EXISTENCE OF MULTIPLE SOLUTIONS TO A DISCRETE $2n$-TH ORDER PERIODIC BOUNDARY VALUE PROBLEM VIA VARIATIONAL METHOD

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Sufficient conditions are given for the existence of multiple solutions to a periodic boundary value problem for a $2n$-th order nonlinear difference equation. Using variational methods and critical point theory for proving our results. One application is included to illustrate the results.

Keywords: Discrete boundary value problems, $2n$-th order, critical point theory, variational methods.
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1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ be the sets of all natural numbers, integers and real numbers, respectively and let $n \geq 1$ be a positive integer. We consider the following nonlinear $2n$-th order periodic boundary value problems:

$$(P)\begin{cases} \sum_{k=0}^{n}(-1)^k\Delta^k(h_k(t-k)\Delta^k u(t-k)) = f(t,u(t)), & t \in [1,N]_{\mathbb{Z}}, \\ \Delta^i u(-(n-1)) = \Delta^i u(N-(n-1)), & i \in [0,2n-1]_{\mathbb{Z}}, \end{cases}$$

where $N \geq n$ is an integer, $[1,N]_{\mathbb{Z}}$ denotes the discrete interval $\{1,2,...,N\}$, Δ is the forward difference operator defined by $\Delta u(t) = u(t+1)-u(t)$, $\Delta^0 u(t) = u(t)$ and $\Delta^i u(t) = \Delta^{i-1}(\Delta u(t))$ for $i = 1,2,3,...,2n$.
The functions h_k, $k \in [0,n]_{\mathbb{Z}}$ and f are assumed to satisfy the following conditions throughout this paper:
i) $h_k : [-(k-1),N]_{\mathbb{Z}} \longrightarrow \mathbb{R}$, $k \in [0,n]_{\mathbb{Z}}$ are some fixed functions such that

$$h_k(-l) = h_k(N-l), \quad \forall k \in [1,n]_{\mathbb{Z}},\ \forall l \in [0,k-1]_{\mathbb{Z}},$$

and

$$\overline{h_k} = \max_{t\in[1,N]_{\mathbb{Z}}} h_k(t) > 0,\ \forall k \in [0,n]_{\mathbb{Z}};$$

ii) $f\colon [1,N]_{\mathbb{Z}} \times \mathbb{R} \longrightarrow \mathbb{R}$, $(t,x) \longrightarrow f(t,x)$ is a continuous function in the second variable for every $t \in [1,N]_{\mathbb{Z}}$.

As usual, a solution of (P) is a function $u : [-(n-1), N+n]_{\mathbb{Z}} \longrightarrow \mathbb{R}$ which satisfies both equations of (P).

Let $\lambda_1, \lambda_2, ..., \lambda_N$ be the eigenvalues of the linear boundary value problem corresponding to the problem (P):

$$(P_0)\begin{cases} \sum_{k=0}^{n}(-1)^k\Delta^k(h_k(t-k)\Delta^k u(t-k)) = \lambda u(t), & t \in [1,N]_{\mathbb{Z}}, \\ \Delta^i u(-(n-1)) = \Delta^i u(N-(n-1)), & i \in [0,2n-1]_{\mathbb{Z}}. \end{cases}$$

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Difference equations appear naturally as discrete analogues and numerical solutions of differential equations and delay differential equations which model various diverse phenomena in statistics, computing, ecology, optimal control, neural network, electrical circuit analysis, population dynamics, economics, biology and other fields; (see, for example [1, 6]).

Recently, discrete fourth order boundary value problems with various boundary conditions has been studied by many researchers. The reader may refer to [2, 4, 7, 8, 10, 11]. As is well known, critical point theory and variational methods are powerful tools to investigate the existence of solutions of various problems on differential equations. For example, John R. Graef, Lingju Kong and Min Wang in [5] studied the existence and multiplicity of solutions of the discrete nonlinear fourth order periodic boundary value problems consisting the following equation

$$\Delta^4 u(t-2) - \Delta(p(t-1)\Delta u(t-1)) + q(t)u(t) = f(t, u(t)), \quad t \in [1, N]_{\mathbb{Z}}, \tag{1}$$

and the boundary condition (BC)

$$\Delta^i u(-1) = \Delta^i u(N-1), \quad i \in [0, 3]_{\mathbb{Z}}, \tag{2}$$

by using the critical point theory and variational methods. Where $N \geq 2$ is an integer, $f\colon [1, N]_{\mathbb{Z}} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function in the second variable and $p, q : [1, N]_{\mathbb{Z}} \longrightarrow \mathbb{R}$ are some fixed functions.

In this paper, we will study the existence and multiplicity of nontrivial solutions of (P) by variational methods and critical point theory.

