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POLYNOMIAL FUNCTION AS AN APPROXIMATION OF THE ANALYTIC SOLUTION OF A LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATION, OBTAINED NUMERICALLY USING A SINGLE CELL

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The author has developed a method to integrate numerically first order Partial Differential Equations (PDE) using complete polynomials of high degree, called Concordant Functions (CF) [1]. The procedure was based on the idea – quasi unanimously accepted - that this can be done by dividing the integration domain (supposed rectangular) into a great number of elements. This approach can ensure the obtaining of results having a good level of accuracy, which improves when the number of elements increases. Recently, the author tackled an ***opposite idea****: solve a nonlinear first order PDE with a satisfactory level of accuracy,* ***using a SINGLE CELL equal to the whole quite large domain of integration*** *[2].*

Here is approached for the first time the subject of the second order linear ELLIPTIC PDEs based on the last idea introduced by [2]: obtain numerically an approximate polynomial solution – valid on the whole domain of integration - using a single cell. Similarly, to an analytic solution, this polynomial can be used to compute at any point - using its coordinates - an approximate value of the function and its derivatives or to draw the graph of the function.

The second order PDE is integrated starting from a general form having, initially, constant coefficients. An example is performed using - besides the unicellular method - a well-known multi-element commercial code that solved the PDE using 20992 triangular elements. The results obtained are close to each other. Finally, is – shortly - presented the extension of the method for variable coefficients.

1. A second order elliptic Partial Differential Equation (PDE)

A complete two-dimensional linear second order PDE, having ϕ as unknown function, includes the following terms

$$PDE = a\frac{\partial^2\phi}{\partial x^2} + b\frac{\partial^2\phi}{\partial x\partial y} + c\frac{\partial^2\phi}{\partial y^2} + M\frac{\partial\phi}{\partial x} + N\frac{\partial\phi}{\partial y} + P\phi + W(x,y) = 0 \qquad (1.1)$$

All the coefficients (a,b,c,M,N,P) are considered, in this first approach, as constants, while the free term W(x,y) is a complete polynomial of any degree. The PDEs with variable coefficients are analyzed later on.

The *PDE* (1.1) is ELLIPTIC if [7] $\qquad b^2 - 4ac < 0 \qquad (1.2)$

This is the first attempt made by the author to numerically integrate an *elliptic* PDE. Until now only the first order PDEs were approached. A first order PDE has only part of the terms mentioned in (1.1)

$$M\frac{\partial\phi}{\partial x} + N\frac{\partial\phi}{\partial y} + P\phi + W(x,y) = 0 \qquad (1.3)$$

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The integration will be performed inside a rectangular domain having the dimensions **B×H**, where B is the base and H is the height. The four corners are noted: *Node 1*(0,0), *Node 2*(B,0), *Node3*(0,H), *Node 4* (B,H). For an elliptic PDE is compulsory to be known the *boundary conditions* along all the four sides of the rectangular domain. These conditions will be considered here as given only by the values of the function ϕ (Dirichlet). For simplicity, the function ϕ will vary on each side either *linear* or *parabolic*, being continuous in the corners.

2. Numeric integration of PDE (1.1)

The numeric integration will be performed according to the methodology described in [1], with some obvious modifications imposed by the increased number of unknown terms depending on ϕ , from 3 (see 1.3) to 6 (see 1.1). Although it is assumed that the reader has downloaded [1] and [2], it is useful to highlight the specific concepts used in the integration procedure: **Concordant Functions** and **Target Point**. Likewise, the procedure based on the Target Residual, used to assess the accuracy of the results will be described.

2.1 The Concordant Function and its first and second derivatives

A **Concordant Function** - noted CF – is a **complete** polynomial of a given degree. For instance a third degree CF - including 10 terms - is given by

$$CF3 = CF310 = C_1 \quad + C_2 x + C_3 y \quad + C_4 x^2 + C_5 xy + C_6 y^2 \quad + C_7 x^3 + C_8 x^2 y + C_9 xy^2 + C_{10} y^3 \tag{2.1}$$

It will be noted either as *CF3* - corresponding to the *third degree* - or *CF310* - including also the number of terms. The function *CF3* can be written as the product between two matrices [1]

$$\phi = CF3 = CF310 = \left[X^{(0)}Y^{(0)}\right] * [Cz]_3 \tag{2.2}$$

where

$$\left[X^{(0)}Y^{(0)}\right] = \left[1 \quad x \;\; y \quad x^2 \;\; xy \;\; y^2 \quad x^3 \;\; x^2 y \;\; xy^2 \;\; y^3\right] \tag{2.3}$$

$$[Cz]_3 = \left[C_1 \;\; C_2 \;\; C_3 \;\; C_4 \;\; C_5 \;\; C_6 \;\; C_7 \;\; C_8 \;\; C_9 \;\; C_{10}\right]^T \tag{2.4}$$

Using this notation, the first derivatives of $[\phi]$ can be written as

$$[\partial\phi/\partial x] = \partial\left[\left[X^{(0)}Y^{(0)}\right] * [Cz]\right]/\partial x = \left[\partial\left[X^{(0)}Y^{(0)}\right]/\partial x\right] * [Cz] = \left[X^{(1)}Y^{(0)}\right] * [Cz] \tag{2.5}$$

$$[\partial\phi/\partial y] = \partial\left[\left[X^{(0)}Y^{(0)}\right] * [Cz]\right]/\partial y = \left[\partial\left[X^{(0)}Y^{(0)}\right]/\partial y\right] * [Cz] = \left[X^{(0)}Y^{(1)}\right] * [Cz] \tag{2.6}$$

where

$$\left[X^{(1)}Y^{(0)}\right] = \partial\left[X^{(0)}Y^{(0)}\right]/\partial x = \left[0 \quad 1 \;\; 0 \quad 2x \;\; y \;\; 0 \quad 3x \;\; 2xy \;\; y^2 \;\; 0\right] \tag{2.7}$$

$$\left[X^{(0)}Y^{(1)}\right] = \partial\left[X^{(0)}Y^{(0)}\right]/\partial y = \left[0 \quad 0 \;\; 1 \quad 0 \;\; x \;\; 2y \quad 0 \;\; x^2 \;\; 2xy \;\; 3y^2\right] \tag{2.8}$$

Similarly, the second partial derivatives will be

$$\left[\partial^2\phi/\partial x^2\right] = \partial^2\left[\left[X^{(0)}Y^{(0)}\right] * [Cz]\right]/\partial x^2 = \left[\partial^2\left[X^{(0)}Y^{(0)}\right]/\partial x^2\right] * [Cz] = \left[X^{(2)}Y^{(0)}\right] * [Cz] \tag{2.9}$$

$$\left[\partial^2\phi/\partial x\partial y\right] = \partial^2\left[\left[X^{(0)}Y^{(0)}\right] * [Cz]\right]/\partial x\partial y = \left[\partial^2\left[X^{(0)}Y^{(0)}\right]/\partial x\partial y\right] * [Cz] = \left[X^{(1)}Y^{(1)}\right] * [Cz] \tag{2.10}$$

$$\left[\partial^2\phi/\partial y^2\right] = \partial^2\left[\left[X^{(0)}Y^{(0)}\right] * [Cz]\right]/\partial y^2 = \left[\partial^2\left[X^{(0)}Y^{(0)}\right]/\partial y^2\right] * [Cz] = \left[X^{(0)}Y^{(2)}\right] * [Cz] \tag{2.11}$$

where $\left[X^{(2)}Y^{(0)}\right], \left[X^{(1)}Y^{(1)}\right], \left[X^{(0)}Y^{(2)}\right]$ are obtained by deriving accordingly $\left[X^{(0)}Y^{(0)}\right]$.

