ON THE MULTIPLIER ALGEBRA OF FRÉCHET ALGEBRAS

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Let $\mathcal{A}$ be a Fréchet algebra and let $M(\mathcal{A})$ denote the algebra of all multipliers of $\mathcal{A}$. In this paper, we show that if $\mathcal{A}$ has a bounded approximate identity, then $M(\mathcal{A})$ is a Fréchet algebra with respect to the strict topology. Under this topology, we then study the spectrum of $M(\mathcal{A})$ and provide some examples in this field. Finally, we introduce and study the notion of φ-multiplier amenability of $\mathcal{A}$, where φ is a complex-valued homomorphism on $\mathcal{A}$.

Keywords: Multiplier, Fréchet algebra, φ-amenability.

MSC2010: 46H05, 46J05, 43A07, 43A22.

1. Introduction and preliminaries

The class of Fréchet algebras which is an important class of locally convex algebras has been widely studied by many authors. For a full understanding of Fréchet algebras, one may refer to [8, 10]. In the class of Banach algebras, there are many concepts which were generalized to the Fréchet case. For example, the notion of amenability of Fréchet algebras was introduced by Helemskii [9] and studied by Pirkovskii [19]. He generalized some theorems about amenability of Banach algebras such as strictly flat Banach $\mathcal{A}$-bimodule, virtual diagonal and approximate diagonal of Banach algebras, to Fréchet algebras. Lawson and Read [15], introduced and studied the notions of approximate amenability and approximate contractibility of Fréchet algebras. Moreover, Abtahi et al. [4] studied the notion of weak amenability of Fréchet algebras. Furthermore, according to the basic definition of Segal algebras and abstract Segal algebras, recently they introduced the notion of Segal Fréchet algebra in the Fréchet algebra $(\mathcal{A}, p_\ell)_{\ell\in\mathbb{N}}$; see [3] for more information. Also, Ranjbari and Rejali generalized the concept of ideal amenability in the class of Fréchet algebras in [20].

In this paper, we study the notion of multiplier algebra of Fréchet algebras. Given a Banach algebra $\mathcal{A}$, denote by $\Delta(\mathcal{A})$ the spectrum of $\mathcal{A}$ consisting of all nonzero characters on $\mathcal{A}$. Let $\varphi \in \Delta(\mathcal{A})$. Following [13], $\mathcal{A}$ is called φ-amenable if there exists $m \in \mathcal{A}^*$ such that $m(\varphi) = 1$ and $m(f \cdot a) = \varphi(a)m(f)$ for all $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$. Also, $\mathcal{A}$ is φ-amenable if and only if for every Banach $\mathcal{A}$-bimodule X with the left module action

$$a \cdot x = \varphi(a)x \qquad (a \in \mathcal{A},\, x \in X),$$

every continuous derivation from $\mathcal{A}$ into X^* is inner. This notion of amenability was recently generalized by Rejali et al. [2] to the Fréchet case.

The multipliers for topological algebras was studied by Johnson; see [11] and [12]. Many other authors investigated this concept in the case of C^*-algebras and Banach algebras. We refer the reader to [5], [7], [21], and [22] for more information.

Let $\mathcal{A}$ be a Fréchet algebra. Consider $M(\mathcal{A})$, the algebra of all multipliers of $\mathcal{A}$. In section 2, we show that if $\mathcal{A}$ has a bounded approximate identity, then $M(\mathcal{A})$ is also a Fréchet

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algebra under the strict topology. In section 3, we study the spectrum of $M(\mathcal{A})$. We prove that

$$\Delta(M(\mathcal{A})) = \{\hat{\varphi} : \varphi \in \Delta(\mathcal{A})\},$$

if $M(\mathcal{A})$ is equipped with the strict topology. Finally, in section 4, we show that φ-amenability of $\mathcal{A}$ is equivalent to $\hat{\varphi}$-amenability of $M(\mathcal{A})$.

