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ON A GENERAL CLASS OF PRECONDITIONERS FOR NONSYMMETRIC GENERALIZED SADDLE POINT PROBLEMS

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This paper deals with applying a class of preconditioners for saddle point problems with nonsymmetric positive definite (1,1) part and symmetric positive semidefinite (2,2) part. The examined class of preconditioners includes the generalized shift-splitting preconditioner and their modified version. The offered class of preconditioner is induced from a stationary iterative method. Numerical experiments for a model Navier-Stokes problem are reported which illustrate the efficiency of the presented preconditioner.

Keywords: Nonsymmetric saddle point problem, preconditioner, shift-splitting, iterative method.
MSC2010: 65F10, 65F50 and 65N22.

1. Introduction

Consider generalized saddle point problems of the form

$$\mathcal{A}u \equiv \begin{pmatrix} A & B^T \\ -B & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix} \equiv b, \tag{1}$$

such that $A \in \mathbb{R}^{n\times n}$ is nonsymmetric positive definite (i.e., the symmetric part $H = \frac{1}{2}(A + A^T)$ is positive definite; $B \in \mathbb{R}^{m\times n}$ is of full row rank $(m \le n)$ and C is symmetric positive semidefinite. In [1, Lemma 1.1], it has been shown that the preceding assumptions guarantee existence and uniqueness of the solution of (1). Linear systems of the form (1) arise in a variety of computational sciences and engineering applications such as constrained optimization, computational fluid dynamics, mixed finite element discretization of the Navier-Stokes equations, the linear elasticity problem, elliptic and parabolic interface problems, constrained least-squares problem and so on; for more details see [1, 2, 3, 9, 10] and references therein.

In the past few years, there has been a growing interest in saddle problems of the form (1) and several types of iterative methods have been proposed; we refer the reader to [2] for a comprehensive survey. More precisely, recently, some research works have focused on variants types of the Uzawa method for the case that (2,2)-block is zero; for more details see [14, 15, 16] and references therein.

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More recently, Liang and Zhang [11] have presented a modified Uzawa method based on the skew-Hermitian triangular splitting (STS) of the (1,1) part of the saddle-point coefficient matrix for solving non-Hermitian saddle-point problems with non-Hermitian positive definite and skew-Hermitian dominant (1,1) and zero (2,2) part.

It is well-known that the Krylov subspace methods are superior to the stationery iterative methods in general. Nevertheless, the Krylov subspace methods converge slowly for solving saddle point problems in practice. To overcome this drawback, several papers have applied different types of preconditioners till now. For instance, Cao et al. [4] have proposed a modified dimensional split preconditioner based on a splitting of the generalized saddle point matrix. In the case that the (2,2)-block C is zero and the (1,1)-block is symmetric positive definite. In [5], Cao et al. have examined the shift-splitting preconditioner of the form

$$\mathcal{P}_{SS} = \frac{1}{2}\begin{pmatrix} \alpha I + A & B^T \\ -B & \alpha I \end{pmatrix}, \qquad \alpha > 0. \tag{2}$$

which is obtained based on shift-splitting of the saddle point coefficient matrix. Also the application of a local preconditioner is studied

$$\mathcal{P}_{LSS} = \frac{1}{2}\begin{pmatrix} A & B^T \\ -B & \alpha I \end{pmatrix}, \qquad \alpha > 0.$$

Recently, Chen and Ma [7] have focused on the saddle point problems with the similar form mentioned in [5] and developed a general class of preconditioners which incorporates (2). More precisely, the following preconditioner has been offered

$$\mathcal{P}_{GSS} = \frac{1}{2}\begin{pmatrix} \alpha I + A & B^T \\ -B & \beta I \end{pmatrix}, \tag{3}$$

where $\alpha \geq 0$ and $\beta > 0$.