Here will be used three higher degree CFs: ***CF6 or CF628*** (six degree – 28 terms) , ***CF7 or CF735*** (seven degree – 36 terms) and ***CF8 or CF845*** (eight degree – 45 terms).

2.2 **The Target Point**

The method proposed by the author in [2] attempts to find the terms of a ***unique*** Concordant Function for the whole domain D that satisfies two types of requirements: the user-defined boundary conditions (as those mentioned above) - on the one hand - and the *PDE* (1.1) considered *at a selected* location called **Target Point** - on the other.

For the first order PDEs, the boundary conditions are imposed only on two sides of the rectangular domain (sides 1-2 and 1-3), the integration developing towards the top of the rectangle, namely *Node 4*, which was named **Target Point [1,2].** This was the point towards which was focused the main information that was spread to the other elements.

When the object of integration is an elliptic *PDE*, for which **the boundary conditions are imposed on all the four sides of the integration domain**, the position – and also the significance - of this special point must change. In fact the results of the integration procedure will be tightly bound especially to this point, which – usually - will be located somewhere towards the middle of the rectangle, not at a corner. The *Target Point* will be usually selected at the intersection of the diagonals or near to it. This location is chosen because if there exists an extremum value of the function-solution (maximum or minimum), it is placed, in most cases, not far from this point. Anyway, is recommendable not to be in the proximity of any one of the four sides, where the influence of the boundary conditions may produce some disturbances.

2.3 The Target Point Residual

As George W. Collins II, wrote in his book [9]: "A numerical solution to a differential equation is of little use if there is no estimate of its accuracy. However, … the formal estimate of the truncation error is often more difficult than finding the solution".

The method developed in [1] avoids the "difficult" estimation of the error by controlling the accuracy of the computation using the **RESIDUAL,** which is **the difference between a result obtained by computation** and **the theoretical result.** In fact, the control is performed at the end of the computation taking the form of the **Target Point Residual (*RES$_T$*)**, which is obtained by replacing the values of the six unknowns that depend on ϕ, in the PDE (1.1)

$$Res_T = a\left(\frac{\partial^2\phi}{\partial x^2}\right)_T + b\left(\frac{\partial^2\phi}{\partial x\partial y}\right)_T + c\left(\frac{\partial^2\phi}{\partial y^2}\right)_T + M\left(\frac{\partial\phi}{\partial x}\right)_T + N\left(\frac{\partial\phi}{\partial y}\right)_T + P(\phi)_T + W(x_T, yT) \tag{2.12}$$

If, incidentally, the *Target Point* unknowns correspond to the exact solution, Res_T is null. Otherwise, the value of the Residual is different from zero, its value being a sure indication of the **global accuracy**, namely of the **error due to all six** *Target Point* unknowns.

3. The basic relations for the numeric integration of elliptic PDEs

3.1 The TWO STEPS strategy

The strategy that will be used is that developed in [2] consisting in two steps, namely:
a. Step 1. The basic idea is to find a CF that could be considered near the accurate (unknown) solution of a given elliptic PDE. Because no one knows which CF is the nearest to the accurate solution, several CFs must be tried namely **CF6, CF7, CF8** and – somewhere else – higher degree CFs, like CF9, CF10… and so on. Here - as a first approach - only the first three will be tried. Specific relationships corresponding to CF6 will be set down below, while for CF7 and CF8 they are briefly pointed out in *Appendix A*. No matter which *CF* is used, it will result an approximate solution that will be revealed by the values of the Target Residuals, more or less remote from zero.
b. Step 2. Based on the *Target Residual* values obtained in *Step 1* it will follow the procedure developed in [2], which modifies the *Concordant Functions* by combining those already obtained in *Step1*: CF6 with CF7 (resulting CF67), CF7 with CF8 (resulting CF78) and CF6 with CF8 (resulting CF68). Thus it will result finally six different Concordant Functions that "claim" to be the best solution. The decision will be taken by the program, based on a computed parameter, without the intervention of the user. The Concordant Function selected can be used, similar to an analytical solution, to calculate quickly the value of the function (or of a derivative) at any point in the integration domain, obviously with a small error.

3.2 First step

The first step begins with the search of an approximate solution - valid on the whole integration rectangular domain - using a *Concordant Function*. The straightforward analysis made below shows that for the elliptic *PDE*s the lowest degree that can be tried is[2] *CF628* (six degree-28 terms). For obtaining the 28 unknown coefficients is necessary to be established 28 equations.

This is possible by using the available information furnished by the *Boundary conditions* and by the *Target Point*. The *Boundary conditions* are known because they are imposed by the user. The particular information that can be obtained from the *Target Point* results by transferring the *PDE* (1.1) to this location.

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[2] A more thorough analysis dedicated to the first order *PDE*s can be found in [2]

$$a\left(\frac{\partial^2\phi}{\partial x^2}\right)_T + b\left(\frac{\partial^2\phi}{\partial x\partial y}\right)_T + c\left(\frac{\partial^2\phi}{\partial y^2}\right)_T + M\left(\frac{\partial\phi}{\partial x}\right)_T + N\left(\frac{\partial\phi}{\partial y}\right)_T + P(\phi)_T + W(x_T, y_T) = 0 \qquad (3.1)$$

The 6 terms depending on ϕ are unknown, but the relation (3.1) is able to furnish enough information that lead to the establishing of 6 equations. Consequently, for finding the 28 coefficients mentioned above, besides these 6 equations, 22 more equations are required; they are furnished by the *Boundary conditions*.

a. Boundary conditions: 22 equations

When the *six-degree* polynomial CF628 is selected, 6 boundary conditions can be used along each side of the rectangular domain [2]. For all the 4 sides it corresponds

6 conditions × 4 sides =24 boundary conditions.

Thus the total number of equations (24 *boundary conditions* + 6 from the *Target Point* = 30) is greater than the 28 coefficients of *CF628*. Consequently, the number of relations resulted from the boundary conditions will be reduced by two, becoming ***24 – 2 = 22 conditions***. This decision is advantageous because it moves away the danger of singularity.

b. Target Point: 6 equations

Choosing the 6 equations connected to the *Target point* follows the path developed in *Chapter 2* of [1], but here it is dedicated to second order elliptic *PDE*s. Some numerical methods perform the solving procedure by transforming the second order *PDE* in a system of first order. **Unlike this last methodology, the method developed here tackles directly the given PDE, without any other intermediary transformation**.

α. *Equations (I) and (II).* As in the case of first order *PDE*s ([1] pag.14), these two equations are represented by the *first order partial derivatives of PDE (1.1).* Because a *Concordant Function* is accepted as an approximate solution, the unknown function $\phi(x, y)$ is replaced in by *z(x,y)*

$$\phi(x, y) \approx z(x, y) = \left[X^{(0)}Y^{(0)}\right] * [Cz] \qquad (3.2)$$

so that the relation (1.1) becomes

$$PDE = a\frac{\partial^2 z}{\partial x^2} + b\frac{\partial^2 z}{\partial x\partial y} + c\frac{\partial^2 z}{\partial y^2} + M\frac{\partial z}{\partial x} + N\frac{\partial z}{\partial y} + Pz + W(x, y) = 0 \qquad (3.3)$$

The two first order partial derivatives of *PDE* (3.3) are considered - like (3.1) - applied in the Target Point

$$\left(\frac{\partial(PDE)}{\partial x}\right)_T = a\left(\frac{\partial^3 z}{\partial x^3}\right)_T + b\left(\frac{\partial^3 z}{\partial x^2\partial y}\right)_T + c\left(\frac{\partial^3 z}{\partial x\partial y^2}\right)_T + M\left(\frac{\partial^2 z}{\partial x^2}\right)_T + N\left(\frac{\partial^2 z}{\partial x\partial y}\right)_T + P\left(\frac{\partial z}{\partial x}\right)_T + \left(\frac{\partial W}{\partial x}\right)_T = 0$$