Before proceeding to the main results, we provide some basic definitions and frameworks, which will be required throughout the paper. Following [8] and [18], a complete topological algebra $\mathcal{A}$ is a Fréchet algebra if its topology is produced by a countable family of increasing submultiplicative seminorms $(p_\ell)_{\ell\in\mathbb{N}}$. Note that $(p_\ell)_{\ell\in\mathbb{N}}$ is a fundamental system of continuous seminorms. In other words, it has the following properties:

(i) for every $a \in \mathcal{A}$ with $a \neq 0$, there exists an $\ell \in \mathbb{N}$ such that $p_\ell(a) > 0$;
(ii) for all $\ell, n \in \mathbb{N}$, there exist $m \in \mathbb{N}$ and also $M > 0$ such that

$$\max(p_\ell(a), p_n(a)) \leq M p_m(a) \qquad (a \in \mathcal{A}).$$

We denote by $(\mathcal{A}, p_\ell)$ the Fréchet algebra $\mathcal{A}$ with the fundamental system of seminorms $(p_\ell)_{\ell\in\mathbb{N}}$. In general, every locally convex Hausdorff space has a fundamental system of seminorms. Throughout the paper, we assume that all locally convex spaces are Hausdorff. Let E and F be locally convex spaces with the fundamental system of seminorms $(p_\alpha)_{\alpha\in A}$ and $(q_\beta)_{\beta\in B}$, respectively. By [18, Proposition 22.6], for every linear mapping $T : E \to F$, the following assertions are equivalent:

(i) $T \in \mathcal{B}(E, F)$, i.e. T is continuous;
(ii) T is continuous at 0;
(iii) for each $\beta \in B$ there exist an $\alpha \in A$ and $M > 0$, such that

$$q_\beta(T(x)) \leq M p_\alpha(x) \qquad (x \in E).$$

In this paper, $\mathcal{B}(E, E)$ is denoted by $\mathcal{B}(E)$.

2. **Multipliers of Fréchet algebras**

Let $(\mathcal{A}, p_\ell)$ be a Fréchet algebra. We recall that the multiplier algebra of $\mathcal{A}$, denoted by $M(\mathcal{A})$, is

$$M(\mathcal{A}) = \{T \in \mathcal{B}(\mathcal{A}) : T(ab) = T(a)b = aT(b) \quad (a, b \in \mathcal{A})\}.$$

Suppose that $\mathcal{A}$ is faithful, that is

$$\{a \in \mathcal{A} : a \cdot \mathcal{A} = \mathcal{A} \cdot a = \{0\}\} = \{0\}.$$

In this case $\mathcal{A} \cong \{L_a : a \in \mathcal{A}\}$, where L_a is defined on $\mathcal{A}$ as $L_a(b) = ab$ $(b \in \mathcal{A})$. Moreover, if $\mathcal{A}$ is commutative, then $\{L_a : a \in \mathcal{A}\} \subseteq M(\mathcal{A})$. Let $\mathcal{A}$ has a bounded approximate identity $(e_\alpha)_{\alpha\in\Lambda}$. Clearly, $\mathcal{A}$ is faithful. Let $\ell \in \mathbb{N}$. Define

$$q_\ell(T) = \lim_\alpha p_\ell(Te_\alpha) \qquad \big(T \in M(\mathcal{A})\big). \tag{1}$$

By applying [18, Remark 23.2], $\sup_{\alpha\in\Lambda} p_\ell(Te_\alpha) < \infty$, and so (1) is well-defined. It is easy to see that q_ℓ is a continuous seminorm on $M(\mathcal{A})$. Also, $q_\ell(L_a) = p_\ell(a)$ for every $a \in \mathcal{A}$. Now, the following theorem is immediate.

Theorem 2.1. *Let $(\mathcal{A}, p_\ell)$ be a commutative Fréchet algebra with a bounded approximate identity $(e_\alpha)_{\alpha\in\Lambda}$. Consider the seminorms $(q_\ell)_{\ell\in\mathbb{N}}$ defined as in (1). Then the following statements hold.*

(i) *$(M(\mathcal{A}), q_\ell)$ is a Fréchet algebra.*
(ii) *$\mathcal{A}$ is dense in $M(\mathcal{A})$ with respect to the topology generated by $(q_\ell)_{\ell\in\mathbb{N}}$.*

