(t) \rangle |^2 \, d\mu (t) \leq \beta' \| x \|^2, \; \forall x \in \mathcal{H},$$

and

$$\alpha\alpha' \| Kx \|^2 \leq \int_{\Omega} | \langle x, F(t) \rangle |^2 \, d\mu (t) \leq \beta' \| x \|^2, \; \forall x \in \mathcal{H}.$$

Therefore, F is a continuous K^*-frame for $\mathcal{H}$.
Other parts of the theorem are given by $R\left(K^\dagger\right) = R(K^*) = R(K)$ and a similar technique. □

Here, we can construct new continuous K-frames by an EP operator and its combinations.

Theorem 3.3. *Let A be an EP operator and let F be a continuous K-frame for $\mathcal{H}$ such that $KA^* = A^*K$. Then $\omega (A) F$ is a continuous K-frame for $R(A)$.*

Proof. Since F is a continuous K-frame for $\mathcal{H}$, the following result is given for positive constants α and β.

$$\alpha \| K^* A^\dagger x \|^2 \leq \int_{\Omega} | \langle A^\dagger x, F(t) \rangle |^2 \, d\mu (t) \leq \beta \| A^\dagger x \|^2 .$$

For $x \in R(A)$, we have

$$\omega (A) \omega (A)^\dagger = P_{R(\omega(A))} = P_{R(A)},$$

and also one obtains that $KA^{*\dagger} = A^{*\dagger} K$ by Proposition 2.1. Therefore,

$$K(x) = K\left(\omega (A) \omega (A)^\dagger x\right) = K\omega (A) (A^* x) = (\omega (A) K) (A^* x), \;\; \forall x \in \mathcal{H}.$$

On the other hand, A is an EP operator, so $A^{*}x \in R(A^{*}) = R(A)$, for every $x \in \mathcal{H}$. Now, by Lemma 2.1, there exists $\alpha > 0$ such that the following statement is satisfying.

$$\alpha \parallel K^{*}x \parallel^{2} \leq \parallel (\omega(A)K)^{*}x \parallel^{2},$$

and

$$\alpha'' \parallel K^{*}x \parallel^{2} \leq \int_{\Omega} | \langle x, \omega(A)F(t)\rangle |^{2} \, d\mu(t) \leq \beta' \parallel x \parallel^{2},$$

which $\alpha'' = \alpha\alpha'$ and $\beta' = \beta \parallel A^{\dagger} \parallel^{2}$. Lemma 3.1 gives that $\omega(A)F$ is weakly-measurable and this completes the proof. □

Corollary 3.2. *Assume F is a continuous K-frame for $\mathcal{H}$. The following statements are valid.*

1. *If A and B are arbitrary elements of $B(\mathcal{H})$ such that $R(A^{*}) \cap R(B^{*}) = R(AB)$, $AB = BA$ and $K \in \{A^{*}, B^{*}\}'$, then $\omega(AB)F$ is a K-continuous frame for $R(AB)$.*
2. *If $A \in \mathcal{B}(\mathcal{H})$ is a normal and closed range such that $KA^{*} = A^{*}K$. Then $\omega(A)F$ is a continuous K-frame for $R(A)$.*

Proof. The proof of 1 is obtained from Theorem 2.1 and Theorem 3.3, and the second statement follows from Theorem 2.1 and Theorem 3.3. □

In the following example, we see samples of continuous K-frames that are constructed by an EP operator and its Cauchy dual.

Example 3.2. *Let $\{e_1, e_2\}$ be an orthonormal basis of $\mathbb{C}^2$. Consider the operators K and A of $\mathcal{B}(\mathbb{C}^2)$ as follows.*

$$K = \begin{bmatrix} 1 - e^{\frac{2\pi i}{3}} & 1 \\ 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 1 & 0 \\ 1 & e^{\frac{-2\pi i}{3}} \end{bmatrix},$$

where $i^2 = -1$. Obviously, we have $A^3 = I$ and then $A^{\dagger} = A^2$. Consequently, A is an EP operator and we can easily show that

$$\omega(A) = A^{*2} = \begin{bmatrix} 1 & e^{\frac{i\pi}{3}} \\ 0 & e^{\frac{4i\pi}{3}} \end{bmatrix}.$$

Now, set

$$F : \mathbb{R} \longrightarrow \mathbb{C}^2; \quad t \longmapsto \left(e^{-\dfrac{(1+i)t^2}{2}} ; 0 \right).$$

For $x = (x_1, x_2) \in \mathbb{C}^2$, we have

$$\int_{\mathbb{R}} | \langle x, F(t)\rangle |^{2} \, d\mu(t) = \int_{\mathbb{R}} | \langle x, \omega(A)F(t)\rangle |^{2} \, d\mu(t) = \sqrt{\pi} \, | x_1 |^{2} .$$