(3.4)

$$\left(\frac{\partial(PDE)}{\partial y}\right)_T = a\left(\frac{\partial^3 z}{\partial x^2\partial y}\right)_T + b\left(\frac{\partial^3 z}{\partial x\partial y^2}\right)_T + c\left(\frac{\partial^3 z}{\partial y^3}\right)_T + M\left(\frac{\partial^2 z}{\partial x\partial y}\right)_T + N\left(\frac{\partial^2 z}{\partial y^2}\right)_T + P\left(\frac{\partial z}{\partial y}\right)_T + \left(\frac{\partial W}{\partial y}\right)_T = 0 \qquad (3.5)$$

Extending the notations (2.5), (2.6), (2.9), (2.10), (2.11) to the third order derivatives, the equations (3.4) and (3.5) can be written as

$$[S_1]*[Cz]+\left(\frac{\partial W}{\partial x}\right)_T=0 \quad \text{(I)}, \qquad [S_2]*[Cz]+\left(\frac{\partial W}{\partial y}\right)_T=0 \quad \text{(II)}, \qquad \text{where}$$

$$[S_1]=a\left[X^{(3)}Y^{(0)}\right]_T+b\left[X^{(2)}Y^{(1)}\right]_T+c\left[X^{(1)}Y^{(2)}\right]_T+M\left[X^{(2)}Y^{(0)}\right]_T+N\left[X^{(1)}Y^{(1)}\right]_T+P\left[X^{(1)}Y^{(0)}\right]_T \quad (3.6)$$

$$[S_2]=a\left[X^{(2)}Y^{(1)}\right]_T+b\left[X^{(1)}Y^{(2)}\right]_T+c\left[X^{(0)}Y^{(3)}\right]_T+M\left[X^{(1)}Y^{(1)}\right]_T+N\left[X^{(0)}Y^{(2)}\right]_T+P\left[X^{(0)}Y^{(1)}\right]_T \quad (3.7)$$

The matrix $[Cz]$ is similar to (2.4), but including 28 terms.

β. *Equations (III), (IV) and (V).* The procedure used for obtaining the equations (3.4) and (3.5) continue, with the *second order partial derivatives of PDE (3.3)*

$$\frac{\partial^2(PDE)}{\partial x^2}=a\frac{\partial^4 z}{\partial x^4}+b\frac{\partial^4 z}{\partial x^3\partial y}+c\frac{\partial^4 z}{\partial x^2\partial y^2}+M\frac{\partial^3 z}{\partial x^3}+N\frac{\partial^3 z}{\partial x^2\partial y}+P\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 W}{\partial x^2}=0 \quad (3.8)$$

$$\frac{\partial^2(PDE)}{\partial x\partial y}=a\frac{\partial^4 z}{\partial x^3\partial y}+b\frac{\partial^4 z}{\partial x^2\partial y^2}+c\frac{\partial^4 z}{\partial x\partial y^3}+M\frac{\partial^3 z}{\partial x^2\partial y}+N\frac{\partial^3 z}{\partial x\partial y^2}+P\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 W}{\partial x\partial y}=0 \quad (3.9)$$

$$\frac{\partial^2(PDE)}{\partial y^2}=a\frac{\partial^4 z}{\partial x^2\partial y^2}+b\frac{\partial^4 z}{\partial x\partial y^3}+c\frac{\partial^4 z}{\partial y^4}+M\frac{\partial^3 z}{\partial x\partial y^2}+N\frac{\partial^3 z}{\partial y^3}+P\frac{\partial^2 z}{\partial y^2}+\frac{\partial^2 W}{\partial y^2}=0 \quad (3.10)$$

Using the same procedure as above, it results - similar to (I) and (II) - other 3 equations

$$[S_3]*[Cz]+\left(\frac{\partial^2 W}{\partial x^2}\right)_T=0 \quad \text{(III)}, \quad [S_4]*[Cz]+\left(\frac{\partial^2 W}{\partial x\partial y}\right)_T=0 \quad \text{(IV)}, \quad [S_5]*[Cz]+\left(\frac{\partial^2 W}{\partial y^2}\right)_T=0 \quad \text{(V)},$$

where

$$[S_3]=a\left[X^{(4)}Y^{(0)}\right]_T+b\left[X^{(3)}Y^{(1)}\right]_T+c\left[X^{(2)}Y^{(2)}\right]_T+M\left[X^{(3)}Y^{(0)}\right]_T+N\left[X^{(2)}Y^{(1)}\right]_T+P\left[X^{(2)}Y^{(0)}\right]_T \quad (3.11)$$

$$[S_4]=a\left[X^{(3)}Y^{(1)}\right]_T+b\left[X^{(2)}Y^{(2)}\right]_T+c\left[X^{(1)}Y^{(3)}\right]_T+M\left[X^{(2)}Y^{(1)}\right]_T+N\left[X^{(1)}Y^{(2)}\right]_T+P\left[X^{(1)}Y^{(1)}\right]_T \quad (3.12)$$

$$[S_5]=a\left[X^{(2)}Y^{(2)}\right]_T+b\left[X^{(1)}Y^{(3)}\right]_T+c\left[X^{(0)}Y^{(4)}\right]_T+M\left[X^{(1)}Y^{(2)}\right]_T+N\left[X^{(0)}Y^{(3)}\right]_T+P\left[X^{(0)}Y^{(2)}\right]_T \quad (3.13)$$

c. Equations (VI). The last equation represents *the integral of (PDE) (3.3)* extended on the whole rectangular area *A* (see [1] ,Par 1.3)

$$Int\,PDE=a\int_A\frac{\partial^2 z}{\partial x^2}dA+b\int_A\frac{\partial^2 z}{\partial x\partial y}dA+c\int_A\frac{\partial^2 z}{\partial y^2}dA+M\int_A\frac{\partial z}{\partial x}dA+N\int_A\frac{\partial z}{\partial y}dA+\int_A z\,dA+\int_A W(x,y)\,dA=0 \quad (3.13)$$

where $dA=dx\times dy$. The last equation becomes

$$[S_6]*[Cz]+\int_A W(x,y)\,dA=0 \qquad \text{(VI)}, \qquad \text{where}$$

$$[S_6]=a\int_A\left[X^{(2)}Y^{(0)}\right]dA+b\int_A\left[X^{(1)}Y^{(1)}\right]dA+c\int_A\left[X^{(0)}Y^{(2)}\right]dA+ +M\int_A\left[X^{(1)}Y^{(0)}\right]dA+N\int_A\left[X^{(0)}Y^{(1)}\right]dA+P\int_A\left[X^{(0)}Y^{(0)}\right]dA \quad (3.14)$$

The solution of the system of equations constituted from the 22 equations imposed by the boundary conditions and the 6 equations (I)… (VI), represents the 28 coefficients of *CF628*. The problem is not yet finished because the values for [*Cz*] obtained using *CF628* could be unsatisfactory. Taking into account this possibility, it is necessary to repeat the procedure developed in this paragraph, for the higher degrees *Concordant Functions CF736* and *CF845.* Being similar to *CF628*, they are only shortly presented in *Appendix A*.

To illustrate, let us integrate - on a SINGLE CELL - a second order elliptic PDE with all the coefficients different from zero, using three different CFs

$$PDE = 2\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial x\partial y} + 3\frac{\partial^2\phi}{\partial y^2} + \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} + 8\phi \quad + 240 + 20x + 30y + 50x^2 + 40xy - 10y^2 + \\ + 40x^3 + 50x^2y + 20xy^2 + 10y^3 + 50x^4 + 20x^3y - 30x^2y^2 + 40xy^3 + 60y^4 = 0 \tag{3.15}$$

The integration takes place on a quite large square domain, with B=1 and H=1. The boundary conditions vary linearly between the four corners, where the values of the function are: $\phi_1(x=0, y=0) = 5$; $\phi_2(1{,}0) = 1$; $\phi_3(0{,}1) = 2$; $\phi_4(1{,}1) = 7$.