Proof. (i). First, we prove that $(q_\ell)_{\ell\in\mathbb{N}}$ is a fundamental system of submultiplicative seminorms. Let T be a nonzero element in $M(\mathcal{A})$. We then have $\lim_\alpha Te_\alpha \neq 0$. Thus, there exists $\ell_0 \in \mathbb{N}$ such that

$$\lim_\alpha q_{\ell_0}(T) = \lim_\alpha p_{\ell_0}(Te_\alpha) \neq 0.$$

Furthermore, if $\ell_1, \ell_2 \in \mathbb{N}$ with $\ell_1 \leq \ell_2$, then $p_{\ell_1}(Te_\alpha) \leq p_{\ell_2}(Te_\alpha)$ for each $\alpha \in \Lambda$. Consequently, $q_{\ell_1}(T) \leq q_{\ell_2}(T)$. Therefore, by [18, Lemma 22.4], $(q_\ell)_{\ell\in\mathbb{N}}$ is a fundamental system of seminorms. Also, if $T, S \in M(\mathcal{A})$ and $\ell \in \mathbb{N}$, then

$$\begin{aligned} q_\ell(T\cdot S) &= \lim_\alpha p_\ell((T\cdot S)e_\alpha) \\ &= \lim_\alpha \lim_\beta p_\ell((T\cdot S)(e_\alpha e_\beta)) \\ &= \lim_\alpha \lim_\beta p_\ell((Te_\alpha)(Se_\beta)) \\ &\leq \lim_\alpha \lim_\beta p_\ell(Te_\alpha)p_\ell(Se_\beta) \\ &= q_\ell(T)q_\ell(S). \end{aligned}$$

It remains to show that $\big(M(\mathcal{A}), q_\ell\big)$ is complete. Let $(T_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $M(\mathcal{A})$. Given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n_1, n_2 \geq n_0$ we have

$$q_\ell(T_{n_1} - T_{n_2}) = \lim_\alpha p_\ell(T_{n_1}e_\alpha - T_{n_2}e_\alpha) < \varepsilon \qquad (\ell \in \mathbb{N}).$$

Since $(\mathcal{A}, p_\ell)$ is complete, there exists $a \in \mathcal{A}$ such that $\lim_\alpha p_\ell(T_ne_\alpha - a) \to_n 0$. Thus, for each $\ell \in \mathbb{N}$ we have

$$\begin{aligned} q_\ell(T_n - L_a) &= \lim_\alpha p_\ell(T_ne_\alpha - ae_\alpha) \\ &\leq \lim_\alpha p_\ell(T_ne_\alpha - a) + \lim_\alpha p_\ell(a - ae_\alpha) \\ &\to_n 0. \end{aligned}$$

(ii). Let $T \in M(\mathcal{A})$. For each $\ell \in \mathbb{N}$,

$$q_\ell(T) = \lim_\alpha p_\ell(Te_\alpha) = \lim_\alpha q_\ell(L_{Te_\alpha}).$$

Therefore, $\mathcal{A}$ is dense in $M(\mathcal{A})$. □

Corollary 2.1. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a Banach algebra with a bounded approximate identity $(e_\alpha)_{\alpha\in\Lambda}$. Then, $M(\mathcal{A})$ is a Fréchet algebra with respect to the topology generated by the seminorm*

$$|||T||| = \lim_\alpha \|Te_\alpha\|_{\mathcal{A}} \qquad (T \in M(\mathcal{A})).$$

Remark 2.1. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a Banach algebra. Following* [14], *$M(\mathcal{A})$ is a Banach algebra with respect to the operator norm $\|\cdot\|_{M(\mathcal{A})}$, defined by*

$$\|T\|_{M(\mathcal{A})} = \sup\big\{\|Ta\| : a \in \mathcal{A},\ \|a\| \leq 1\big\} \qquad (T \in M(\mathcal{A})).$$

If $\mathcal{A}$ has a bounded approximate identity, then there exists $M > 0$ such that

$$|||T||| \leq M\|T\|_{M(\mathcal{A})} \qquad (T \in M(\mathcal{A})),$$

where $|||\cdot|||$ is defined as in Corollary 2.1.