For convenience, we introduce the following notations:

$$F_0 = \lim_{x\to 0} \inf \min_{t\in[1,N]_{\mathbb{Z}}} \frac{2F(t,x)}{x^2} \quad , \quad F^\infty = \lim_{|x|\to\infty} \sup \max_{t\in[1,N]_{\mathbb{Z}}} \frac{2F(t,x)}{x^2},$$

$$F^0 = \lim_{x\to 0} \sup \max_{t\in[1,N]_{\mathbb{Z}}} \frac{2F(t,x)}{x^2} \quad , \quad F_\infty = \lim_{|x|\to\infty} \inf \min_{t\in[1,N]_{\mathbb{Z}}} \frac{2F(t,x)}{x^2},$$

where $F(t, x) = \int_0^x f(t, s)ds$ for all $(t, x) \in [1, N]_{\mathbb{Z}} \times \mathbb{R}$.
Let us impose the following hypotheses throughout this paper:

(H_1) there exists η with $\eta < \lambda_1$ such that

$$F^\infty \leq \eta < \lambda_1 < F_0 \ ;$$

(H_2) there exists δ with $\delta > \lambda_N$ such that

$$F^0 < \lambda_1 < \lambda_N < \delta \leq F_\infty;$$

(H_2') there exists δ' with $\delta' > \gamma = \sum_{k=0}^{n} 4^k \overline{h_k}$ such that

$$F^0 < \lambda_1 < \gamma < \delta' \leq F_\infty;$$

(H_3) there exists δ'' with $\delta'' > \lambda_N$ and $p \in \{1, 2, ..., N\}$ such that

$$F^0 < \lambda_p \leq \lambda_N < \delta'' \leq F_\infty \ ;$$

(H_3') there exists δ''' with $\delta''' > \gamma = \sum_{k=0}^{n} 4^k \overline{h_k}$ and $p \in \{1, 2, ..., N\}$ such that

$$F^0 < \lambda_p \leq \gamma < \delta''' \leq F_\infty \ ;$$

(H_4) $f(t, x)$ is odd in x, i.e., $f(t, -x) = -f(t, x)$ for $(t, x) \in [1, N]_{\mathbb{Z}} \times \mathbb{R}$.

The main results of this paper are the following theorems.

Theorem 1.1. *Assume that (H_1) hold, then the problem (P) has at least one nontrivial solution.*

Theorem 1.2. *Assume that* (H_2) *hold, then the problem* (P) *has at least one nontrivial solution.*

Corollary 1.1. *Assume that* (H_2') *hold, then the problem* (P) *has at least one nontrivial solution.*

Theorem 1.3. *Assume that* (H_3) *and* (H_4) *hold, then the problem* (P) *has at least* $2(N-p+1)$ *nontrivial solutions.*

Corollary 1.2. *Assume that* (H_3') *and* (H_4) *hold, then the problem* (P) *has at least* $2(N-p+1)$ *nontrivial solutions.*

The rest of this paper is organized as follows. Section 2, contains some preliminary lemmas. Section 3, we introduce the eigenvalue problem (P_0) associated to (P). The main results are proved in Section 4.

2. **Preliminary lemmas**

In the present paper, we define a vector space E by

$$E = \{u : [-(n-1), N+n]_{\mathbb{Z}} \longrightarrow \mathbb{R} \mid \Delta^i u(-(n-1)) = \Delta^i u(N-(n-1)),\ i = 0, 1, 2, 3, ..., 2n-1\},$$

E can be equipped with inner product $\langle ., . \rangle$ and norm $\|.\|$ as follows:

$$\langle u, v\rangle = \sum_{t=1}^{N} u(t)v(t), \quad \|u\| = \left(\sum_{t=1}^{N} |u(t)|^2\right)^{1/2} \quad \text{for all } u, v \in E.$$

Remark 2.1. *It is easy to see that, for any* $u \in E$, *we have*

$$\begin{aligned} u(-(n-1)) &= u(N-(n-1)) \\ u(-(n-1)+1) &= u(N-(n-1)+1) \\ u(-(n-1)+2) &= u(N-(n-1)+2) \\ \vdots \quad & \quad \vdots \\ u(0) &= u(N) \\ u(1) &= u(N+1) \\ \vdots \quad & \quad \vdots \\ u(n) &= u(N+n). \end{aligned} \tag{3}$$

Clearly, $(E, \|.\|)$ is an N-dimensional reflexive Banach space, since it is isomorphic to the finite dimensional space $\mathbb{R}^N$. When we say that the vector $u = (u(1), ..., u(N)) \in \mathbb{R}^N$, we understand that u can be extended to a vector in E so that (3) holds, that is, u can be extended to the vector

$$(u(N-(n-1)), u(N-(n-1)+1), ..., u(N), u(1), u(2), ..., u(N), u(1), ..., u(n)) \in E$$

and when we write $E = \mathbb{R}^N$, we mean the elements in $\mathbb{R}^N$ which have been extended in the above sense.

Lemma 2.1. (*see* [1]) *Let* $u(t)$ *be defined on* $\mathbb{Z}$. *Then, for all* $k \in \mathbb{N}^*$ *we have*

$$\Delta^k u(t) = \sum_{i=0}^{k} (-1)^{k-i} C_k^i u(t+i), \quad t \in \mathbb{Z}.$$

Lemma 2.2. *Let* $n \in \mathbb{N}^*$. *For all* $u, v \in E$ *we have*

$$\sum_{t=1}^{N} h_k(t-k)\Delta^k u(t-k)\Delta^k v(t-k) = (-1)^k \sum_{t=1}^{N} \Delta^k(h_k(t-k)\Delta^k u(t-k))v(t), \quad k \in [0, n]_{\mathbb{Z}}. \tag{4}$$

Proof. For $k = 0$, it is easy to check the conclusion is true. We suppose that (4) is true for $k \in [0, n-1]_{\mathbb{Z}}$ and we prove that it is true for $k+1$, i.e.,

$$\sum_{t=1}^{N} h_{k+1}(t-(k+1))\Delta^{k+1}u(t-(k+1))\Delta^{k+1}v(t-(k+1)) =$$

$$(-1)^{k+1}\sum_{t=1}^{N} \Delta^{k+1}(h_{k+1}(t-(k+1))\Delta^{k+1}u(t-(k+1))v(t).$$