Lately, Salkuyeh et al. [13] have applied a modified generalized shift-splitting (MGSS) preconditioner

$$\mathcal{P}_{MGSS} = \frac{1}{2}\begin{pmatrix} \alpha I + A & B^T \\ -B & \beta I + C \end{pmatrix}, \qquad \alpha, \beta > 0, \tag{4}$$

for the saddle point problems of the form (1) so that the (1,1)-block is symmetric positive definite and $C \neq 0$. The MGSS preconditioner is based on a splitting of the saddle point coefficient matrix which results in an unconditionally convergent stationary iterative method. In addition, the following relaxed version of the MGSS preconditioner is examined

$$\mathcal{P}_{RMGSS} = \frac{1}{2}\begin{pmatrix} A & B^T \\ -B & \beta I + C \end{pmatrix}, \qquad \beta > 0.$$

More recently, Cao et al. [6] have studied the application of the generalized shift-splitting preconditioner (3) for saddle point problem (1) whose (2,2)-block is zero and (1,1)-block is a nonsymmetric positive definite matrix.

In this paper, we introduce a class of general preconditioners which are obtained by splitting the diagonal blocks of the coefficient matrix $\mathcal{A}$ in (1). To do this, first, the mixed splitting (MS) iterative method is presented and its convergence properties are discussed. The offered MS iterative method includes the modified generalized shift-splitting (MGSS) iterative method [13]. Notice that the convergence properties of the MGSS iterative method and the application of the preconditioners of the form (4) have not been studied for the case that the (1,1)-block A in (1) is nonsymmetric. As seen, the induced preconditioner by the MS method incorporates the preconditioners (2), (3) and (4).

The rest of this paper is organized as follows. Before ending this section, some notations are introduced which are exploited throughout the paper. In Section 2, first, we present the mixed splitting (MS) iterative method and give some sufficient conditions under which the method is convergent. Then based on the presented splitting, a general class of preconditioners is offered to improve the speed of convergence of the Krylov subspace methods to solve the saddle point problems of the form (1). In Section 3, some numerical experiments are reported for a model of Navier-Stokes problem to illustrate the efficiency of the proposed new class of preconditioners. Finally the paper is finished with a brief conclusion in Section 4.

Notations: The real and conjugate parts of a complex number a are denoted by $\mathrm{Re}(a)$ and $\bar{a}$, respectively. For a square matrix A, the symbol $\rho(A)$ stands for the spectral radius of A. For two given symmetric positive definite matrices A and B, the notation $A \succ B$ means that $A - B$ is positive definite.

2. Main results

Consider the following mixed splitting for the coefficient matrix $\mathcal{A}$,

$$\mathcal{A} = \mathcal{P}_{MS} - \mathcal{Q}_{MS} = \begin{pmatrix} \hat{M} & \frac{1}{2}B^T \\ -\frac{1}{2}B & \tilde{M} \end{pmatrix} - \begin{pmatrix} \hat{N} & -\frac{1}{2}B^T \\ \frac{1}{2}B & \tilde{N} \end{pmatrix}, \tag{5}$$

in which $\hat{M}$ and $\tilde{M}$ are both nonsingular matrices such that $A = \hat{M} - \hat{N}$ and $C = \tilde{M} - \tilde{N}$.

Remark 2.1. *Assume that* $C = 0$. *Consider a special case of (5) such that*

$$\hat{M} = \frac{1}{2}(\alpha I + A), \quad \hat{N} = \frac{1}{2}(\alpha I - A), \quad \tilde{M} = \beta I \quad \text{and} \quad \tilde{N} = \beta I,$$

where $\alpha > (\geq) 0$ *and* $\beta > 0$. *Evidently, in this case, the mixed splitting reduces to the generalized shift-splitting considered in [6]. Note that it is not difficult to verify that* $\hat{M} + \hat{N}$ *is symmetric (semi) positive definite,* $\tilde{M} + \tilde{N}$ *is nonsingular,* $\tilde{M}$ *is symmetric positive definite and* $\rho(\hat{M}^{-1}\hat{N}) < (\leq)\ 1$.

Remark 2.2. *Suppose that* $\alpha > (\geq)\ 0$ *and* $\beta > 0$ *are given and* A *is symmetric. We comment here that setting*

$$\hat{M} = \frac{1}{2}(\alpha I + A), \quad \hat{N} = \frac{1}{2}(\alpha I - A), \quad \tilde{M} = \beta I + C \quad \text{and} \quad \tilde{N} = \beta I - C,$$

in (5), we obtain the modified generalized shift-splitting splitting discussed in [13]. Similar to the previous remark, we point out that here $\hat{M} + \hat{N}$ *is symmetric (semi) positive definite,* $\tilde{M} + \tilde{N}$ *is nonsingular,* $\tilde{M}$ *is symmetric positive definite and* $\rho(\hat{M}^{-1}\hat{N}) < (\leq)\ 1$.

