NUMERICAL APPROXIMATION OF HUNTER-SAXTON EQUATION BY AN EFFICIENT ACCURATE APPROACH ON LONG TIME DOMAINS

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This paper aimed to propose a combined effective technique for obtaining an approximate solution to the Hunter-Saxton equation arising in the modelling of the direct field of a nematic liquid crystal. The time-marching algorithm is based on the linearized Taylor expansion series while a collocation method based on novel Bessel polynomials is utilized for the space variable. The main advantage of this method is that, in each time step, it converts the problem into a fundamental matrix equation so that the computation is effective and straightforward. Through numerical simulations, the efficiency of the combined scheme is compared with exact solutions as well as existing available numerical models. The results of comparisons indicate that the combined method developed by a large time step and over a large time domain is an efficient approach.

Keywords: Hunter-Saxton equation, Bessel functions, Collocation points, Taylor expansion.

MSC2020: 26A33 65L60 42C05 65L05.

1. Introduction

The main goal of this work is to propose an effective approximation algorithm to solve the nonlinear Hunter-Saxton equation [6]

$$\begin{cases} w_{xt} + w\, w_{xx} + \frac{1}{2}\, w_x^2 = 0 \\ w|_{t=0} = w_0(x), \end{cases} \tag{1}$$

subjected to the boundary condition [3]

$$\lim_{x\to\infty} w(x,t) = 0. \tag{2}$$

In addition, enforcing the boundary condition $w(x,t) \to 0$ as $t \to \infty$ and using the separable solution approach give rise to a class of solutions with algebraic decay [3]

$$w(x,t) = \frac{w_0(x)}{1+\lambda t}, \tag{3}$$

where $w_0(x)$ satisfies an appropriate second-order ordinary differential equation and $\lambda > 0$. Historically, the origin of Hunter-Saxton (HS) equation dates backed to Hunter and Saxton who first suggested it as a simplified model for stuying a nematic liquid crystal [6]. Over the past decade, various analytical techniques and computational procedures have been proposed to investigate the model problem (1)-(2).

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Among analytic methods, various kinds of separable as well as self-similar solutions have been found in [3] to the equation. Rational approximate solutions using the Padé approach are obtained in [1] for the generalized HS equation. On the other hand, numerous finite difference methods with convergence results were proposed and analyzed in [7]. The development of a local discontinuous Galerkin (DG) method as well as a new dissipative DG method to solve the HS-type equations were considered in [15, 16], see also [8, 9]. The combination of a finite difference method in time and the technique of quasilinearization along with Haar wavelet basis functions in space was used in [2]. The collocation method based on cubic trigonometric B-spline and the collocation finite element approach based on quintic B-spline basis functions were developed for solving the HS equation [5, 11]. A hybrid computational approach based on the generalized Chebyshev polynomials along with quasilinearization technique was applied in [14]. In addition, two H^1-preserving Galerkin methods were presented for the HS equation in [13].

In this research work, the chief aim is to devise a new hybrid approximation technique for the nonlinear HS equation. Due to the fact that the HS equation is a time-dependent problem, developing an accurate algorithm for the time advancement is of interest. For this purpose, the Taylor approach with second-order accuracy is utilized. Hence, in each time level, we employ the novel Bessel functions in conjunction with collocation points to approximate the solution with respect to space variable. In fact, the Bessel polynomial of order l is defined explicitly as [12, 4, 10]

$$\mathbb{B}_l(x) = \sum_{k=0}^{l} \frac{1}{k!} \frac{(l+k)!}{(l-k)!} \left(\frac{x}{2}\right)^k, \quad l = 0, 1, \dots. \tag{4}$$

The main benefits of these polynomials can be summarized as follows. First, all the coefficients of these polynomials are positive integers. The second fact is about the strictly totally positiveness of any collocation matrix of these polynomials at positive points, which has recently been proved in [4].