The first attempt using a single cell will be made by employing *CF628*, based on the relations established above. The results obtained for the *Target Point* - namely the values of the function, together with the first and second order derivatives - are given in *Row 1* of *Table 1*. Based on them it results the residual (Res_T), whose value (-58.799) is far from those considered acceptable [1]. Therefore the operation continues in the same way using *CF736* and *CF845,* which lead to the results given the *Rows 2 and 3* of the same *Table 1*. The values of Res_T for the last computations are similar to the first, all three being unsatisfactory. The graphs of the three functions obtained as solutions, which are given in *Fig.1a, Fig.1b, Fig.1c*, are quite similar and bring no hint for accepting one of them as being "the best".

Table 1

Row ↓	Source ↓	z_T	$\left(\frac{\partial z}{\partial x}\right)_T$	$\left(\frac{\partial z}{\partial y}\right)_T$	$\left(\frac{\partial^2 z}{\partial x^2}\right)_T$	$\left(\frac{\partial^2 z}{\partial x\partial y}\right)_T$	$\left(\frac{\partial^2 z}{\partial y^2}\right)_T$	Res_T
Step1: Results from "Single Cell" integration using different Concordant Functions								
1	CF628 (CF6)	17.705	2.0185	1.8947	-94.213	4.2374	-106.81	-58.799
2	CF736 (CF7)	17.703	2.4598	2.1452	-94.212	4.3228	-106.76	-58.055
3	CF845 (CF8)	17.127	2.5562	2.2037	-87.281	2.8209	-101.71	-31.996
Step2: Results from "Single Cell" integration using a COMBINATION of ***two*** *Concordant Functions*								
4	$(CF)^{\{6/7\}}_{Combined}$	26.491	16.255	7.5258	-300.19	61.324	-239.96	*
5	$(CF)^{\{7/8\}}_{Combined}$	14.151	-11.654	-13.715	-91.322	36.403	-87.530	*
6	$(CF)^{\{6/8\}}_{Combined}$	14.340	-11.227	-13.391	-94.517	36.784	-89.862	*

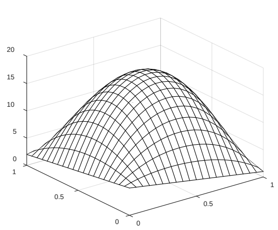


Fig1a

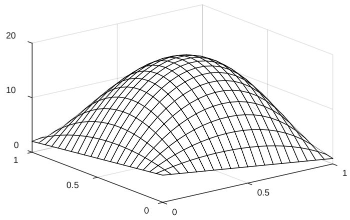


Fig1b

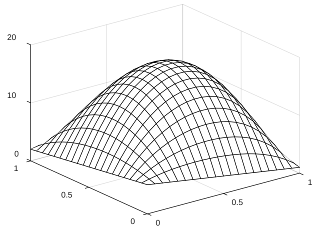


Fig1c

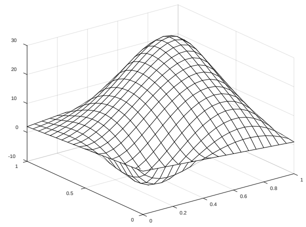


Fig1d

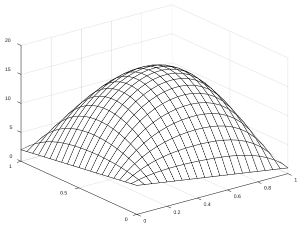


Fig1e

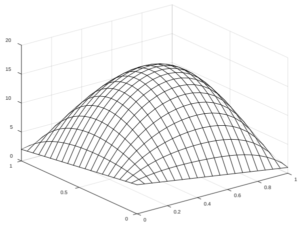


Fig1f

Fig1

3.3 Step 2: Integration with a single cell using combinations of different *CF*s

Because the attempt made until now, with the three *CF*s gave not a clear answer, a further try that give better results and request a little computation time, will be made. In the paper [2] the results obtained in *Step 1* using a single cell for the integration of nonlinear first order PDEs, were improved in *Step 2* by combining different CFs. As criterion for obtaining the combined CFs was used the *Target Residual*, which is - as seen before - a global accuracy parameter.

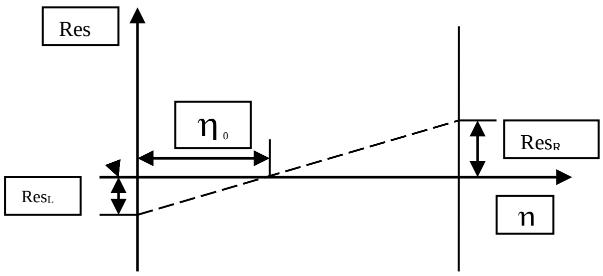


Fig.2 Fictitious linear variation between two residuals

Let be two *Concordant Functions*, noted CF_L (index L for Left) and CF_R (index R for Right). If they are used to integrate a *PDE*, it will result - as seen in the previous paragraph - two different *Target Residuals*, here noted as Res_L and Res_R. Suppose that they have different signs. Represented using a *fictitious* *abscissa* noted η ($\eta_L = 0, \eta_R = 1$), it may result a graph like that given in *Fig.2* [2]. Between these two values, we assume that **the Residuals vary continuously**, following an unknown path, so *somewhere* the path goes through Res_T=0. Because there is no information concerning the supposed trail, **we accept the rough hypothesis that the residual varies linearly** with η, according to

$$\mathrm{Res}(\eta) = \mathrm{Res_L} + (\mathrm{Res_R} - \mathrm{Res_L})\eta \qquad (3.16)$$

The *Target Residual* becomes null for $\boxed{\eta_0 = \dfrac{\mathrm{Res_L}}{\mathrm{Res_L} - \mathrm{Res_R}}}$ (3.17)

A new Concordant Function may be now obtained by combining CF_L and CF_R, according also to a linear relation similar to (4.2), where η is replaced by η_0

$$\boxed{[Cz]^{\{L/R\}}_{combined} = [Cz]_L + \left[\ [Cz]_R - [Cz]_L\ \right]\eta_0} \qquad (3.18)$$

Using $[Cz]_{combined}$ all the parameters can be computed straightforwardly, according to the relations given in Par.2.1. For instance the function results from (2.2) as

$$z_T = \left[X^{(0)}Y^{(0)}\right]_T [Cz]^{\{L/R\}}_{combined} \qquad (3.19)$$

The values obtained with this relation proved in [2] to be much better than those obtained using the two initial *Concordant Functions* CF_L and CF_R. Let us apply this procedure using the results obtained above in *Step 1*, considering CF_L= *CF7* and CF_R= *CF8*. Using the values given in *Table 1* – namely Res_{CP7}= Res_L= -58.055 and Res_{CP8}= Res_R= -31.996 – one obtains from (3.17)

$$(\eta_0)_{78} = \frac{\mathrm{Res_L}}{\mathrm{Res_L} - \mathrm{Res_R}} = \frac{-58.055}{-58.055 - (-31.996)} = 2.2278$$

It result similarly, $(\eta_0)_{68} = 2.1938$ (3.20) and $(\eta_0)_{67} = 79.057$ (3.21)

Onward, using (3.18), it results $[Cz]^{\{6/7\}}_{combined}$, $[Cz]^{\{7/8\}}_{combined}$, $[Cz]^{\{6/8\}}_{combined}$ that lead to the values given in the rows *4,5,6* of Table 1. Based on the three $[Cz]_{combined}$ it results the three graphs given in *Fig.1d, Fig.1e* and *Fig.1f.*

Now emerges the issue mentioned above: “which graph is the closest to the exact variation of the function-solution (which remains, in fact, unknown)?”. More than that, it arise a new problem: “why is the *Fig.1d* so different from the 5 others that are quite similar?”.