Let $(\mathcal{A}, p_\ell)$ be a faithful, commutative Fréchet algebra. Clearly,

$$L_a \cdot T = T \cdot L_a = L_{Ta} \qquad (a \in \mathcal{A},\ T \in M(\mathcal{A})).$$

Consider the seminorms $(\bar{p}_\ell)_{\ell\in\mathbb{N}}$ on $\{L_a : a \in \mathcal{A}\}$, defined by

$$\bar{p}_\ell(L_a) = p_\ell(a) \qquad (a \in \mathcal{A}).$$

Following [1], for every $a \in \mathcal{A}$ and $\ell \in \mathbb{N}$, $q_{L_a,\ell}$ is a continuous seminorm on $M(\mathcal{A})$, where

$$q_{L_a,\ell}(T) := \bar{p}_\ell(L_a \cdot T) + \bar{p}_\ell(T \cdot L_a) = 2\bar{p}_\ell(L_{Ta}) = 2p_\ell(Ta) \qquad (T \in M(\mathcal{A})).$$

Also, the strict topology on $M(\mathcal{A})$, with respect to $\mathcal{A}$, is defined by the system of seminorms

$$\bar{q}_{F,\ell} = \max_{a\in F} q_{L_a,\ell},$$

for every finite subset F of $\mathcal{A}$ and $\ell \in \mathbb{N}$. We state here the main result of this section.

Proposition 2.1. *Let $(\mathcal{A}, p_\ell)$ be a commutative Fréchet algebra with a bounded approximate identity $(e_\alpha)_{\alpha\in\Lambda}$. Consider the seminorms $(q_\ell)_{\ell\in\mathbb{N}}$ defined as in (1). Then, $(M(\mathcal{A}), q_\ell)$ is homeomorphic to $(M(\mathcal{A}), \bar{q}_{F,\ell})$.*

Proof. Let a net $(T_\gamma)_{\gamma\in\Gamma} \subseteq M(\mathcal{A})$ be convergent to some T in $(M(\mathcal{A}), q_\ell)$. Thus,

$$\lim_\gamma q_\ell(T_\gamma - T) = \lim_\gamma \lim_\alpha p_\ell(T_\gamma e_\alpha - Te_\alpha) = 0 \qquad (\ell \in \mathbb{N}).$$

If F is a finite subset of $\mathcal{A}$, then for every $a \in F$ we have

$$\begin{aligned} q_{L_a,\ell}(T_\gamma - T) &= 2p_\ell(T_\gamma a - Ta) \\ &= \lim_\alpha 2p_\ell(T_\gamma(e_\alpha a) - T(e_\alpha a)) \\ &\leq \lim_\alpha 2p_\ell(T_\gamma e_\alpha - Te_\alpha)p_\ell(a). \end{aligned}$$

Therefore, $\lim_\gamma q_{L_a,\ell}(T_\gamma - T) = 0$ for every $a \in F$ and so

$$\lim_\gamma \bar{q}_{F,\ell}(T_\gamma - T) = 0.$$

Conversely, suppose that $T_\gamma \to_\gamma T$ in $(M(\mathcal{A}), \bar{q}_{F,\ell})$. If $F \subseteq \mathcal{A}$ is finite, then by assumption, $\lim_\gamma p_\ell(T_\gamma a - Ta) = 0$ for every $a \in F$ and $\ell \in \mathbb{N}$. Consequently,

$$\lim_\gamma q_\ell(T_\gamma - T) = \lim_\gamma \lim_\alpha p_\ell(T_\gamma e_\alpha - Te_\alpha) = 0,$$

which completes the proof. □

Corollary 2.2. *Let $(\mathcal{A}, p_\ell)$ be a commutative Fréchet algebra. If $\mathcal{A}$ has a bounded approximate identity, then $M(\mathcal{A})$ is a Fréchet algebra with respect to the strict topology.*

3. On the spectrum of $M(\mathcal{A})$

Let $(\mathcal{A}, p_\ell)$ be a faithful, commutative Fréchet algebra and $a \in \mathcal{A}$. Following [14], define

$$\hat{a} : \Delta(\mathcal{A}) \to \mathbb{C}, \quad \hat{a}(\varphi) = \varphi(a) \qquad (\varphi \in \Delta(\mathcal{A})).$$

Consider $\hat{\mathcal{A}} = \{\hat{a} : a \in \mathcal{A}\}$ and $\mathtt{M}(\mathcal{A}) = \{\theta \in \mathcal{B}(\Delta(\mathcal{A})) : \theta\hat{\mathcal{A}} \subseteq \hat{\mathcal{A}}\}$. Similar to the proof of [14, Theorem 1.2.2], one can show that for every $T \in M(\mathcal{A})$ there exists a unique element $\hat{T} \in \mathtt{M}(\mathcal{A})$ such that

$$\hat{T}(\varphi) = \frac{\varphi \circ T(a)}{\varphi(a)} \qquad (\varphi \in \Delta(\mathcal{A})),$$

for some $a \in \mathcal{A}$ with $\varphi(a) \neq 0$. Clearly, this definition of $\hat{T}$ is independent of the choice of a.