On the other hand, for $x = (x_1, x_2) \in \mathbb{C}^2$, one obtaines that the following equality.

$$\parallel K^{*}x \parallel^{2} = 4 \, | x_1 |^{2} .$$

Then, the main result is given by the above relations as follows.

$$\frac{1}{4} \parallel K^{*}x \parallel^{2} \leq \int_{\mathbb{R}} | \langle x, F(t)\rangle |^{2} \, d\mu(t) \leq \sqrt{\pi} \parallel x \parallel^{2} .$$

Therefore, F and $\omega(A)F$ are continuous K-frames for $\mathcal{H}$.

The unitary operators also generate a family of continuous K-frames. The following theorem illustrates this fact.

Theorem 3.4. *Let U be a unitary operator and let F be a continuous K-frame for $\mathcal{H}$. Then UF is a continuous $(U\omega(K)U^{*})$-frame for $\mathcal{H}$.*

Proof. For $x \in \mathcal{H}$, we have

$$\| (UKU^*)^* x \|^2 \leq \| K^* U^* x \|^2 .$$

Therefore, there exist positive constants α and β such that the following statemets are valid for $x \in \mathcal{H}$.

$$\alpha \| (UKU^*)^* x \|^2 \leq \int_\Omega | \langle x, UF(t) \rangle |^2 \, d\mu(t) \leq \beta \| U \|^2 \| x \|^2 .$$

On the other hand, UF is weakly-measurable by Lemma 3.1. Then UF is a continuous (UKU^*)-frame for $\mathcal{H}$. Moreover, Proposition 2.2 and Theorem 3.1 conclude that UF is also a continuous $(U\omega(K)U^*)$-frame for $\mathcal{H}$. □

In the final part, we present another family of continuous K-frame by the combinations of operator K and some operators of $\mathcal{B}(\mathcal{H})$ and see some samples of this family in the next example.

Theorem 3.5. *Let F be a continuous K-frame for $\mathcal{H}$ and $L \in \mathcal{B}(\mathcal{H})$ such that $KLK = K$. Then $(KL)F$ is a continuous K-frame for $\mathcal{H}$.*

Proof. By the property of F, there are positive constants α and β such that for all $x \in \mathcal{H}$, the following inequalities hold.

$$\alpha \| K^* x \|^2 \leq \int_\Omega | \langle x, F(t) \rangle |^2 \, d\mu(t) \leq \beta \| x \|^2 .$$

Now, let L be a bounded operator on $\mathcal{H}$ such that $KLK = K$. Then

$$\alpha \| K^* L^* K^* x \|^2 \leq \int_\Omega \langle L^* K^* x, F_\omega \rangle \leq \beta \| L^* K^* x \|^2, \quad \forall x \in \mathcal{H}.$$

By taking $\beta' = \beta \| KL \|^2$, we obtain

$$\alpha \| K^* x \|^2 \leq \int_\Omega | \langle x, (KL) F(t) \rangle |^2 \leq \beta' \| x \|^2, \quad \forall x \in \mathcal{H}.$$

Also, Lemma 3.1 concludes that $(KL)F$ is weakly-measurable. So $\{(KL)x_j\}_{j \in \mathbb{J}}$ is a continuous K-frame for $\mathcal{H}$.

□

Example 3.3. *Assume that $z \in \mathbb{C}$ and the operators K and L are defined on $\mathcal{B}(\mathbb{C}^2)$ as follows.*

$$K = \begin{pmatrix} e^{iz} & e^{-iz} \\ 0 & 0 \end{pmatrix}, \; L = \begin{pmatrix} e^{-iz} & 0 \\ 0 & 0 \end{pmatrix}.$$

Consider $F : \mathbb{R} \longrightarrow \mathbb{C}^2; \quad t \longmapsto \left(ie^{-t^2}, 0\right)$, where $i^2 = -1$. For $x = (x_1, x_2) \in \mathbb{C}$, we have

$$\| K^* x \|^2 = 2 | x_1 |^2 \quad \text{and} \quad \int_\mathbb{R} | \langle x, F(t) \rangle |^2 \, d\mu(t) = \sqrt{\frac{\pi}{2}} | x_1 |^2 .$$

Then

$$\frac{1}{2} \| K^* x \|^2 \leq \int_\mathbb{R} | \langle x, F(t) \rangle |^2 \, d\mu(t) \leq \sqrt{\pi} \| x \|^2 .$$

Since

$$KL = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

the following inequalities are obtained for $x \in \mathbb{C}$.

$$\frac{1}{2} \| K^* x \|^2 \leq \int_\mathbb{R} | \langle x, (KL) F(t) \rangle |^2 \, d\mu(t) \leq \sqrt{\pi} \| x \|^2 .$$

Therefore, $(KL)F$ is a continuous K-frames for $\mathbb{C}$.