4. A parameter for selecting the results: the Mean Square Root Residual

In [1] the purpose of the computation was to use an increasing number of elements, so that the *Target Residual* becomes small enough to be considered as acceptable. Consequently, the objective was to compute the *Residual* in ONE

POINT. The methodology used in the previous paragraph was connected - seemingly- to the calculation of the *Residual* also in a SINGLE POINT.
The aim of the present paper is different: to find out **for the whole domain**, a polynomial that is *the closest to the exact variation of the function-solution*, which - in fact - remains unknown. Consequently, the values of the function and of its derivatives at the *Target Point* are,in fact, not important. Instead, the computation of the *Residual* will be extended to great number of points spread on a large part of a *Test Domain*. In [2] has been shown that the *Residual* tends to zero near the *Target* and increases, when the point in which it is calculated moves away from the *Target*. Its value may become unacceptable near the edges. Therefore, it is proposed the following strategy:

a. From the integration domain will be extracted a large part, avoiding the portions near to the edges and, therefore, the influence of the *Boundary Conditions*. This *Test Domain* may be limited inside

$$(x_{min}\times B) < x < (x_{max}\times B) \quad \text{and} \quad (y_{min}\times B) < y < (y_{max}\times B) \qquad (4.1)$$

b. The *Test Domain* can be divided in *Ix* intervals along *x* axis and *Iy* intervals along *y* axis, leading to a mesh of N_P intermediary points

$$N_P = (Ix+1)*(Iy+1) \qquad (4.2)$$

c. For each *CF*, in the intermediary points $k=1..N_P$ can be computed the *Residual* - Res_{Int} - using (2.12), where *T* is replaced by *k*. A special parameter, the ***Mean Square Root Residual,*** may be defined as

$$Res_{Mean} = \sqrt{\sum_{k=1}^{k=N_P} (\mathrm{Re}\, s_{Int})^2} \Big/ \left((Ix+1)*(Iy+1)\right) \qquad (4.3)$$

Its value constitutes a criterion that indicates *the Residual deviation, inside the Test Domain of the*
selected solution as against the exact solution. To illustrate, let calculate Res_{Mean} for all the 6 cases analyzed in Table 1. Taking into account that B=1, H=1, the *Test Domain* is chosen as : $x_{min}= y_{min}=0.3$ and $x_{max}= y_{max}=0.7$. The results of the computations are given in Table 2.

Table 2

Case	1	2	3	4	5	6
CF	CF6	CF7	CF8	CF67	**CF78**	CF68
Res_{Mean}	2.7208	2.6846	1.5156	8.0204	**0.1502**	0.1859

From the 6 posibilities, has to be selected the case corresponding to **the minimum value of ResMean**, being considered the closest to the unknown exact solution. The case selected from Table 2 is number 5 (CF78).

The computation of the 6 values of Res_{Mean} eliminates also the problem risen by Fig.1d. This problem was signaled, in [2], but not completely elucidated. It is owed to the methodology itself , namely to the relation (3.17). In fact, when

Res_L and Res_R have close values, the denominator of (3.17) tends to zero and may have great values. This falsifies the $[Cz]_{combined}^{\{L/R\}}$, the consequence being a distorted graph like in Fig.1d. and great values of the *Residual*. For instance Res_{Mean} for the Case 4 (CF67) is more then 50 times greater than that corresponding to Case 5 (CF78). This is enough for the program to eliminate the Case 4 from the "contest", in behalf of the Case 5 that has been already selected as "winner". Obviously this happens always, without the intervention of the user, which is not even informed.

It is interesting to follow - in Table 3 - the variation of Res_{Mean} corresponding to *CF78*, with the modification of the dimensions of the Testing domain.

Table 3

$x_{min}=y_{min}$	0.19	0.24	0.29	0.34	0.39	0.44	0.49
$x_{max}=y_{max}$	0.81	0.76	0.71	0.66	0.61	0.56	0.51
Res_{Mean}	5.50×10^{-1}	3.50×10^{-1}	1.77×10^{-1}	6.75×10^{-2}	1.63×10^{-2}	1.52×10^{-3}	1.19×10^{-6}

When the *Test Domain* is extended near the edges ($x_{min}=y_{min}=0.19$), the value of Res_{Mean} increases, while in proximity of the Target ($x_{min}=y_{min}=0.49$) it drops abruptly to 1.19×10^{-6}.

5. The integration of an elliptic PDE using a SINGLE CELL compared with a multi-cell integration based on a well-known commercial code

The following PDE will be used as a benchmark test

$$3\frac{\partial^2\phi}{\partial x^2}+2\frac{\partial^2\phi}{\partial x\partial y}+4\frac{\partial^2\phi}{\partial y^2}-5\phi\ -144-12x-18y-30x^2-24xy+6y^2-24x^3-30x^2y-12xy^2-6y^3=0 \qquad (5.1)$$

The integration takes place on a square domain having *B=1* and *H=1*. The boundary conditions vary *linearly* between the four corners, where the values of the function are:

$$\phi_1(x=0,y=0)=7\ ;\quad \phi_2(1,0)=9\ ;\quad \phi_3(0,1)=10\ ;\quad \phi_4(1,1)=14\ .$$

According to (1.2), the PDE (6.1) is elliptic because $b^2-4ac=2^2-4\times3\times4=-44<0$.

a. Integration using MATLAB

Because (5.1) is be considered as a benchmark, a fine triangular mesh was generated automatically by the program. It is realized using **20992 triangles** and **10657 nodes**. Nevertheless, the nearest point from the intersection of the diagonals is at x=0.5049, y=0.5049, for which the value of the function results as being $\mathbf{F_T=7.018}$.

b. Integration using with the method described in this paper

The program, written by the author, is based on the steps established in Par.3. The Target Point is considered T ($x_T=0.5049$, $y_T=0.5049$). The output

1. The value of the function, which is **z_T= 6.9498**
2. The combined *Concordant Function [Cz]*.

The program can provide, upon request:

- the values of the *Mean Square Root Residuals;*
- the maximum value of the function and the coordinates of the point in which it appears.

Based on Table 3, the Concordant Function selected by the programme is $[Cz]_{combined}^{\{7/8\}}$.

Table 4

Case	1	2	3	4	5	6
CF	CF6	CF7	CF8	CF67	**CF78**	CF68
Res_{Mean}	1.0379	1.0458	0.6045	4.4676	**0.07571**	0.08613

From the values found using the two fundamentally different methods, it results (considering F_T as the reference value)

Target Function Error=(z_T- F_T)/ F_T=(6.9498-7.018)/7.018= -9.71×10^{-3} that means -0. 971 % .

The variation of the function computed by MATLAB is given in Fig.3a, while that based on $[Cz]_{combined}^{\{7/8\}}$ in Fig.3b. A visual comparison shows no difference between them.

The Single-cell method has passed successfully the benchmark test.