From Lemma 2.1 and (3), we get that

$$\Delta^{k}(h_{k+1}(N-k)\Delta^{k+1}u(N-k)) = \Delta^{k}(h_{k+1}(-k)\Delta^{k+1}u(-k)). \tag{5}$$

Using the summation by parts formula and the fact that $v(N+1) = v(1)$ and (5), it follows that

$$\sum_{t=1}^{N} \Delta^{k+1}(h_{k+1}(t-(k+1))\Delta^{k+1}u(t-(k+1))v(t) =$$

$$[\Delta^{k}(h_{k+1}(t-(k+1))\Delta^{k+1}u(t-(k+1)))v(t)]_{1}^{N+1} - \sum_{t=1}^{N} \Delta^{k}(h_{k+1}(t-k)\Delta^{k+1}u(t-k))\Delta v(t)$$

$$= (-1)^{k+1}\sum_{t=1}^{N} h_{k+1}(t-k)\Delta^{k+1}u(t-k)\Delta^{k+1}v(t-k)$$

$$= (-1)^{k+1}\sum_{t=1}^{N} h_{k+1}(t-(k+1))\Delta^{k+1}u(t-(k+1))\Delta^{k+1}v(t-(k+1)).$$

Which means that

$$\sum_{t=1}^{N} h_{k+1}(t-(k+1))\Delta^{k+1}u(t-(k+1))\Delta^{k+1}v(t-(k+1)) =$$

$$(-1)^{k+1}\sum_{t=1}^{N} \Delta^{k+1}(h_{k+1}(t-(k+1))\Delta^{k+1}u(t-(k+1))v(t).$$

The proof is complete. □

For $u \in E$, let Φ be the functional defined by

$$\Phi(u) = \frac{1}{2}\sum_{t=1}^{N}\sum_{k=0}^{n} h_k(t-k)|\Delta^{k}u(t-k)|^2 - \sum_{t=1}^{N} F(t, u(t)). \tag{6}$$

Then, it is easy to see that $\Phi \in C^1(E, \mathbb{R})$, i.e., Φ is a continuously Fréchet-differentiable and its derivative $\Phi'(u)$ at $u \in E$ is given by

$$\Phi'(u).v = \sum_{t=1}^{N}\left[\sum_{k=0}^{n} h_k(t-k)\Delta^{k}u(t-k)\Delta^{k}v(t-k) - f(t, u(t))v(t)\right] \quad \text{for any } v \in E. \tag{7}$$

By Lemma 2.2, Φ' can be written as

$$\Phi'(u).v = \sum_{t=1}^{N}\left[\sum_{k=0}^{n}(-1)^{k}\Delta^{k}(h_k(t-k)\Delta^{k}u(t-k)) - f(t, u(t))\right] v(t) \quad \text{for any } v \in E.$$

Thus, finding solution of (P) is equivalent to finding critical point of the functional Φ.

We denote by $C^1(E,\mathbb{R})$ the space of continuously fréchet-differentiable functionals from E into $\mathbb{R}$.

Definition 2.1. *Let E be a real Banach space, and $\Phi \in C^1(E,\mathbb{R})$ be a continuously Fréchet-differentiable functional defined on E. Recall that Φ is said to satisfy the Palais-Smale (PS) condition if every sequence $(u_m) \subset E$, such that $\Phi(u_m)$ is bounded and $\Phi'(u_m) \to 0$ as $m \to \infty$, has a convergent subsequence. Here, the sequence (u_m) is called a PS sequence.*

Lemma 2.3. (*see* [3, 9]) *Let E be a real Banach space and $\Phi \in C^1(E,\mathbb{R})$ be even, bounded from below, and satisfy the (PS) condition. Suppose that $\Phi(0) = 0$ and there is a set $K \subset E$ such that K is homeomorphic to S^{r-1} by an odd map, where S^{r-1} is the $r-1$ dimensional unit sphere, and $\sup_{u\in K}\Phi(u) < 0$.*
Then, Φ has at least r disjoint pairs of nontrivial critical points.

3. **Eigenvalue problem**

We consider the linear eigenvalue problem (P_0) corresponding to the problem (P):

$$(P_0)\quad \begin{cases} \sum\limits_{k=0}^{n}(-1)^k\Delta^k(h_k(t-k)\Delta^k u(t-k)) &= \lambda u(t),\ \ t\in[1,N]_{\mathbb{Z}},\\ \Delta^i u(-(n-1)) &= \Delta^i u(N-(n-1)),\ \ i\in[0,2n-1]_{\mathbb{Z}}.\end{cases}$$

Definition 3.1. *λ is called eigenvalue of (P_0) if there exists $u \in E \setminus \{0\}$ such that*

$$\sum_{t=1}^{N}\sum_{k=0}^{n}(-1)^k\Delta^k(h_k(t-k)\Delta^k u(t-k))v(t) = \lambda\sum_{t=1}^{N}u(t)v(t)\quad \textit{for every } v\in E.$$