Mixed splitting (MS) iterative method:

Exploiting the mixed splitting (5), we may define the MS iterative scheme to solve (1) as follows:

$$u^{(k+1)} = \Gamma_{MS} u^{(k)} + c, \quad k = 0,1,2,\ldots, \tag{6}$$

where $\Gamma_{MS} = \mathsf{P}_{MS}^{-1}\mathsf{Q}_{MS}$ is the iteration matrix, $c = \mathsf{P}_{MS}^{-1}b$, $u^{(k)} \equiv ((x^{(k)})^T, (y^{(k)})^T)^T$ and the initial guess $u^{(0)}$ is given. It is well-known that (6) is convergent for any arbitrary initial guess $u^{(0)}$ iff $\rho(\Gamma_{MS}) < 1$; for further details one may refer to [12, Chapter 4].

In the sequel we discuss the convergence of the MS method. Let λ be an arbitrary eigenvalue of Γ_{MS} and $u^T = (x^T, y^T)$ be its corresponding eigenvector. That is, $\mathsf{P}_{MS}^{-1}\mathsf{Q}_{MS}u = \lambda u$ which is equivalent to say that $\mathsf{Q}_{MS}u = \lambda \mathsf{P}_{MS}u$. Hence, we obtain

$$\hat{N}x - \frac{1}{2}B^T y \ = \ \lambda \hat{M}x + \frac{\lambda}{2}B^T y, \tag{7}$$

and

$$\frac{1}{2}Bx + \tilde{N}y \quad = \quad -\frac{\lambda}{2}Bx + \lambda \tilde{M}y. \tag{8}$$

Lemma 2.1. *Suppose that* $\tilde{M} + \tilde{N}$ *is nonsingular. If* λ *is an eigenvalue of* Γ_{MS}, *then* $\lambda \neq \pm 1$.

Proof. Assume that u is the corresponding eigenvector of λ, hence $u \neq 0$. In view of the fact that A is nonsingular then $\lambda \neq 1$. Since $\lambda = 1$ implies that $\mathsf{A}u = 0$. From (8), we find that if $\lambda = -1$ then $(\tilde{M} + \tilde{N})u = 0$ which is a contradiction by our assumption that $\tilde{M} + \tilde{N}$ is nonsingular. □

Remark 2.3. *In view of Remarks 2.1 and 2.2, throughout this work, we assume that $\hat{M} + \hat{N}$ is symmetric positive semidefinite, $\tilde{M} + \tilde{N}$ is nonsingular, $\tilde{M}$ is symmetric positive definite, $\tilde{M} \succ C$ and $\rho(\hat{M}^{-1}\hat{N}) < 1$.*

Theorem 2.1. *Assume that Γ_{MS} denotes the iteration matrix of the MS method. Suppose that $\hat{M} + \hat{N}$ is symmetric positive semidefinite, $\tilde{M} + \tilde{N}$ is nonsingular, $\tilde{M}$ is symmetric positive definite, $\tilde{M} \succ C$ and $\rho(\hat{M}^{-1}\hat{N}) < 1$. Then $\rho\left(\Gamma_{MS}\right) < 1$.*

Proof. Let (λ, u) be an arbitrary eigenpair of Γ_{MS} so that $\lambda \neq 0$ where $u^H = (x^H, y^H)$.

We first show that $x \neq 0$. If $x = 0$ then from (7), it is seen that $(1+\lambda)B^T y = 0$. Now Lemma 2.1 implies that $y = 0$, hence $u = 0$ which cannot be true invoking the fact that u is an eigenvector.

Notice if $y = 0$ then from (7), we have $\hat{N}x = \lambda\hat{M}x$. That is λ is an eigenvalue of $\hat{M}^{-1}\hat{N}$. Therefore, by the assumption $|\lambda| < 1$.