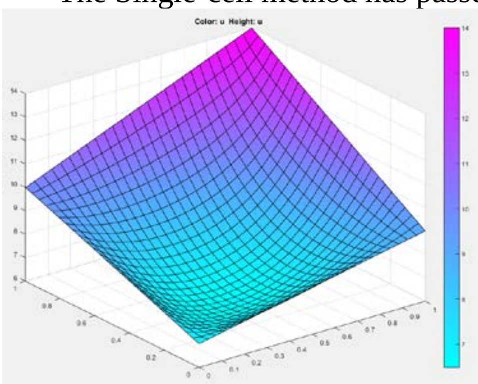


Fig.3a

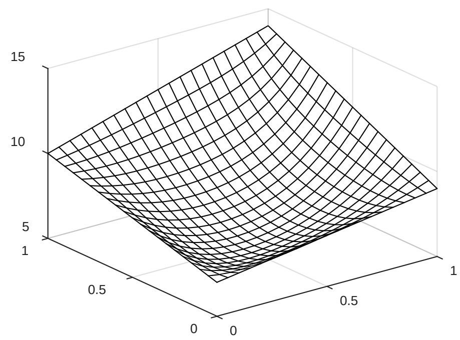


Fig.3b

Remark. a. For those readers who desire to verify the value of z_T or to represent the Fig.3b, the matrix *[Cz]* is given in APPENDIX B. For this purpose it is also given a short MATLAB [function]. The reader will be able to observe that ***[Cz]*** **can be employed similarly to an *exact analytic solution***. The difference between them consists in the fact that [Cz] leads to an approximate solution, but which is **obtained quickly using a numerical method**. In fact *[Cz]* may be considered a *quasi-analytic solution* of the PDE.

b. From Fig.3b results that the function has a minimum, whose value is z_{min}=6.3969, at a point having the coordinates *x=0.35, y=0.35*.

6. Examples

Example 1.

PDE:
$$2\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial x\partial y}+3\frac{\partial^2\phi}{\partial y^2}+2\phi \quad -384-32x-48y-80x^2-64xy+16y^2- -64x^3-80x^2y-32xy^2-16y^3-80x^4-32x^3y+48x^2y^2-64xy^3-96y^4=0$$

Square dimensions: B=1 ; H=1. Boundary conditions: $\phi_{12}(x, y=0)=2+13x-12x^2$,

$\phi_{13}(x=0, y)=2+11y-8y^2$, $\phi_{24}(x=1, y)=3+y+4y^2$, $\phi_{34}(x, y=1)=5-5x+8x^2$.

Results. *Target Point*: $x_T=y_T=0.5$. *Test Domain* : $x_{min}=y_{min}=0.3$; $x_{max}=y_{max}=0.7$. The values of Res_{Mean}

Case	1	2	3	4	5	6
CF	CF6	CF7	CF8	CF67	**CF78**	CF68
Res_{Mean}	4.4657	4.4630	2.4182	6.28×10^7	**0.2355**	0.2744

The selected case is number 5 (CF78). The graph of the function is given in Fig.4
.
Function value at T= -11.861.Minimum value of the function z_{Min}= -12.036, at x=0.55,y=0.55.

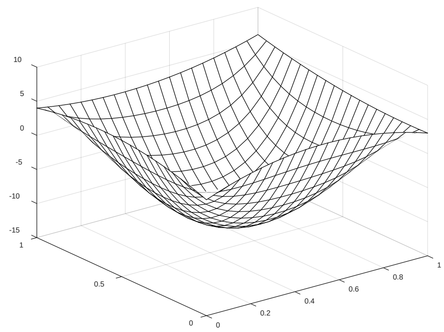


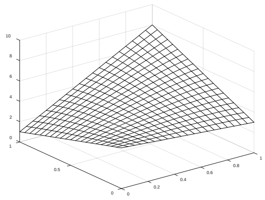
Fig.4

Example 2.

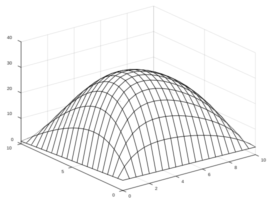
PDE:
$$2\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial x\partial y}+3\frac{\partial^2\phi}{\partial y^2}+2\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y} \quad +15+0.5x-y=0$$

Boundary conditions: $\phi_1(x=0, y=0)=4$, $\phi_2(1,0)=3$, $\phi_3(0,1)=1$, $\phi_4(1,1)=8$, with linear variation between corners.

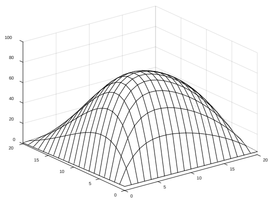
Tests will be made with three *Square dimensions*: *a.* **B=1 , H=1** ; the two others have very high dimensions *b.* **B=10 , H=10** ; *c.* **B=20 , H=20** .



B=H=1
$z_T(0.5,0.5)$= 4.4179
$Res_{Mean}=5.32\times10^{-3}$
Fig.4a



B=H=5
$z_T(5,5)$= 30.914
$Res_{Mean}=6.32\times10^{-3}$
Fig.4b



B=H=10
$z_T(10,10)$= 73.900
$Res_{Mean}=1.52\times10^{-2}$
Fig.4c

Results. For all three tests was selected $[Cz]^{\{7/8\}}_{combined}$. The graphs of the function are given in Fig.4a,b,c. When *B* and *H* increase, the influence of *W(x,y)* produce a great modification of the solution, leading to extreme values in the tests ***Fig4b***. (z_{Max}= 35.226 at x=3.5, y=3.5) and ***Fig4c***. (z_{Max}= 90.830 at x=6, y=6), which do not exist in the test ***Figa***. If the dimensions of the square increase to the large value ***B=H=30***, MATLAB delivers for the *Concordant Function CF8* the warning: *"Matrix is close to singular or badly scaled"*.

Example 3.

PDE:
$$a\frac{\partial^2\phi}{\partial x^2}-b\frac{\partial^2\phi}{\partial x\partial y}+c\frac{\partial^2\phi}{\partial y^2}+M\frac{\partial\phi}{\partial x}+N\frac{\partial\phi}{\partial y}+P\phi \quad +192+16x+24y+40x^2+32xy-8y^2+ \\ +32x^3+40x^2y+16xy^2+8y^3+40x^4+16x^3y-24x^2y^2+32xy^3+48y^4=0 \quad (6.1)$$

where a=2 ; b=-3.4 ; c=3 ; M=-4 ; N=3 ; P=2 .

Square dimensions: B=1 ; H=1. Boundary conditions:

$\phi_{12}(x,y=0)=2+13x-12x^2$;

$\phi_{13}(x=0,y)=2+10y-8y^2$; $\phi_{24}(x=1,y)=3-2y+4y^2$; $\phi_{34}(x,y=1)=4-7x+8x^2$

Results. *Target Point*: $x_T=y_T=0.5$. *Test Domain* : $x_{min}=y_{min}=0.3$; $x_{max}=y_{max}=0.7$.

The values of Res_{Mean}:

Case	1	2	3	4	5	6
CF	CF6	CF7	CF8	CF67	**CF78**	CF68
Res_{Mean}	3.1977	3.3515	2.1400	4.5220	0.3477	0.5361

The selected case is number 5 (CF78). The graph of the function is given in Fig.5a Function value at T: z_T= 13.210. Maximum value of the function z_{Max}= 13.346, at x=0.55,y=0.45.

Remark. The duration of the entire procedure for these examples (except the drawings) is 1.4...1.7 seconds, on an outdated desktop, using a program written by the (unprofessionally) author.

7. PDEs with variable coefficients

The "two steps" method can be used also for the integration of elliptic PDEs with variable coefficients. Such PDE can be written, similarly to (1.1), as

$$\boxed{PDE = a(x,y)\frac{\partial^2\phi}{\partial x^2} + b(x,y)\frac{\partial^2\phi}{\partial x\partial y} + c(x,y)\frac{\partial^2\phi}{\partial y^2} + M(x,y)\frac{\partial\phi}{\partial x} + N(x,y)\frac{\partial\phi}{\partial y} + P(x,y)\phi + W(x,y) = 0} \quad (7.1)$$

The main difference, as compared to *PDE* with constant coefficients, consists in a more difficult establishing of the PDE derivatives. For instance, the first term of (3.4), namely $a\frac{\partial^2 z}{\partial x^2}$ that is designate as $f = a(x,y)\frac{\partial^2 z}{\partial x^2}$ (7.2) illustrates this difference. The first and second order derivatives of (7.2) are

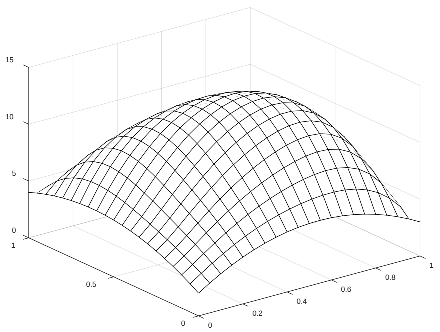


Fig.5a.