Lemma 3.1. *Problem (P_0) has exactly N real eigenvalues λ_i, $i \in [1,N]_{\mathbb{Z}}$, which satisfies*

$$\lambda_1 \le \lambda_2 \le ... \le \lambda_N.$$

Proof. Let $n \in \mathbb{N}^*$, $k \in [0,n]_{\mathbb{Z}}$ and $u,v \in E$. It is clear to see that the application $L_k : (u,v) \longrightarrow \sum_{t=1}^{N}(-1)^k\Delta^k(h_k(t-k)\Delta^k u(t-k))v(t)$, is bilinear and symmetric.
From Riesz theorem, there exists a unique symmetric matrix A_k such that

$$L_k(u,v) = \langle A_k u, v\rangle \quad \text{for all } u,v\in E. \tag{8}$$

Therefore,

$$\begin{aligned}\langle\sum_{k=0}^{n}A_k u, v\rangle &= \sum_{k=0}^{n}L_k(u,v)\\ &= \sum_{k=0}^{n}\sum_{t=1}^{N}(-1)^k\Delta^k(h_k(t-k)\Delta^k u(t-k))v(t)\\ &= \sum_{t=1}^{N}\sum_{k=0}^{n}(-1)^k\Delta^k(h_k(t-k)\Delta^k u(t-k))v(t).\end{aligned}$$

Thus the eigenvalues of (P_0) are exactly the eigenvalues of the matrix $\sum_{k=0}^{n}A_k$.
Then, the problem (P_0) has exactly N real eigenvalues λ_i, $i \in [1,N]_{\mathbb{Z}}$, which satisfies $\lambda_1 \le \lambda_2 \le ... \le \lambda_N$. The proof is complete. □

Remark 3.1. *Using Lemma 2.2, if v is replaced by u in (8), we get*

$$\langle A_k u, u\rangle = \sum_{t=1}^{N}(-1)^k\Delta^k(h_k(t-k)\Delta^k u(t-k))u(t) = \sum_{t=1}^{N}h_k(t-k)|\Delta^k u(t-k)|^2,\ k\in[0,n]_{\mathbb{Z}}. \tag{9}$$

It is easy to see that, there exists a unique symmetric matrix B_k such that

$$\langle B_k u, v\rangle = \sum_{t=1}^{N} (-1)^k \Delta^{2k} u(t-k) v(t),\ \ k \in [0,n]_{\mathbb{Z}}\,. \tag{10}$$

Lemma 3.2. *Let $k \in [0,n]_{\mathbb{Z}}$. The general form of the matrix B_k for $N \geq 2k+1$ is $B_k = [b_{ij}]_{1\leq i,j\leq N}$, where*

$$\begin{array}{lcl}
b_{ii} = C_{2k}^{k} & ; & i \in [1,N]_{\mathbb{Z}}\,, \\
b_{ii+j} = (-1)^j C_{2k}^{k+j} & ; & j \in [1,k]_{\mathbb{Z}}\,, i \in [1,N-j]_{\mathbb{Z}}\,, \\
b_{ii+j} = 0 & ; & j \in [k+1, N-(k+1)]_{\mathbb{Z}}\,, i \in [1,N-j]_{\mathbb{Z}}\,, \\
b_{ii+j} = (-1)^{N-j} C_{2k}^{k+N-j} & ; & j \in [N-k, N-1]_{\mathbb{Z}}\,, i \in [1,N-j]_{\mathbb{Z}}\,,
\end{array}$$

that is,

$$B_k = \begin{pmatrix}
C_{2k}^{k} & (-1)^1 C_{2k}^{k+1} & \cdots & 0 & \cdots & (-1)^k C_{2k}^{2k} & \cdots & (-1)^1 C_{2k}^{k+1} \\
(-1)^1 C_{2k}^{k+1} & C_{2k}^{k} & (-1)^1 C_{2k}^{k+1} & \cdots & 0 & \cdots & (-1)^3 C_{2k}^{k+3} & (-1)^2 C_{2k}^{k+2} \\
\vdots & (-1)^1 C_{2k}^{k+1} & C_{2k}^{k} & (-1)^1 C_{2k}^{k+1} & \cdots & 0 & \cdots & (-1)^3 C_{2k}^{k+3} \\
0 & \vdots & \ddots & \ddots & \ddots & \cdots & \ddots & \vdots \\
\vdots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
(-1)^k C_{2k}^{k} & \vdots & 0 & \vdots & \ddots & \ddots & (-1)^1 C_{2k}^{k+1} & (-1)^2 C_{2k}^{k+2} \\
\vdots & (-1)^3 C_{2k}^{k+3} & \vdots & \ddots & \cdots & \ddots & C_{2k}^{k} & (-1)^1 C_{2k}^{k+1} \\
(-1)^1 C_{2k}^{k+1} & (-1)^2 C_{2k}^{k+2} & (-1)^3 C_{2k}^{k+3} & \cdots & \cdots & \cdots & (-1)^1 C_{2k}^{k+1} & C_{2k}^{k}
\end{pmatrix}_{N\times N}.$$