In the rest of the proof without loss of generality, it is assumed that $x \neq 0$ and $y \neq 0$. From Eqs (7) and (8), we have

$$x^H B^T y = -2x^H \hat{M}x + \frac{2}{1+\lambda} x^H (\hat{M} + \hat{N})x, \tag{9}$$

and

$$y^H B^T x = 2\frac{\lambda - 1}{1+\lambda} y^H \tilde{M}y + \frac{2}{1+\lambda} y^H Cy. \tag{10}$$

Note that the left hand side of (9) is equal to the conjugate of the left hand side of (10), hence

$$-2x^H \hat{M}^T x + \frac{2}{1+\bar{\lambda}} x^H (\hat{M} + \hat{N})x = 2\frac{\lambda - 1}{1+\lambda} y^H \tilde{M}y + \frac{2}{1+\lambda} y^H Cy.$$

Or equivalently,

$$-2x^H \hat{M}^T x + \left(-\overline{\left(\frac{\lambda - 1}{1+\lambda}\right)} + 1 \right) x^H (\hat{M} + \hat{N})x = 2\frac{\lambda - 1}{1+\lambda} y^H \tilde{M}y + \left(-\frac{\lambda - 1}{1+\lambda} + 1 \right) y^H Cy.$$

For simplicity, we set $q = x^H (\hat{M} + \hat{N})x$, $r = y^H \tilde{M}y$ and $s = y^H Cy$. Now from the above relation, it is seen that

$$\operatorname{Re}\left(\frac{\lambda-1}{1+\lambda}\right)=\frac{-2\operatorname{Re}\left(x^H\hat{M}^T x\right)+q-s}{2r-s+q}. \tag{11}$$

We point out that by the assumptions, r, s and q are positive and $r>s$. On the other hand, straightforward computations reveal that

$$2\operatorname{Re}\left(x^H\hat{M}^T x\right)-q=2\operatorname{Re}\left(x^H\hat{M}x\right)-x^H(\hat{M}+\hat{N})x$$

$$=\operatorname{Re}(x^H A x)>0.$$

Consequently, from (11), we deduce that

$$\operatorname{Re}\left(\frac{\lambda-1}{1+\lambda}\right)<0.$$

Now the result follows immediately from the following fact

$$\operatorname{Re}\left(\frac{\lambda-1}{1+\lambda}\right)=\frac{|\lambda|^2-1}{|1+\lambda|^2}. \qquad \square$$

Remark 2.4. *As pointed earlier, Salkuyeh et al. [13] has developed the MGSS method for saddle point problems with symmetric positive definite (1,1)-block. It is pointed in Remark 2.2 that the MGSS method is a special case of the MS method. Therefore, Theorem 2.1 turns out that all of the results established by Salkuyeh et al. are valid when (1,1)-block A is nonsymmetric positive definite.*

Remark 2.5. *Assume that A is a nonsymmetric positive definite matrix. Let P and Q be two arbitrary symmetric positive definite matrices. If we set*

$$\hat{M}=\frac{1}{2}(P+A),\quad \hat{N}=\frac{1}{2}(P-A),\quad \tilde{M}=\frac{1}{2}(Q+C)\quad \text{and}\quad \tilde{N}=\frac{1}{2}(Q-C),$$

then it can be seen that all of the assumptions in Theorem 2.1 hold.

Our numerical experiments illustrate that the matrix $\mathcal{P}_{MS}$ can be served as an effective preconditioner to speed up the convergence of the Krylov subspace methods (e.g., restarted GMRES method) although its corresponding stationery iterative method may converge slowly. The same ideas are utilized in [5, 6, 7, 13] which focus on the shift-splitting types of preconditioners. To apply the preconditioner induced from the mixed splitting, in fact, we need to handle the Krylov subspace methods for solving the following linear system $\mathcal{P}_{MS}^{-1}\mathcal{A}u=\mathcal{P}_{MS}^{-1}b.$

Remark 2.6. *In view of Remark 2.5, in our numerical experiments, we set*

$$\hat{M}=\frac{1}{2}(\gamma H+A),\quad \hat{N}=\frac{1}{2}(\gamma H-A),\quad \tilde{M}=\frac{1}{2}(\beta I+C)\quad \text{and}\quad \tilde{N}=\frac{1}{2}(\beta I-C),$$

where γ and β are two given positive parameters and H is the symmetric part of A. That is, we use the following preconditioner

$$\mathcal{P}_{MS}^{*} = \frac{1}{2}\begin{pmatrix} \gamma H + A & B^T \\ -B & \beta I + C \end{pmatrix}. \tag{12}$$

The following proposition presents the eigenvalue distribution of the $(\mathcal{P}_{MS}^{*})^{-1}\mathcal{A}$.