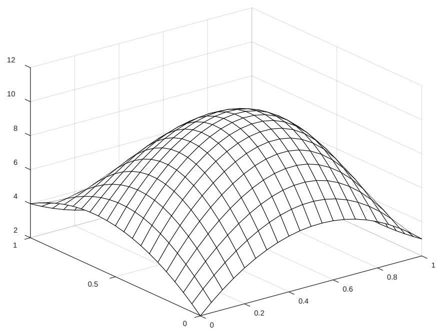


Fig.5b.

First order: $\frac{\partial f}{\partial x} = \frac{\partial a(x,y)}{\partial x}\frac{\partial^2 z}{\partial x^2} + a(x,y)\frac{\partial^3 z}{\partial x^3}$; $\frac{\partial f}{\partial y} = \frac{\partial a(x,y)}{\partial y}\frac{\partial^2 z}{\partial x^2} + a(x,y)\frac{\partial^3 z}{\partial x^2\partial y}$

Second order: $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 a(x,y)}{\partial x^2}\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial a(x,y)}{\partial x}\frac{\partial^3 z}{\partial x^3} + a(x,y)\frac{\partial^4 z}{\partial x^4}$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 a(x,y)}{\partial x\partial y}\frac{\partial^2 z}{\partial x^2} + \frac{\partial a(x,y)}{\partial x}\frac{\partial^3 z}{\partial x^2\partial y} + \frac{\partial a(x,y)}{\partial y}\frac{\partial^3 z}{\partial x^3} + a(x,y)\frac{\partial^4 z}{\partial x^3\partial y}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 a(x,y)}{\partial y^2}\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial a(x,y)}{\partial y}\frac{\partial^3 z}{\partial x^2\partial y} + a(x,y)\frac{\partial^4 z}{\partial x^2\partial y^2}$$

The other terms follow the same procedure, leading to a great number of terms that are not written here. The *Integral* equation (VI) follows the procedure described in [1], taking into account that the coefficients are variable.

Example 4. The *PDE* (6.1) from *Example 3* has the following variable coefficients

$a = 2 + 0.4x + 0.2y + 0.4x^2 + 0.5xy + 0.2y^2$; $b = -3.4 + 0.2x + 0.3y + 0.1x^2 + 0.3xy + 0.2y^2$

$c = 3 + 0.4x + 0.3y + 0.2x^2 + 0.3xy + 0.2y^2$; $M = -4 + 0.1x + 0.3y + 0.1x^2 + 0.3xy + 0.2y^2$

$N = 3 + 0.1x + 0.3y + 0.4x^2 + 0.1xy + 0.2y^2$; $P = 2 + 0.2x + 0.1y + 0.4x^2 + 0.3xy + 0.3y^2$

The selected case is also number 5 (CF78) with Res_{Mean}=0.6150. The graph of the function is given in Fig.5b. Function value at T: z_T= 9.9316e. Maximum value of the function z_{Max}=10.042, at x=0.5,y=0.45

Acknowledgement. The author wants to thank professor Stefan Sorohan, from the University Politehnica Bucarest, Department of Strength of Materials, who has computed - using the part dedicated to *PDE*s from MATLAB - several examples that have been used as benchmarks. One example was included in paragraph 5.

Appendix A

A.1 Selecting the equations for CF736

The procedure follows the same path as in Par 3.1, taking into account the increased number of unknown coefficients from 28 to 36.

a. Boundary conditions: 26 equations

As compared to CF628, the degree of CF736 has raised from 6 to 7. This means that an additional condition must be added to each side, which raises the number of limit conditions from 22 to 26. Consequently there is enough information to write 26 equations.

b. **Target Point: 10 equations**

There are necessary other 10 equations that added to the 26 already found, to complete the number of 36. They result by adding to the 6 equations (I)… (VI) written above, the 4 equations obtained from the *third order partial derivatives of PDE (1.1).*

$$[S_7]*[Cz]+\left(\frac{\partial^3 W}{\partial x^3}\right)_T=0\,(\text{VII}),\ [S_8]*[Cz]+\left(\frac{\partial^3 W}{\partial x^2\partial y}\right)_T=0\,(\text{ VIII}),$$

$$[S_9]*[Cz]+\left(\frac{\partial^3 W}{\partial x\partial y^2}\right)_T=0\,(\text{IX}),\qquad [S_{10}]*[Cz]+\left(\frac{\partial^3 W}{\partial y^3}\right)_T=0\,(\text{X}),\qquad \text{where}$$

$$[S_7]=a\lfloor X^{(5)}Y^{(0)}\rfloor_T+b\lfloor X^{(4)}Y^{(1)}\rfloor_T+c\lfloor X^{(3)}Y^{(2)}\rfloor_T+M\lfloor X^{(4)}Y^{(0)}\rfloor_T+N\lfloor X^{(3)}Y^{(1)}\rfloor_T+P\lfloor X^{(3)}Y^{(0)}\rfloor_T$$

$$[S_8]=a\lfloor X^{(4)}Y^{(1)}\rfloor_T+b\lfloor X^{(3)}Y^{(2)}\rfloor_T+c\lfloor X^{(2)}Y^{(3)}\rfloor_T+M\lfloor X^{(3)}Y^{(1)}\rfloor_T+N\lfloor X^{(2)}Y^{(2)}\rfloor_T+P\lfloor X^{(2)}Y^{(1)}\rfloor_T$$

$$[S_9]=a\lfloor X^{(3)}Y^{(2)}\rfloor_T+b\lfloor X^{(2)}Y^{(3)}\rfloor_T+c\lfloor X^{(1)}Y^{(4)}\rfloor_T+M\lfloor X^{(2)}Y^{(2)}\rfloor_T+N\lfloor X^{(1)}Y^{(3)}\rfloor_T+P\lfloor X^{(1)}Y^{(2)}\rfloor_T$$

$$[S_{10}]=a\lfloor X^{(2)}Y^{(3)}\rfloor_T+b\lfloor X^{(1)}Y^{(4)}\rfloor_T+c\lfloor X^{(0)}Y^{(5)}\rfloor_T+M\lfloor X^{(1)}Y^{(3)}\rfloor_T+N\lfloor X^{(0)}Y^{(4)}\rfloor_T+P\lfloor X^{(0)}Y^{(3)}\rfloor_T$$

A.2 Selecting the equations for CF845

The procedure follows the same path, taking into account the increased number of unknown coefficients from 36 to 45.

a. Boundary conditions: 30 equations

By adding one boundary condition on each side, the number of equations raises from 26 to 30.