Proof. Using (10) and Lemma 2.1, we get

$$\begin{aligned}
\langle B_k u, u\rangle &= \sum_{t=1}^{N} (-1)^k \left[\sum_{i=0}^{2k} (-1)^{2k-i} C_{2k}^{i} u(t-k+i)\right] u(t) \\
&= \sum_{t=1}^{N} (-1)^k C_{2k}^{0} u(t-k)u(t) + ... + (-1)^1 C_{2k}^{k-1} u(t-1)u(t) + (-1)^0 C_{2k}^{k} u^2(t) \\
&+ (-1)^1 C_{2k}^{k+1} u(t+1)u(t) + ... + (-1)^k C_{2k}^{2k} u(t+k)u(t) \\
&= C_{2k}^{k} \sum_{t=1}^{N} u^2(t) + (-1)^1 C_{2k}^{k-1} \sum_{t=1}^{N} u(t-1)u(t) + ... + (-1)^k C_{2k}^{0} \sum_{t=1}^{N} u(t-k)u(t) \\
&+ (-1)^1 C_{2k}^{k+1} \sum_{t=1}^{N} u(t+1)u(t) + ... + (-1)^k C_{2k}^{2k} \sum_{t=1}^{N} u(t+k)u(t) \\
&= C_{2k}^{k} \sum_{t=1}^{N} u^2(t) + (-1)^1 C_{2k}^{k+1} \sum_{t=1}^{N} u(t)u(t+1) + ... + (-1)^k C_{2k}^{2k} \sum_{t=1}^{N} u(t)u(t+k) \\
&+ (-1)^1 C_{2k}^{k+1} \sum_{t=1}^{N} u(t+1)u(t) + ... + (-1)^k C_{2k}^{2k} \sum_{t=1}^{N} u(t+k)u(t) \\
&= C_{2k}^{k} \sum_{t=1}^{N} u^2(t) + 2\times(-1)^1 C_{2k}^{k+1} \sum_{t=1}^{N} u(t)u(t+1) + ... \\
&+ 2\times(-1)^k C_{2k}^{2k} \sum_{t=1}^{N} u(t)u(t+k).
\end{aligned}$$

Finally, we conclude that

$$B_k = \begin{pmatrix} C_{2k}^{k} & (-1)^1C_{2k}^{k+1} & \cdots & 0 & \cdots & (-1)^kC_{2k}^{2k} & \cdots & (-1)^1C_{2k}^{k+1} \\ (-1)^1C_{2k}^{k+1} & C_{2k}^{k} & (-1)^1C_{2k}^{k+1} & \cdots & 0 & \cdots & (-1)^3C_{2k}^{k+3} & (-1)^2C_{2k}^{k+2} \\ \vdots & (-1)^1C_{2k}^{k+1} & C_{2k}^{k} & (-1)^1C_{2k}^{k+1} & \cdots & 0 & \cdots & (-1)^3C_{2k}^{k+3} \\ 0 & \vdots & \ddots & \ddots & \ddots & \cdots & \ddots & \vdots \\ \vdots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ (-1)^kC_{2k}^{k} & \vdots & 0 & \vdots & \ddots & \ddots & (-1)^1C_{2k}^{k+1} & (-1)^2C_{2k}^{k+2} \\ \vdots & (-1)^3C_{2k}^{k+3} & \vdots & \ddots & \cdots & \ddots & C_{2k}^{k} & (-1)^1C_{2k}^{k+1} \\ (-1)^1C_{2k}^{k+1} & (-1)^2C_{2k}^{k+2} & (-1)^3C_{2k}^{k+3} & \cdots & \cdots & \cdots & (-1)^1C_{2k}^{k+1} & C_{2k}^{k} \end{pmatrix}_{N\times N}.$$

The proof of Lemma 3.2 is complete. $\square$

Lemma 3.3. *Let $n \in \mathbb{N}^*$. For any $u \in E$ we have*

$$\sum_{t=1}^{N} |\Delta^k u(t-k)|^2 \le 4^k \|u\|^2, \quad k \in [0,n]_{\mathbb{Z}}. \tag{11}$$

Proof. From (10), Lemma 2.2 and Cauchy inequality, we obtain

$$\sum_{t=1}^{N} |\Delta^k u(t-k)|^2 = \sum_{t=1}^{N} (-1)^k \Delta^{2k} u(t-k)u(t) = \langle B_k u, u\rangle \le \|B_k u\|\|u\| \le \|B_k\|_\infty \|u\|^2$$

for any $k \in [0,n]_{\mathbb{Z}}$, where $\|B_k\|_\infty = \max\limits_{i\in[1,N]_{\mathbb{Z}}} \left[\sum\limits_{j=1}^{N} |b_{ij}|\right]$.

According to the general form of the matrix B_k, we get

$$\begin{aligned} \sum_{j=1}^{N} |b_{ij}| &= C_{2k}^{k} + 2C_{2k}^{k+1} + 2C_{2k}^{k+2} + ... + 2C_{2k}^{2k} \\ &= C_{2k}^{0} + C_{2k}^{1} + ... + C_{2k}^{k-1} + C_{2k}^{k} + C_{2k}^{k+1} + ... + C_{2k}^{2k} \\ &= \sum_{l=0}^{2k} C_{2k}^{l} = 2^{2k} = 4^k \end{aligned}$$

for any $i \in [1,N]_{\mathbb{Z}}$.

Thus, one has $\sum\limits_{t=1}^{N} |\Delta^k u(t-k)|^2 \le 4^k\|u\|^2$ for any $k \in [0,n]_{\mathbb{Z}}$.

The proof of Lemma 3.3 is complete. $\square$

Let ξ_i, $i \in \{1,2,...,N\}$ be an eigenvector of (P_0) associated with λ_i such that

$$\langle \xi_i, \xi_j\rangle = \begin{cases} 0, & i \ne j \\ 1, & i = j. \end{cases}$$

Then, for any $u = (u(1), u(2), u(3)....., u(N))^T \in \mathbb{R}^N$, it is easy to see that

$$\lambda_1 \|u\|^2 \le \langle \sum_{k=0}^{n} A_k u, u\rangle \le \lambda_N \|u\|^2.$$

4. **Proofs of the main results**

From (9), Φ can be rewritten as

$$\Phi(u) = \frac{1}{2}\langle \sum_{k=0}^{n} A_k u, u\rangle - \sum_{t=1}^{N} F(t, u(t)).$$