2. Time discretization: Taylor approach

Assuming that $T > 0$ is a given final time. To discretize the Hunter-Saxton equation in time, we partition $[0, T]$ into M uniform subintervals with grid points $t_0 = 0 < t_1 = \Delta t < \ldots < t_M = M\Delta t = T$ and $\Delta t = t_n - t_{n-1}$ is the time step. For obtaining a higher-order approximation in time, the idea of Taylor expansion is employed for $v_t^n = v(x, t_n)$ to get

$$v_t^n = \frac{v^{n+1} - v^n}{\Delta t} - \frac{\Delta t}{2} v_{tt}^n + O(\Delta t^2). \tag{5}$$

Using $v = w_x$ in (5) we obtain

$$w_{xt}^n = \frac{w_x^{n+1} - w_x^n}{\Delta t} - \frac{\Delta t}{2} w_{xtt}^n + O(\Delta t^2). \tag{6}$$

Differentiation of both sides (1) with respect to time yields

$$w_{xtt}^n = -(w^n\, w_{xx}^n + \frac{1}{2}\,(w_x^n)^2)_t = -w_t^n\, w_{xx}^n - w^n\,(w_t^n)_{xx} - w_x^n\,(w_t^n)_x.$$

Replacing the first order derivatives w_t^n by the aid of forward difference approximation $(w^{n+1} - w^n)/\Delta t$, we may write w_{xtt}^n as

$$\Delta t\, w_{xtt}^n = 2w^n\, w_{xx}^n - w_{xx}^n\, w^{n+1} - w^n\, w_{xx}^{n+1} + (w_x^n)^2 - w_x^n\, w_x^{n+1}. \tag{7}$$

Next, we insert (7) into (6) and then equating to $w_{xt}^n = -w^n\, w_{xx}^n - \frac{1}{2}\,(w_x^n)^2$. After rearranging the terms, the following time discretized equation for (1) is obtained

$$\Delta t\, w_{xx}^n\, w^{n+1} + (2 + \Delta t\, w_x^n)\, w_x^{n+1} + \Delta t\, w^n\, w_{xx}^{n+1} = 2w_x^n, \quad n = 0, 1, \ldots. \tag{8}$$

According to (1), the initial condition becomes $w^0 = w_0(x)$. Furthermore, the boundary conditions, which obtained from (3) at $x = 0, 1$ are expressed as

$$w^{n+1}(0) = b_0^{n+1} := \frac{w_0(0)}{1 + \lambda t_{n+1}}, \quad w^{n+1}(1) = b_1^{n+1} := \frac{w_0(1)}{1 + \lambda t_{n+1}}, \quad n = 0, 1, \ldots. \tag{9}$$

Similarly, we may impose the following initial conditions

$$w^{n+1}(0) = b_0^{n+1} := \frac{w_0(0)}{1 + \lambda t_{n+1}}, \quad \frac{d}{dx} w^{n+1}(0) = b_1^{n+1} := \frac{w_0'(0)}{1 + \lambda t_{n+1}}, \quad n = 0, 1, \ldots. \tag{10}$$

3. **Bessel functions: Basic matrix relations**

After discretizing the Hunter-Saxton equation in time by using (8), our next aim is to approximate the solution in space variable. For this purpose, we approximate w^{n+1} in terms of $\mathbb{B}_l(x)$. Obviously, for $n = 0$ the value w^0 is known from the initial condition $w_0(x)$. Assuming that the approximate solution $\mathcal{W}_{n,N}(x)$ of w^n is at hand in the time level t_n, we seek $\mathcal{W}_{n+1,N}(x)$ at the next time level t_{n+1}, $n = 0, 1, \ldots, M$ in the form

$$\mathcal{W}_{n+1,N}(x) = \sum_{l=0}^{N} a_{l,n}\, \mathbb{B}_l(x), \quad x \in [0, 1], \tag{11}$$

where the unknown coefficients $a_{l,n}$, $l = 0, 1, \ldots, N$ to be determined. We can rewrite the finite series (11) in a matrix form compactly as

$$\mathcal{W}_{n+1,N}(x) = \boldsymbol{B}_N(x)\, \boldsymbol{A}_{n,N}, \tag{12}$$

where the unknown vector $\boldsymbol{A}_{n,N}$ and known vector $\boldsymbol{B}_N(x)$ are defined as

$$\boldsymbol{A}_{n,N} = \begin{bmatrix} a_{0,n} & a_{1,n} & \ldots & a_{n,N} \end{bmatrix}^T, \quad \boldsymbol{B}_N(x) = \begin{bmatrix} \mathbb{B}_0(x) & \mathbb{B}_1(x) & \ldots & \mathbb{B}_N(x) \end{bmatrix}.$$