Theorem 3.1. *Let $(\mathcal{A}, p_\ell)$ be a commutative Fréchet algebra. For every $\varphi \in \Delta(\mathcal{A})$ there exists a unique homomorphism $\hat{\varphi}$ on $M(\mathcal{A})$ such that the following hold.*

(i) *For every $a \in \mathcal{A}$, $\hat{\varphi}(L_a) = \varphi(a)$.*
(ii) *If $\mathcal{A}$ is faithful, then $\hat{\varphi}|_\mathcal{A} = \varphi$.*

Proof. Let $\varphi \in \Delta(\mathcal{A})$. Define $\hat{\varphi} : M(\mathcal{A}) \to \mathbb{C}$ by $\hat{\varphi}(T) = \hat{T}(\varphi)$. The details of the proof are similar to the Banach algebra case; see [14, Theorem 1.4.1]. □

Throughout the paper, $\hat{\varphi}$ is the nonzero homomorphism which is defined as in the proof of Theorem 3.1. Clearly, if $\mathcal{A}$ is a Banach algebra, then $\hat{\varphi} \in \Delta(M(\mathcal{A}))$. However, the following lemma is immediate for a Fréchet case.

Lemma 3.1. *Let $(\mathcal{A}, p_\ell)$ be a commutative Fréchet algebra with a bounded approximate identity $(e_\alpha)_{\alpha\in\Lambda}$. If $\varphi \in \Delta(\mathcal{A})$, then $\hat{\varphi} \in \Delta(M(\mathcal{A}))$.*

Proof. Let $\varphi \in \Delta(\mathcal{A})$. It is enough to show that $\hat{\varphi}$ is continuous at zero. To prove this, consider the net $(T_\gamma)_{\gamma\in\Gamma}$ such that

$$\lim_\gamma q_\ell(T_\gamma) = \lim_\gamma \lim_\alpha p_\ell(T_\gamma e_\alpha) = 0 \qquad (\ell \in \mathbb{N});$$

see Theorem 2.1. Without loss of generality we may assume that $\varphi(e_\alpha) \neq 0$ for each $\alpha \in \Lambda$. Now, set $f_\alpha := \frac{e_\alpha}{\varphi(e_\alpha)}$ $(\alpha \in \Lambda)$. Clearly, $\varphi(f_\alpha) = 1$ $(\alpha \in \Lambda)$. Since φ is continuous, there exist an $\ell_0 \in \mathbb{N}$ and $M > 0$ such that

$$\varphi(T_\gamma f_\alpha) \leq M p_{\ell_0}(T_\gamma f_\alpha) \qquad (\alpha \in \Lambda,\ \gamma \in \Gamma).$$

Thus, we have

$$\lim_\gamma \lim_\alpha \varphi(T_\gamma f_\alpha) \leq \lim_\gamma \lim_\alpha M p_{\ell_0}(T_\gamma f_\alpha).$$

Consequently, $\lim_\gamma \hat{\varphi}(T_\gamma) = 0$, which implies that $\hat{\varphi}$ is continuous. □

Theorem 3.2. *Let $(\mathcal{A}, p_\ell)$ be a commutative Fréchet algebra. If $\mathcal{A}$ has a bounded approximate identity, then*

$$\{\hat{\varphi} : \varphi \in \Delta(\mathcal{A})\} = \Delta\big((M(\mathcal{A}), q_\ell)\big).$$

Proof. Let $(e_\alpha)_{\alpha\in\Lambda}$ be a bounded approximate identity for $\mathcal{A}$, and $\xi \in \Delta(M(\mathcal{A}))$. Define $\varphi(a) := \xi(L_a)$ for every $a \in \mathcal{A}$. Thus, we have

$$\varphi(ab) = \xi(L_{ab}) = \xi(L_a)\xi(L_b) = \varphi(a)\varphi(b),$$

for every $a, b \in \mathcal{A}$. In addition, there exists $T \in M(\mathcal{A})$ such that $\xi(T) \neq 0$. Now, by applying Theorem 2.1, $\lim_\alpha \xi(L_{Te_\alpha}) \neq 0$. For this reason, there exists $a \in \mathcal{A}$ such that $\varphi(a) \neq 0$. To show $\xi = \hat{\varphi}$, let $T \in M(\mathcal{A})$. Therefore,

$$\xi(T) = \lim_\alpha \xi(L_{Te_\alpha}) = \lim_\alpha \varphi(Te_\alpha) = \lim_\alpha \frac{\varphi((Te_\alpha)a)}{\varphi(a)} = \frac{\varphi(Ta)}{\varphi(a)} = \hat{\varphi}(T).$$