Proof of Theorem 1.1. Since $F^\infty \leq \eta$, there exists $R_1 > 0$ such that

$$\frac{2F(t,x)}{x^2} \leq \eta + \varepsilon \quad \text{for} \quad (t,|x|) \in [1,N]_{\mathbb{Z}} \times]R_1, +\infty[\,,$$

where $\varepsilon > 0$ satisfies

$$\varepsilon < \lambda_1 - \eta, \tag{12}$$

i.e., $$F(t,x) \leq \frac{1}{2}(\eta+\varepsilon)x^2 \quad \text{for } (t,|x|) \in [1,N]_{\mathbb{Z}} \times]R_1, +\infty[\,. \tag{13}$$

On the other hand, by continuity of *f*, there exists $\Psi_1 : [1,N]_{\mathbb{Z}} \longrightarrow \mathbb{R}^+$ such that

$$|F(t,x)| \leq \Psi_1(t) \quad \text{for } (t,|x|) \in [1,N]_{\mathbb{Z}} \times [0,R_1]\,. \tag{14}$$

For any $u \in E$, let $S_1 = \{t \in [1,N]_{\mathbb{Z}} : |u(t)| \leq R_1\}$ and $S_2 = \{t \in [1,N]_{\mathbb{Z}} : |u(t)| > R_1\}$.
From (13) and (14), we have

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\langle \sum_{k=0}^{n} A_k u, u\rangle - \sum_{t=1}^{N} F(t,u(t)) \\
&\geqslant \frac{1}{2}\lambda_1\|u\|^2 - \sum_{t\in S_1} F(t,u(t)) - \sum_{t \in S_2} F(t,u(t)) \\
&\geqslant \frac{1}{2}\lambda_1\|u\|^2 - \sum_{t=1}^{N}\Psi_1(t) - \frac{1}{2}(\eta+\varepsilon)\|u\|^2 \\
&= \frac{1}{2}(\lambda_1 - \eta - \varepsilon)\|u\|^2 - \sum_{t=1}^{N}\Psi_1(t).
\end{aligned}$$

Then, in view of (12), $\Phi(u) \to \infty$ as $\|u\| \to \infty$. Thus, Φ is coercive and bounded from below, hence there is a minimum point of Φ at some $u_0 \in X$ i.e., $\Phi(u_0) = \inf_{u\in E} \Phi(u)$, which is a critical point of Φ and it is a solution of the problem (P).
Since $F_0 > \lambda_1$, there exists $\rho_1 > 0$ such that

$$\frac{2F(t,x)}{x^2} \geqslant F_0 - \varepsilon \ \text{ for } (t,|x|) \in [1,N]_{\mathbb{Z}} \times [0,\rho_1]\,,$$

where $\varepsilon > 0$ satisfies $\varepsilon < F_0 - \lambda_1$.
Thus,

$$F(t,x) \geqslant \frac{1}{2}(F_0 - \varepsilon)|x|^2. \tag{15}$$

Define $K_1 = span\{\xi_1\} \cap S_{\rho_1}$ where S_{ρ_1} is the sphere of center 0 and radius ρ_1.
For any $u \in K_1$, $|u(t)| \leq \|u\| = \rho_1$.
From (15), we have

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\langle \sum_{k=0}^{n} A_k u, u\rangle - \sum_{t=1}^{N} F(t,u(t)) \\
&\leqslant \frac{1}{2}\lambda_1\|u\|^2 - \frac{1}{2}(F_0 - \varepsilon)\|u\|^2 \\
&= \frac{1}{2}(\lambda_1 - F_0 + \varepsilon)\rho_1^2 < 0.
\end{aligned}$$

Thus, we obtain $\sup_{u\in K_1} \Phi(u) < 0$.
Consequently, $\Phi(u_0) = \inf_{u\in E} \Phi(u) \leq \inf_{u\in K_1} \Phi(u) \leq \sup_{u \in K_1} \Phi(u) < 0$.

Then the problem (P) has at least one nontrivial solution.
The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. Since $F_\infty \geq \delta$, there exists $R_2 > 0$ such that

$$\frac{2F(t,x)}{x^2} \geq \delta - \varepsilon \quad \text{for} \quad (t,|x|) \in [1,N]_{\mathbb{Z}} \times]R_2, +\infty[\,,$$

where $\varepsilon > 0$ satisfies

$$\varepsilon < \delta - \lambda_N, \tag{16}$$

which means that $\quad F(t,x) \geq \dfrac{1}{2}(\delta - \varepsilon)x^2 \;$ for $(t,|x|) \in [1,N]_{\mathbb{Z}} \times]R_2, +\infty[\,.$ (17)

On the other hand, by continuity of f, there exists $\Psi_2 : [1,N]_{\mathbb{Z}} \longrightarrow \mathbb{R}^+$ such that

$$|F(t,x)| \leq \Psi_2(t) \quad \text{ for } (t,|x|) \in [1,N]_{\mathbb{Z}} \times [0,R_2]\,. \tag{18}$$

For any $u \in E$, let $S_3 = \{t \in [1,N]_{\mathbb{Z}} : |u(t)| \leq R_2\}$ and $S_4 = \{t \in [1,N]_{\mathbb{Z}} : |u(t)| > R_2\}$.
Then, from (17) and (18), we have