Moreover, we can write $\boldsymbol{B}_N(x)$ in the matrix representation as follows

$$\boldsymbol{B}_N(x) = \boldsymbol{X}_N(x)\, \boldsymbol{D}^T, \tag{13}$$

where

$$\boldsymbol{X}_N(x) = \begin{bmatrix} 1 & x & x^2 & \ldots & x^N \end{bmatrix},$$

and the lower triangular matrix $\boldsymbol{D}$ of size $(N+1)\times(N+1)$ takes the form

$$\boldsymbol{D} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ 1 & 3 & 3 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \dfrac{2^{-1}\,N!}{(N-2)!\,1!} & \dfrac{2^{-2}\,(N+1)!}{(N-3)!\,2!} & \dots & \dfrac{2^{1-N}\,(2N-2)!}{0!\,(N-1)!} & 0 \\ 1 & \dfrac{2^{-1}\,(N+1)!}{(N-1)!\,1!} & \dfrac{2^{-2}\,(N+2)!}{(N-2)!\,2!} & \dots & \dfrac{2^{1-N}\,(2N-1)!}{1!\,(N-1)!} & \dfrac{2^{-N}\,(2N)!}{0!\,N!} \end{bmatrix}.$$

On the other hand, an easy calculation shows that the following relationship between $\boldsymbol{X}_N(x)$ and its first derivative holds

$$\frac{d}{dx}\boldsymbol{X}_N(x) = \boldsymbol{X}_N(x)\,\boldsymbol{M}^T, \quad \boldsymbol{M}^T = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & N \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{(N+1)\times(N+1)}. \tag{14}$$

Ultimately, to obtaining an approximate solution of the form (11) for the solution of equation (8), the following collocation points are used on $0 \leqslant x \leqslant 1$,

$$x_i = \frac{i}{N}, \quad i = 0, 1, \dots, N. \tag{15}$$

4. Taylor-Bessel collocation method

Now, we are able to complete the process of finding an approximate solution of the form (11) for the discretized model (8). To this end, we represent the unknown w^{n+1}, w_x^{n+1}, and w_{xx}^{n+1} in (8) in the matrix representation forms and then use the collocation points (15) to find the unknown coefficients in (11).

To proceed, we combine the relation (12) and (13) to express (11) in the matrix form

$$\mathcal{W}_{n+1,N}(x) = \boldsymbol{X}_N(x)\,\boldsymbol{D}^T\,\boldsymbol{A}_{n,N}. \tag{16}$$

With the help of collocation points (15) and replacing them into the relation (16), we get

$$\boldsymbol{W}_{n+1} = \boldsymbol{Y}\,\boldsymbol{D}^T\,\boldsymbol{A}_{n,N}, \qquad \boldsymbol{W}_{n+1} = \begin{bmatrix} \mathcal{W}_{n+1,N}(x_0) \\ \mathcal{W}_{n+1,N}(x_1) \\ \vdots \\ \mathcal{W}_{n+1,N}(x_N) \end{bmatrix}, \quad \boldsymbol{Y} = \begin{bmatrix} \boldsymbol{X}_N(x_0) \\ \boldsymbol{X}_N(x_1) \\ \vdots \\ \boldsymbol{X}_N(x_N) \end{bmatrix}, \tag{17}$$

With the aid of relations (16) and (14) we can represent the first and second-order derivatives involved in (8) in the matrix forms

$$\begin{cases} w_x^{n+1} \approx \mathcal{W}^{(1)}_{n+1,N}(x) = \boldsymbol{X}_N(x)\,\boldsymbol{M}^T\,\boldsymbol{D}^T\,\boldsymbol{A}_{n,N}, \\ w_{xx}^{n+1} \approx \mathcal{W}^{(2)}_{n+1,N}(x) = \boldsymbol{X}_N(x)\,(\boldsymbol{M}^T)^2\,\boldsymbol{D}^T\,\boldsymbol{A}_{n,N}. \end{cases} \tag{18}$$

Analogously, by exploiting the collocation points, the first and second derivatives in (18) can be expressed as

$$\dot{\boldsymbol{W}}_{n+1} = \boldsymbol{Y}\,\boldsymbol{M}^T\,\boldsymbol{D}^T\,\boldsymbol{A}_{n,N}, \quad \ddot{\boldsymbol{W}}_{n+1} = \boldsymbol{Y}\,(\boldsymbol{M}^T)^2\,\boldsymbol{D}^T\,\boldsymbol{A}_{n,N}, \tag{19}$$

where

$$\dot{\boldsymbol{W}}_{n+1} = \begin{bmatrix} \mathcal{W}^{(1)}_{n+1,N}(x_0) \\ \mathcal{W}^{(1)}_{n+1,N}(x_1) \\ \vdots \\ \mathcal{W}^{(1)}_{n+1,N}(x_N) \end{bmatrix}, \qquad \ddot{\boldsymbol{W}}_{n+1} = \begin{bmatrix} \mathcal{W}^{(2)}_{n+1,N}(x_0) \\ \mathcal{W}^{(2)}_{n+1,N}(x_1) \\ \vdots \\ \mathcal{W}^{(2)}_{n+1,N}(x_N) \end{bmatrix}.$$