Consequently, $\Delta\big((M(\mathcal{A}), q_\ell)\big) \subseteq \{\hat{\varphi} : \varphi \in \Delta(\mathcal{A})\}$. As the reverse inclusion is trivially true,

$$\{\hat{\varphi} : \varphi \in \Delta(\mathcal{A})\} = \Delta\big((M(\mathcal{A}), q_\ell)\big).$$

□

Example 3.1. *(i). Let G be a locally compact Abelian group. Due to Wendel* [23], *$M(L^1(G)) = \mathrm{M}(G)$, the Banach algebra of bounded regular complex valued Borel measures on G. Indeed, for every $T \in M(L^1(G))$ there exists a uniqe element $\mu \in \mathrm{M}(G)$ such that*

$$T(f) = f * \mu \qquad (f \in L^1(G)).$$

Also, $\Delta(L^1(G)) = \hat{G}$, the dual group of G, that is the group of continuous characters on G. In addition, for every $\chi \in \hat{G}$ we have

$$\begin{aligned}\chi(f*\mu) &= \int f*\mu(x)\chi(x)\mathrm{d}\lambda(x) = \int\int f(xy^{-1})\chi(x)\mathrm{d}\mu(y)\mathrm{d}\lambda(x)\\ &= \int\int f(x)\chi(x)\chi(y)\mathrm{d}\mu(y)\mathrm{d}\lambda(x) = \chi(f)\int \chi(y)\mathrm{d}\mu(y),\end{aligned}$$

where λ is the Haar measure on G. Let $(f_\alpha)_{\alpha\in\Lambda}$ be a bounded approximate identity for $L^1(G)$. By applying Corollary 2.1, consider the seminorm

$$|||\mu||| = \lim_\alpha \|f_\alpha * \mu\|_1$$

on $\mathrm{M}(G)$. *Then, from Theorem 3.2, it follows that*

$$\Delta\big((\mathrm{M}(G), |||\cdot|||)\big) = \big\{\hat{\chi} : \chi \in \hat{G}\big\},$$

where for every $\mu \in \mathrm{M}(G)$ *and* $f \in L^1(G)$ *with* $\chi(f) \neq 0$ *we have*

$$\hat{\chi}(\mu) = \frac{\chi(f * \mu)}{\chi(f)} = \int \chi(y)\mathrm{d}\mu(y).$$

(ii). Let G *be any locally compact Abelian group, and let* $A(G)$ *denote the Fourier algebra with pointwise product. By* [6, chapter 3], $\Delta(A(G)) = G$, *where*

$$\delta_x(f) = f(x) \qquad (x \in G,\ f \in A(G)).$$

Also, if G *is amenable, then* $A(G)$ *has a bounded approximate identity and* $M(A(G)) = B(G)$, *the Fourier-Stieltjes algebra; see* [16] *and* [17]. *Indeed, for every* $T \in M(A(G))$ *there exists a unique element* $u \in B(G)$ *such that*

$$Tf = uf \qquad (f \in A(G)).$$

Now, let $(e_\alpha)_{\alpha\in\Lambda}$ *be a bounded approximate identity for* $A(G)$. *Consider the seminorm* $|||u||| = \lim_\alpha \|ue_\alpha\|$ *on* $B(G)$. *By using Theorem 3.2, we have*

$$\Delta\big((B(G), |||\cdot|||)\big) = \{\hat{\delta}_x : x \in G\},$$

where if $x \in G$ *and* $f \in A(G)$ *with* $f(x) \neq 0$, *then*

$$\hat{\delta}_x(u) = \frac{\delta_x(uf)}{\delta_x(f)} = \frac{u(x)f(x)}{f(x)} = u(x) \qquad (u \in B(G)).$$