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\langle \sum_{k=0}^{n} A_k u, u\rangle - \sum_{t=1}^{N} F(t,u(t)) \\
&\leq \frac{1}{2}\lambda_N \|u\|^2 - \sum_{t\in S_3} F(t,u(t)) - \sum_{t \in S_4} F(t,u(t)) \\
&\leq \frac{1}{2}\lambda_N \|u\|^2 + \sum_{t=1}^{N} \Psi_2(t) - \frac{1}{2}(\delta-\varepsilon)\|u\|^2 + c \\
&\leq \frac{1}{2}(\lambda_N - (\delta - \varepsilon))\|u\|^2 + \sum_{t=1}^{N} \Psi_2(t) + c,
\end{aligned}$$

where c is a positive constant. Then, in view of (16), $[-\Phi](u) \to \infty$ as $\|u\| \to \infty$. Thus, $[-\Phi]$ is coercive and bounded from below, hence there is a minimum point of $[-\Phi]$ at some $u_1 \in E$ i.e., $[-\Phi](u_1) = \inf\limits_{u \in E}[-\Phi](u)$, which is a critical point of $[-\Phi]$ and it is a solution of the problem (P).
Since $F^0 < \lambda_1$, there exists $\rho_2 > 0$ such that

$$\frac{2F(t,x)}{x^2} \leq F^0 + \varepsilon \quad \text{for} \quad (t,|x|) \in [1,N]_{\mathbb{Z}} \times [0,\rho_2]\,,$$

where $\varepsilon > 0$ satisfies $\varepsilon < \lambda_1 - F^0$, i.e.,

$$F(t,x) \leq \frac{1}{2}(F^0 + \varepsilon)|x|^2. \tag{19}$$

Define $K_2 = span\{\xi_1\} \cap S_{\rho_2}$ where S_{ρ_2} is the sphere of center 0 and radius ρ_2.
For any $u \in K_2$, $|u(t)| \leq \|u\| = \rho_2$.
Therefore, by (19) one has

$$\begin{aligned}
[-\Phi](u) &= -\frac{1}{2}\langle \sum_{k=0}^{n} A_k u, u\rangle + \sum_{t=1}^{N} F(t,u(t)) \\
&\leq -\frac{1}{2}\lambda_1 \|u\|^2 + \frac{1}{2}(F^0 + \varepsilon)\|u\|^2 \\
&= \frac{1}{2}(F^0 + \varepsilon - \lambda_1)\rho_2^2 < 0.
\end{aligned}$$

Thus, we obtain $\sup\limits_{u\in K_2}[-\Phi](u)<0$.
Hence, we deduce that $[-\Phi](u_1)=\inf\limits_{u\in E}[-\Phi](u)\leq \inf\limits_{u\in K_2}[-\Phi](u)\leq \sup\limits_{u\in K_2}[-\Phi](u)<0$.
So the problem (P) has at least one nontrivial solution.
The proof of Theorem 1.2 is complete. $\square$

Proof of Corollary 1.1. We apply Lemma 3.3 to prove the Corollary.
We know that,

$$\begin{aligned}
\lambda_N &= \max_{u\in E\setminus\{0\}} \frac{\langle \sum\limits_{k=0}^{n} A_k u, u\rangle}{\|u\|^2} \\
&= \max_{u\in E\setminus\{0\}} \frac{\sum\limits_{k=0}^{n}\sum\limits_{t=1}^{N} h_k(t-k)|\Delta^k u(t-k)|^2}{\|u\|^2} \\
&\leq \max_{u\in E\setminus\{0\}} \sum_{k=0}^{n} \overline{h_k}\frac{\sum\limits_{t=1}^{N}|\Delta^k u(t-k)|^2}{\|u\|^2} \\
&\leq \sum_{k=0}^{n} 4^k\overline{h_k}=\gamma.
\end{aligned}$$

Then (H_2') implies (H_2). The conclusion now follows from Theorem 1.2.
The proof is complete. $\square$

Proof of Theorem 1.3. Let Φ be defined by (6). Then, it is clear that $[-\Phi](0)=0$ and $[-\Phi]$ is even by (H_4). Since $F_\infty\geq\delta''$, from the proof of Theorem 1.2, $[-\Phi]$ is bounded from below, coercive and any PS sequence (u_m) is bounded. In view of the fact that the dimension of E is finite, we see that $[-\Phi]$ satisfies the (PS) condition.
Since $F^0<\lambda_p$, there exists $\rho_3>0$ such that

$$\frac{2F(t,x)}{x^2}\leq F^0+\varepsilon \quad \text{for } (t,|x|)\in[1,N]_{\mathbb{Z}}\times[0,\rho_3],$$

where $\varepsilon>0$ satisfies $\varepsilon<\lambda_p-F^0$, i.e.,

$$F(t,x)\leq\frac{1}{2}(F^0+\varepsilon)x^2 \quad \text{for } (t,|x|)\in[1,N]_{\mathbb{Z}}\times[0,\rho_3]. \tag{20}$$

Put $K_3=\left\{u=\sum\limits_{i=p}^{N}c_i\xi_i \ \ / \ \ \sum\limits_{i=p}^{N}c_i^2=\rho_3^2\right\}$.
Let S^{N-p} be the unit sphere in $\mathbb{R}^{N-p+1}$, and define $T:K_3\longrightarrow S^{N-p}$ by

$$T(u)=\frac{1}{\rho_3}(c_p,c_{p+1},...,c_N).$$

Then, T is an odd homeomorphism between K_3 and S^{N-p}.
For any $u\in K_3$, $|u(t)|\leq\|u\|=\rho_3$.