b. **Target Point: 15 equations**

There are necessary other 15 equations that added to the 30 already found, to complete the number of 45. They result by adding to the 10 equations (I)… (X) written above, the 5 equations obtained from the *fourth order partial derivatives of PDE (1.1).*

$$[S_{11}]*[Cz]+\left(\frac{\partial^4 W}{\partial x^4}\right)_T=0\ \text{(XI)},\ [S_{12}]*[Cz]+\left(\frac{\partial^4 W}{\partial x^3\partial y}\right)_T=0\ \text{(XII)},\ [S_{13}]*[Cz]+\left(\frac{\partial^4 W}{\partial x^2\partial y^2}\right)_T=0\ \text{(XIII)},$$

$$[S_{14}]*[Cz]+\left(\frac{\partial^4 W}{\partial x\partial y^3}\right)_T=0\ \text{(XIV)},\quad [S_{15}]*[Cz]+\left(\frac{\partial^4 W}{\partial y^4}\right)_T=0\ \text{(XV)},\quad \text{where}$$

$$[S_{11}]=a\lfloor X^{(6)}Y^{(0)}\rfloor_T+b\lfloor X^{(5)}Y^{(1)}\rfloor_T+c\lfloor X^{(4)}Y^{(2)}\rfloor_T+M\lfloor X^{(5)}Y^{(0)}\rfloor_T+N\lfloor X^{(4)}Y^{(1)}\rfloor_T+P\lfloor X^{(4)}Y^{(0)}\rfloor_T$$

$$[S_{12}]=a\lfloor X^{(5)}Y^{(1)}\rfloor_T+b\lfloor X^{(4)}Y^{(2)}\rfloor_T+c\lfloor X^{(3)}Y^{(3)}\rfloor_T+M\lfloor X^{(4)}Y^{(1)}\rfloor_T+N\lfloor X^{(3)}Y^{(2)}\rfloor_T+P\lfloor X^{(3)}Y^{(1)}\rfloor_T$$

$$[S_{13}]=a\lfloor X^{(4)}Y^{(2)}\rfloor_T+b\lfloor X^{(3)}Y^{(3)}\rfloor_T+c\lfloor X^{(2)}Y^{(4)}\rfloor_T+M\lfloor X^{(3)}Y^{(2)}\rfloor_T+N\lfloor X^{(2)}Y^{(3)}\rfloor_T+P\lfloor X^{(2)}Y^{(2)}\rfloor_T$$

$$[S_{14}]=a\lfloor X^{(3)}Y^{(3)}\rfloor_T+b\lfloor X^{(2)}Y^{(4)}\rfloor_T+c\lfloor X^{(1)}Y^{(5)}\rfloor_T+M\lfloor X^{(2)}Y^{(3)}\rfloor_T+N\lfloor X^{(1)}Y^{(4)}\rfloor_T+P\lfloor X^{(1)}Y^{(3)}\rfloor_T$$

$$[S_{15}]=a\lfloor X^{(2)}Y^{(4)}\rfloor_T+b\lfloor X^{(1)}Y^{(5)}\rfloor_T+c\lfloor X^{(0)}Y^{(6)}\rfloor_T+M\lfloor X^{(1)}Y^{(4)}\rfloor_T+N\lfloor X^{(0)}Y^{(5)}\rfloor_T+P\lfloor X^{(0)}Y^{(4)}\rfloor_T$$

Appendix B

The following *function* can be used in MATLAB to compute the value of the function. Input data: the coordinates of the point, the degree of the Concordant Function (usually 8) and the matrix [Cz] that that can be found below.

```
function[FZ]=FunctionZ(x,y,Degree,Cz)
NT=(Degree+1)*(Degree+2)/2;
RF=[0 1 0 2 1 0 3 2 1 0 4 3 2 1 0 5 4 3 2 1 0 6 5 4 3 2 1 0 7 6 5 4 3 2 1 0 ...
    8 7 6 5 4 3 2 1 0  9 8 7 6 5 4 3 2 1 0 ];
SF=[0 0 1 0 1 2 0 1 2 3 0 1 2 3 4 0 1 2 3 4 5 0 1 2 3 4 5 6 ...
        0 1 2 3 4 5 6 7  0 1 2 3 4 5 6 7 8  0 1 2 3 4 5 6 7 8 9 ];
for i=1:NT
    R=RF(i);S=SF(i);     MF(i)=x^R*y^S;
end
FZ=MF*Cz';
```

[Cz]↓	PDE (5.1)	*Example 1*	*Example 3*	*Example 4*
1	7	2	2	2
x	2	13	13	13
y	3	10	10	10
x^2	0	-12	-12	-12
xy	-94.739e+01	-546.39	11.179	30.410
y^2	0	-8	-8	-8
x^3	0	0	0	0
x^2y	27.256	1471.2	217.88	-20.097
xy^2	23.3282	1178.6	-226.26	47.247
y^3	0	0	0	0
x^4	0	0	0	0
x^3y	-362.98	-1991.9	-891.48	161.59
x^2y^2	-892.22	-4347.9	997.60	-426.06
xy^3	-224.51	-1127.1	905.96	-208.64
y^4	0	0	0	0
x^5	0	0	0	0
x^4y	217.37	1316	1705.8	-415.25
x^3y^2	1356.5	6636.8	-601.43	315.51
x^2y^3	1057.7	5069.4	-4370.1	1116.8
xy^4	75.729	437.90	-881.97	79.382
y^5	0	0	0	0
x^6	0	0	0	0
x^5y	-14.298	-269.32	-1287.1	298.61

x^4y^2	-859.20	-4268.7	-2130.7	553.89
x^3y^3	-1669.0	-7952.2	5829.9	-1513.0
x^2y^4	-450.28	-2222.3	3711.7	-613.75
xy^5	26.034	72.397	152.87	51.815
y^6	0	0	0	0
x^7	0	0	0	0
x^6y	-15.915	9.4792	231.76	-67.260
x^5y^2	146.19	822.74	2204.6	-545.84
x^4y^3	967.64	4563.3	-1448.3	357.61
x^3y^4	700.36	3395.0	-4703.0	1030.6
x^2y^5	-1.1363	15.252	-518.85	-57.130
xy^6	-13.335	-34.396	18.226	-20.218
y^7	0	0	0	0
x^8	0	0	0	0
x^7y	0	0	0	0
x^6y^2	15.915	-9.4792	-231.76	67.260
x^5y^3	-131.89	-553.42	-917.45	247.23
x^4y^4	-325.81	-1610.6	1873.3	-496.24
x^3y^5	-24.898	-87.650	365.97	5.3150
x^2y^6	13.335	34.396	-18.226	20.218
xy^7	0	0	0	0
y^8	0	0	0	0

R E F E R E N C E S

[1]. *M.Blumenfeld*, A Consistent Numerical Method Monitored by Residuals for Solving the Partial or Ordinary Differential Equations of First Order, Editura PRINTECH, Bucharest, 2015.

Can be free downloaded from the site **blumenfeld.ro**

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[2]. *M. Blumenfeld*, University Polytehnica Bucharest Sci.Bull., Series A, Vol. 80, ISS2, 2018.

[3]. *M. Blumenfeld*, Quasi-analytic solutions of first-order Partial Differential Equations using the Accurate Element Method, University Polytehnica Bucharest Sci.Bull., Series A, Vol. 72, ISS2, 2010.

[4]. *M. Blumenfeld*, Verification of the quasi-analytic solutions of Ordinary Differential Equations using the Accurate Element Method, University Politehnica Bucharest Sci.Bull., Series A, Vol. 71, ISS 2/2009.

[5]. *M. Blumenfeld*, Accurate Element Method strategy for finding quasi-analytic solutions of Ordinary Differential Equations with variable coefficients, University Polytehnica Bucharest Sci.Bull., Vol. 72, ISS4, 2010.

[6]. *C. Berbente, S.Mitran, S. Zancu*, Metode Numerice (Numerical Methods), Editura Tehnica, Bucharest, 1997.

[7*]. S. Danaila, C.Berbente*, Metode Numerice in Dinamica Fluidelor (Numerical Methods in Fluid Dynamics), Editura Academiei Romane, Bucharest, 2003

[8]. *S. C.Chapra, R. P. Canale*, Numerical Methods for Engineers, McGraw-Hill, 2002.

[9]. *G.W.Collins II*, Fundamental Numerical Methods and Data Analysis, Internet Edition, 2003.

[10]. *Thir Raj Dangal*, Numerical Solution of Partial Differential Equations Using Polynomial Particular Solutions, University of Southern Missisippi, 2017.