4. φ-multiplier amenability

Let $\mathcal{A}$ be a Fréchet algebra and $\varphi \in \Delta(\mathcal{A})$. Following [2], $\mathcal{A}$ is left φ-amenable if there exists $m \in \mathcal{A}^{**}$ such that

$$m(\varphi) = 1 \quad \text{and} \quad m(f \cdot a) = \varphi(a)m(f),$$

for every $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$. Now, consider the left locally convex $M(\mathcal{A})$-module X with the module action

$$T \cdot x := \hat{\varphi}(T)x \qquad (T \in M(\mathcal{A}),\ x \in X).$$

Note that if $T, S \in M(\mathcal{A})$ and $x, y \in X$, then clearly

$$T \cdot (x + y) = T \cdot x + T \cdot y \quad \text{and} \quad (T + S) \cdot x = T \cdot x + S \cdot x.$$

Also, by Theorem 3.1, we have

$$(T \cdot S) \cdot x = \hat{\varphi}(T \cdot S)x = \hat{\varphi}(T)\hat{\varphi}(S)x = \hat{\varphi}(T)(S \cdot x) = T \cdot (S \cdot x).$$

Therefore, X^* is a right locally convex $M(\mathcal{A})$-module with the module action

$$(f \cdot T)(x) = f(T \cdot x) \qquad (T \in M(\mathcal{A}),\ f \in X^*,\ x \in X).$$

In the following, we define the concept of φ-multiplier amenability. Then, we investigate the relation between φ-multiplier amenability and φ-amenability.

Definition 4.1. *Let* $\mathcal{A}$ *be a commutative Fréchet algebra and* $\varphi \in \Delta(\mathcal{A})$. *We say that* $\mathcal{A}$ *is left* φ*-multiplier amenable if there exists* $\mathrm{M} \in M(\mathcal{A})^{**}$ *such that*

$$\mathrm{M}(\hat{\varphi}) = 1 \quad \textit{and} \quad \mathrm{M}(F \cdot T) = \hat{\varphi}(T)\mathrm{M}(F),$$

for all $T \in M(\mathcal{A})$ *and* $F \in M(\mathcal{A})^*$. *In other words,* $\mathcal{A}$ *is left* φ*-multiplier amenable if and only if* $M(\mathcal{A})$ *is left* $\hat{\varphi}$*-amenable.*

Consider the commutative Fréchet algebra $\mathcal{A}$ with a bounded approximate identity $(e_\alpha)_{\alpha\in\Lambda}$. For every $f \in \mathcal{A}^*$ define $\bar{f} \in M(\mathcal{A})^*$, by

$$\bar{f}(T) := \lim_\alpha f(Te_\alpha) \qquad (T \in M(\mathcal{A})).$$

Obviously, $\bar{f}|_{\mathcal{A}} = f$. Now, we can conclude the following lemma.

Lemma 4.1. *Let $\mathcal{A}$ be a commutative Fréchet algebra with a bounded approximate identity and let $\varphi \in \Delta(\mathcal{A})$. Then, $\mathcal{A}$ is left φ-multiplier amenable if and only if it is left φ-amenable.*

Proof. Let $(e_\alpha)_{\alpha\in\Lambda}$ be a bounded approximate identity for $\mathcal{A}$. Consider $\mathrm{M} \in M(\mathcal{A})^{**}$ such that

$$\mathrm{M}(\hat{\varphi}) = 1 \quad \text{and} \quad \mathrm{M}(F \cdot T) = \hat{\varphi}(T)\mathrm{M}(F),$$

for every $T \in M(\mathcal{A})$ and $F \in M(\mathcal{A})^*$. According to the previous arguments, define

$$m(f) := \mathrm{M}(\bar{f}) \qquad (f \in \mathcal{A}^*).$$

If $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$, then

$$m(f \cdot a) = \mathrm{M}(\overline{f \cdot a}) = \mathrm{M}(\bar{f} \cdot L_a) = \hat{\varphi}(L_a)\mathrm{M}(\bar{f}) = \varphi(a)m(f).$$

Let $T \in M(\mathcal{A})$. Therefore, by Theorem 3.1, we have

$$\overline{\hat{\varphi}|_{\mathcal{A}}}(T) = \lim_\alpha \hat{\varphi}(L_{Te_\alpha}) = \lim_\alpha \frac{\varphi(L_{Te_\alpha}a^2)}{\varphi(a^2)} = \lim_\alpha \frac{\varphi(L_{Te_\alpha}a)}{\varphi(a)} = \frac{\varphi(Ta)}{\varphi(a)} = \hat{\varphi}(T),$$

for some $a \in \mathcal{A}$ with $\varphi(a) \neq 0$. For this reason,

$$m(\varphi) = m(\hat{\varphi}|_{\mathcal{A}}) = \mathrm{M}(\overline{\hat{\varphi}|_{\mathcal{A}}}) = \mathrm{M}(\hat{\varphi}) = 1.$$

Thus, $\mathcal{A}$ is left φ-amenable.