Proposition 2.1. *Assume that* $\mathcal{P}_{MS}^{*}$ *is defined as in (12). If* A *is nonsymmetric positive definite, then the eigenvalues of* $(\mathcal{P}_{MS}^{*})^{-1}\mathcal{A}$ *belongs to the following subset of* X,

$$\mathcal{S} = \{z + iw \mid z > 0,\ z^2 + w^2 < 4 \quad \text{and} \quad w < \sqrt{2z}\}.$$

Proof. Let $\lambda = \lambda_1 + i\lambda_2$ *be an arbitrary eigenvalue of* $(\mathcal{P}_{MS}^{*})^{-1}\mathcal{A}$. *Therefore,*

$$\mathcal{A}u = \lambda \mathcal{P}_{MS}^{*} u, \tag{13}$$

so that $u = (x^H, y^H)^H$. It is not difficult to verify that $\lambda \neq 0$ and $x \neq 0$. From (13), it can be observed that

$$x^H Ax + x^H B^T y \ = \ \mu x^H Ax + \alpha\mu x^H Hx + \mu x^H B^T y,$$

and

$$-y^H Bx + y^H Cy \ = \ -\mu y^H Bx + \mu y^H Cy + \beta\mu y^H y,$$

where $\mu = \lambda / 2$. Consequently, we have

$$(1-\mu)\, x^H B^T y \ = \ (\mu - 1)\, x^H Ax + \alpha \frac{\lambda}{2} x^H Hx,$$

and

$$(\mu - 1)\, y^H Bx \ = \ (\mu - 1)\, y^H Cy + \beta\mu y^H y.$$

For simplicity, assume that $x^H Ax = p_1 + ip_2$, $q = y^H Cy$ and $r = \beta\mu y^H y$. The above two relations implies that

$$-(p_1 + ip_2) + \alpha\left(\frac{\mu - |\mu|^2}{|1-\mu|^2}\right)p_1 = q + \left(\frac{|\mu|^2 - \mu}{|1-\mu|^2}\right)r.$$

Since A is positive definite ($p_1 > 0$), it can be concluded that

$$\left(\frac{\operatorname{Re}(\mu) - |\mu|^2}{|1-\mu|^2}\right)(\alpha p_1 + r) = p_1 > 0.$$

The preceding relation shows that $\operatorname{Re}(\mu) > |\mu|^2$. In view of the fact that $|\mu|^2 = \operatorname{Re}(\mu)^2 + \operatorname{Im}(\mu)^2$, we deduce that $\operatorname{Re}(\mu) > \operatorname{Im}(\mu)^2$ and $|\mu| < 1$ which complete the proof. $\square$

3. Numerical experiments

In this section, we give an example to demonstrate the applicability of the proposed preconditioner (in the form given in Remark 2.6) and compare its performance with the shift-splitting types. All of the reported experiments were performed on a 64-bit 2.45 GHz core i7 processor and 8.00GB RAM using some matlab codes on MATLAB version 8.3.0532.

In matrices P_{GSS}, P_{DSS}, P_{MGSS}, P_{RMGSS} and P_{MS}^{*}, the corresponding parameters α, β and γ are taken to be 0.001.

Example 3.1. *Consider the Oseen equation as follows:*

$$\begin{cases} -\nu\Delta\mathbf{w} + (\mathbf{v}.\nabla)\mathbf{w} + \nabla p = \mathbf{f} \\ \nabla.\mathbf{w} = 0. \end{cases} \quad in\ \Omega \tag{14}$$

which is derived when the steady-state Navier-Stokes equation is linearized by Picard iteration. Here the vector field $\mathbf{v}$ *is the approximation of* $\mathbf{w}$ *from the previous Picard iteration, the vector* $\mathbf{w}$ *represents the velocity,* p *denotes the pressure and* Ω *is a bounded domain. The parameter* $\nu > 0$ *stands for the viscosity.*

Conservative discretizations of the Oseen problem (14) leads to a generalized saddle point system of the form (1) with nonsymmetric positive definite (1,1)-block. Our examined test problems are constructed using IFISS software package written by Elman et al. [8]. The package is used to generate discretizations of leaky lid driven cavity problem using stabilized Q1-P0 (Q2-Q1) finite elements method (FEM) to study the performance of the offered preconditioner for the case that (2,2)-block in (1) is nonzero (zero).