According to (20), we obtain
$$\begin{aligned}
[-\Phi](u) &= -\frac{1}{2}\langle\sum_{k=0}^{n}A_k u,u\rangle+\sum_{t=1}^{N}F(t,u(t)) \\
&\leq -\frac{1}{2}\lambda_p\|u\|^2+\frac{1}{2}(F^0+\varepsilon)\|u\|^2 \\
&= \frac{1}{2}(F^0+\varepsilon-\lambda_p)\rho_3^2<0.
\end{aligned}$$

Thus, we get $\sup_{u\in K_3}[-\Phi](u) < 0$.
Hence, all the conditions of Lemma 2.3 are satisfied, so $[-\Phi]$ has at least $2(N-p+1)$ nontrivial critical points, which are nontrivial solutions of the problem (P).
This completes the proof of the Theorem 1.3. □

Proof of Corollary 1.2. From the proof of Corollary 1.1, it is easy to see that (H_3') implies (H_3). The conclusion now follows from Theorem 1.3. □

We end this paper with the following Application.

5. Application

In problem (P), let $n=3$, $N=8$, $h_0(t)=4-t^2$, $h_1(0)=64$, $h_1(t)=t^2$, $h_2(t)=2+sin(\frac{\pi}{4}t)$, $h_3(t)=1$ and

$$f(t,x)=220t\begin{cases} x, & x\in\,]-\infty,-1[\,\cup\,]1,\infty[,\\ x^3, & x\in[-1,1],\end{cases}$$

for any $t\in[1,8]_{\mathbb{Z}}$.
It is clear that $f(t,x)$ is odd in x. Then, (H_4) holds.
With the above n and N, from Lemma 2.1 we can determine the matrix A_k, $k\in[0,3]_{\mathbb{Z}}$ by the formula

$$\langle A_k u,u\rangle=\sum_{t=1}^{8}(-1)^k\Delta^k(h_k(t-k)\Delta^k u(t-k))u(t),\ k\in[0,3]_{\mathbb{Z}}.$$

After a simple calculation, we get:

$$A_3=\begin{pmatrix} 20 & -15 & 6 & -1 & 0 & -1 & 6 & -15\\ -15 & 20 & -15 & 6 & -1 & 0 & -1 & 6\\ 6 & -15 & 20 & -15 & 6 & -1 & 0 & -1\\ -1 & 6 & -15 & 20 & -15 & 6 & -1 & 0\\ 0 & -1 & 6 & -15 & 20 & -15 & 6 & -1\\ -1 & 0 & -1 & 6 & -15 & 20 & -15 & 6\\ 6 & -1 & 0 & -1 & 6 & -15 & 20 & -15\\ -15 & 6 & -1 & 0 & -1 & 6 & -15 & 20\end{pmatrix},\tag{21}$$

$$A_2=\begin{pmatrix} a_{11} & a_{12} & h_2(1) & 0 & 0 & 0 & h_2(7) & a_{18}\\ a_{21} & a_{22} & a_{23} & h_2(2) & 0 & 0 & 0 & h_2(8)\\ h_2(1) & a_{32} & a_{33} & a_{34} & h_2(3) & 0 & 0 & 0\\ 0 & h_2(2) & a_{43} & a_{44} & a_{45} & h_2(4) & 0 & 0\\ 0 & 0 & h_2(3) & a_{54} & a_{55} & a_{56} & h_2(5) & 0\\ 0 & 0 & 0 & h_2(4) & a_{65} & a_{66} & a_{67} & h_2(6)\\ h_2(7) & 0 & 0 & 0 & h_2(5) & a_{76} & a_{77} & a_{78}\\ a_{81} & h_2(8) & 0 & 0 & 0 & h_2(6) & a_{87} & a_{88}\end{pmatrix},\tag{22}$$

where $a_{ii}=h_2(i)+4h_2(i-1)+h_2(i-2)$, $i=1,2,...8$,
$a_{ii+1}=a_{i+1i}=-2(h_2(i-1)+h_2(i))$, for $i=1,2,...,7$,
and $a_{18}=a_{81}=-2(h_2(7)+h_2(8))$,

$$A_1=\begin{pmatrix} h_1(0)+h_1(1) & -h_1(1) & 0 & \cdots & -h_1(0)\\ -h_1(1) & h_1(1)+h_1(2) & -h_1(2) & \cdots & 0\\ 0 & -h_1(2) & h_1(2)+h_1(3) & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & -h_1(7)\\ -h_1(0) & 0 & 0 & \cdots & h_1(7)+h_1(0)\end{pmatrix},$$

and

$$A_0 = \begin{pmatrix} h_0(1) & 0 & 0 & ... & 0 & 0 \\ 0 & h_0(2) & 0 & ... & 0 & 0 \\ 0 & 0 & h_0(3) & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & h_0(7) & 0 \\ 0 & 0 & 0 & \cdots & 0 & h_0(8) \end{pmatrix}. \tag{23}$$

By using MATLAB, we find that the eigenvalues of $A_0 + A_1 + A_2 + A_3$ are given by

$$\lambda_1 = -41.567, \quad \lambda_2 = -9.763, \quad \lambda_3 = 5.779, \quad \lambda_4 = 36.006,$$

$$\lambda_5 = 52.441, \quad \lambda_6 = 101.010, \quad \lambda_7 = 118.448, \quad \lambda_8 = 219.158.$$

Since $F^0 = 0$ and $F_\infty = 220$, then (H_3) holds with $p = 3$. The hypotheses of Theorem 1.3 are satisfied and so the problem (P) has at least 12 nontrivial solutions.

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