Acknowledgment

The author would like to express her sincere gratitude to anonymous referees for their helpful comments and recommendations which improved the quality of the paper.

REFERENCES

[1] *Alijani, A., Dehghan, M. A.*, $*$-frames in Hilbert C^*-modules, U.P.B. Sci. Bull., Series A, **73** (4) (2011), 89-106.

[2] *Ali, S.T., Antoine, J.P., Gazeau, J.P.*, Continuous frames in Hilbert spaces, Ann. Phys., **222** (1993), 1-37.

[3] *Christensen, O.*, An Introduction to Frames and Riesz Bases. Applied and Numerical Harmonic Analysis, Boston Inc, Boston. 2003.

[4] *Daubechies, I., Grossmann, A., Meyer, Y.*, Painless non orthogonal expansions, J. Math. Phys., **27** (1986), 1271-1283.

[5] *Djordjevic, D.S.*, Product of EP operators on Hilbert spaces, Proc. Am. Math. Soc., **129** (6) (2001), 1727-1731.

[6] *Douglas, R.G.*, On majorization, factorization, and range inclusion of operators on Hilbert space, Proc. Am. Math. Soc., **17** (1966), 413-415.

[7] *Duffin, R.J., Schaeer, A.C.*, A class of nonharmonic fourier series, Trans. Am. Math. Soc., **72** (1952), 341-366.

[8] *Ezzahraoui, H., Mbekhta, M., Zerouali, E. H.*, On the Cauchy dual of closed range operators, Acta Sci. Math. (Szeged), **85** (2019), 231-248.

[9] *Ferreira, P.*, Mathematics for multimedia signal processing II: Discrete finite frames and signal reconstruction, Byrnes, J.S. (ed.), **17** (1999), 35-54.

[10] *Găvruţa, L.*, Frames for operators, Appl. Comput. Harmonic Anal., **32** (1) (2012), 139-144.

[11] *Harte, R., Mbekhta, M.*, On generalized inverses in C^*-algebras. Studia Math., **103** (1992).

[12] *Jalaeian, M., Karizaki, M. M., Mahmoud, H.*, Conditions that the product of operators is an EP operator in Hilbert C^*-module, Linear Multilinear Algebra, **68** (10) (2020), 1990-2004.

[13] *Kaiser, G.*, A Friendly Guide to Wavelets, Birkhäuser, Boston. 1994.

[14] *Karizaki, M. M., Djordjević, D.S.*, Commuting C^*-modular operators, Aequationes Math., **90** (2016), 1103-1114.

[15] *Koliha, J.J.*, The Drazin and Moore-Penrose inverse in $\mathbb{C}^*$-algebras, Math. Proc. R. Ir. Acad., **99** A (1) (1999), 17-27.

[16] *Mirzaee Azandaryani, M., Javadi, Z.*, Pseudo-duals of continuous frames in Hilbert spaces, J. Pseudo-Differ. Oper. Appl., **13** (4) (2022), 52.

[17] *Rahimlou, GH., Ahmadi, R., Jafarizadeh, M. A., Nami, S.*, Continuous K-frames an their duals, Sahand Commun. Math. Anal., **17** (3) (2020), 145-160.

[18] *Rahimi, A., Najati, A., Deghan,Y. N.*, Continuous frames in Hilbert spaces, Methods Funct. Anal. Topology, **12** (2) (2006), 170-182.

[19] *Schwerdtfeger, H.*, Introduction to Linear Algebra and the Theory of Matrices, P. Noordhoff, N.V., Groningen-Holland. 1961.

[20] *Sharifi, K.*, The product of operators with closed range in Hilbert C*-modules, Linear Algebra Appl.,**435** (2011), 1122-1130.

[21] *Stephen, L., Campbell, Carl Meyer, D.*, EP operators and generalized inverses, Canad. Math. Bull., **18** (3) (1975), 327-333.

[22] *Strohmer, T., Heath, R.W., Grassmannian, Jr.*, Frames with applications to coding and communication, Appl. Comput. Harmon. Anal., **14** (2003), 257-275.

[23] *Shimorin, S.*, Wold-type decompositions and wandering subspaces for operators close to isometries, J. Reine Angew. Math., **531** (2001), 147-189.

[24] *Yosida, K.*, Functional Analysis of Grundlehren der Mathematischen Wissenschaften, Springer, Berlin, Germany, 6 edition. 1980.