Conversely, let $m \in \mathcal{A}^{**}$ such that

$$m(\varphi) = 1 \quad \text{and} \quad m(f \cdot a) = \varphi(a)m(f),$$

for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$. Consider the map $\iota : \mathcal{A} \hookrightarrow M(\mathcal{A})$ defined by $\iota(a) = L_a$. Set $\mathrm{M} := m \circ \iota^* \in M(\mathcal{A})^{**}$. Thus,

$$\mathrm{M}(\hat{\varphi}) = m \circ \iota^*(\hat{\varphi}) = m(\hat{\varphi} \circ \iota) = m(\varphi) = 1.$$

Also, by Theorem 2.1, for every $F \in M(\mathcal{A})^*$ and $T \in M(\mathcal{A})$ we have

$$\begin{aligned}
\mathrm{M}(F \cdot T) &= m \circ \iota^*(F \cdot T) = \lim_\alpha m \circ \iota^*(F \cdot L_{Te_\alpha}) \\
&= \lim_\alpha m((F \cdot L_{Te_\alpha}) \circ \iota) = \lim_\alpha m((F \circ \iota) \cdot Te_\alpha) \\
&= \lim_\alpha \varphi(Te_\alpha)m(F \circ \iota) = \lim_\alpha \hat{\varphi}(L_{Te_\alpha})m(F \circ \iota) \\
&= \hat{\varphi}(T)\mathrm{M}(F).
\end{aligned}$$

Consequently, $\mathcal{A}$ is left φ-multiplier amenable. □

Theorem 4.1. *Let $\mathcal{A}$ be a commutative Fréchet algebra with a bounded approximate identity and let $\varphi \in \Delta(\mathcal{A})$. Then, the following statements are equivalent:*

(i) *$\mathcal{A}$ is left φ-amenable;*

(ii) *$\mathcal{A}$ is left φ-multiplier amenable;*

(iii) *every continuous derivation $D : M(\mathcal{A}) \to (\frac{M(\mathcal{A})^*}{\mathbb{C}\hat{\varphi}})^*$ is inner, where*

$$T \cdot (F + \mathbb{C}\hat{\varphi}) = \hat{\varphi}(T)F + \mathbb{C}\hat{\varphi} \qquad (T \in M(\mathcal{A}),\ F \in M(\mathcal{A})^*);$$

(iv) *for every left locally convex $M(\mathcal{A})$-module X, with the module action*

$$T \cdot x := \hat{\varphi}(T)x \qquad (T \in M(\mathcal{A}),\ x \in X),$$

every continuous derivation $D : M(\mathcal{A}) \to X^*$ is inner.

Proof. Due to Corollary 2.2, $M(\mathcal{A})$ is a Fréchet algebra with respect to the strict topology. Also, we have

$$T \cdot \hat{\varphi} = \hat{\varphi} \cdot T = \hat{\varphi}(T)\hat{\varphi} \qquad (T \in M(\mathcal{A})).$$

Now, in [2, Theorem 3.2], it is enough to use $M(\mathcal{A})$ and $\hat{\varphi}$ instead of $\mathcal{A}$ and φ, respectively. □

Corollary 4.1. *If $\mathcal{A}$ is a commutative Banach algebra with a bounded approximate identity and $\varphi \in \Delta(\mathcal{A})$, then (i), (ii), and (iii) in Theorem 4.1, are equivalent to*

- *for every left Banach $M(\mathcal{A})$-module X, with the module action*

$$T \cdot x := \hat{\varphi}(T)x \qquad (T \in M(\mathcal{A}),\ x \in X),$$

every continuous derivation $D : M(\mathcal{A}) \to X^*$ is inner.

Acknowledgment
The authors would like to thank the Banach algebra center of Excel-lence for Mathematics, University of Isfahan.

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