Four grids are exploited, 16×16, 32×32, 64×64 and 128×128 which Q2-Q1 FEM corresponds $(578,81)$, $(2178,289)$, $(8450,1089)$ and $(33282,4225)$ to (n,m), respectively and Q1-P0 FEM corresponds $(578,289)$, $(2178,1024)$, $(8450,4096)$ and $(33282,16384)$ to (n,m), respectively.

In all of the tests, we apply the GMRES(5) method such that the initial guess is taken to be zero and the iterations are terminated as soon as

$$\|b - \mathrm{A}u^{(k)}\| \le 10^{-6}\ \|b\|,$$

or if the number of iterations exceeds from $k_{\max} = 5000$ where $u^{(k)}$ denotes the k th approximate solution. The corresponding results are disclosed in Tables 1 and 2.

We end this section with a remark given as follows.

Remark 3.1. *In [13], the application of the MGSS and the relaxed MGSS (RMGSS) preconditioners only studied under the assumption that the (1,1)-block* A *is symmetric positive definite. Our numerical tests show that this* *preconditioners are also effective when A is nonsymmetrical positive definite. Here, we point out that the proposed MGSS iterative scheme is not convergence in the case that the parameter $\alpha = 0$, i.e., when P_{MGSS} reduces to P_{RMGSS}. On the hand, the RMGSS preconditioner outperforms the MGSS preconditioner which the reason can be expressed as in [13, Remark 2]. As seen, P^*_{MS} is as effective as P_{RMGSS} and this may be a motivation to study the other kind of preconditioners in the form of P_{MS} in the future works with details.*

Table 1

Numerical results for Oseen problem with ν = 0.01 (Q2-Q1 FEM)

Preconditioner	I		P_{MGSS}		P_{RMGSS}		P^*_{MS}	
Grid	IT	CPU	IT	CPU	IT	CPU	IT	CPU
16×16	215	0.143	2	0.026	1	0.036	1	0.020
32×32	533	0.918	3	0.259	1	0.121	1	0.107
64×64	1214	5.407	11	4.657	2	0.79	2	0.764
128×128	2164	48.935	82	162.299	3	5.438	3	5.438

Table 2

Numerical results for Oseen problem with ν = 0.01 (Q1-P0 FEM)

Preconditioner	I		P_{GSS}		P_{DSS}		P^*_{MS}	
Grid	IT	CPU	IT	CPU	IT	CPU	IT	CPU
16×16	179	0.131	2	0.040	1	0.036	1	0.023
32×32	369	0.486	3	0.263	1	0.114	1	0.099
64×64	628	2.960	6	3.346	2	0.806	2	0.788
128×128	1397	29.809	23	66.72	3	6.902	3	6.939

4. Conclusions

Recently, Cao et al. [Appl. Math. Lett. 49 (2015), 20-27] and Salkuyeh et al. [Appl. Math. Lett. 48 (2015), 55-61] have focused on applying shift-splitting preconditioners for saddle point problems. This paper has been concerned with employing a general class of preconditioners incorporates shift-splitting preconditioners. Our established results have revealed that the results proved by Salkuyeh et al. are valid for the case that (1,1) part of the coefficient matrix of the saddle point problem is nonsymmetric. The reported numerical results by Cao et al. (Salkuyeh et al.) shown that the proposed DSS (RMGSS) preconditioner outperforms their other handled preconditioners although the corresponding iterative method to DSS (RMGSS) preconditioner is not convergent. Our examined numerical experiments have demonstrated that a special preconditioner of the offered class is as effective as DSS and RMGSS preconditioners. Further work can be focused on studying the performance of other possible instances of the introduced class of the preconditioners with detail.

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