According to (8), we introduce

$$s_{n,0}(x) = \Delta t\, w_{xx}^n, \quad s_{n,1}(x) = 2 + \Delta t\, w_x^n, \quad s_{n,2}(x) = \Delta t\, w^n, \quad f_n(x) = 2w_x^n.$$

Utilizing the approximations $\mathcal{W}_{n+1,N}(x), \mathcal{W}^{(1)}_{n+1,N}(x), \mathcal{W}^{(2)}_{n+1,N}(x)$ we may rewrite (8) as

$$s_{n,2}(x)\,\mathcal{W}^{(2)}_{n+1,N}(x) + s_{n,1}(x)\,\mathcal{W}^{(1)}_{n+1,N}(x) + s_{n,0}(x)\,\mathcal{W}_{n+1,N}(x) = f_n(x), \quad 0 \leqslant x \leqslant 1. \tag{20}$$

By substituting the collocation points into (20) to get the system

$$\boldsymbol{S}_{n,2}\,\ddot{\boldsymbol{W}}_{n+1} + \boldsymbol{S}_{n,1}\,\dot{\boldsymbol{W}}_{n+1} + \boldsymbol{S}_{n,0}\,\boldsymbol{W}_{n+1} = \boldsymbol{F}_n. \tag{21}$$

In (21), the matrices $\boldsymbol{S}_{n,l}$, and the vector $\boldsymbol{F}_n$ take the forms

$$\boldsymbol{S}_{n,l} = \begin{bmatrix} s_{n,l}(x_0) & 0 & \dots & 0 \\ 0 & s_{n,l}(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s_{n,l}(x_N) \end{bmatrix}_{(N+1)\times(N+1)}, \quad \boldsymbol{F}_n = \begin{bmatrix} f_n(x_0) \\ f_n(x_1) \\ \vdots \\ f_n(x_N) \end{bmatrix}_{(N+1)\times 1},$$

for $l = 0, 1, 2$. Let us put the relations (17) and (19) into (21). This yields the fundamental matrix equation

$$\boldsymbol{U}_n\,\boldsymbol{A}_{n,N} = \boldsymbol{F}_n, \qquad \text{or} \qquad [\boldsymbol{U}_n; \boldsymbol{F}_n], \tag{22}$$

where

$$\boldsymbol{U}_n := \left\{\boldsymbol{S}_{n,2}\,\boldsymbol{Y}\,(\boldsymbol{M}^T)^2 + \boldsymbol{S}_{n,1}\,\boldsymbol{Y}\,\boldsymbol{M}^T + \boldsymbol{S}_{n,0}\,\boldsymbol{Y}\right\}\,\boldsymbol{D}^T.$$

Clearly, the fundamental matrix equation (22) is a set of $(N+1)$ linear equations in terms of $(N+1)$ unknown coefficients $a_{0,n}, a_{1,n}, \dots, a_{N,n}$ to be found.

In order to implement the boundary conditions (9), we must also convert them into a matrix form. Based on the representation (16), these conditions i.e., $\mathcal{W}_{n+1,N}(0) = b_0^{n+1}$ and $\mathcal{W}_{n+1,N}(1) = b_1^{n+1}$ can be expressed in the matrix notations

$$\bar{\boldsymbol{U}}_{n,0}\,\boldsymbol{A}_{n,N} = b_0^{n+1}, \qquad \bar{\boldsymbol{U}}_{n,0} := \boldsymbol{X}_N(0)\,\boldsymbol{D}^T = [\bar{u}_{0,0} \quad \bar{u}_{0,1} \quad \dots \quad \bar{u}_{0,N}],$$

$$\bar{\boldsymbol{U}}_{n,1}\,\boldsymbol{A}_{n,N} = b_1^{n+1}, \qquad \bar{\boldsymbol{U}}_{n,1} := \boldsymbol{X}_N(1)\,\boldsymbol{D}^T = [\bar{u}_{1,0} \quad \bar{u}_{1,1} \quad \dots \quad \bar{u}_{1,N}].$$

In the case of the initial conditions (10), we need to replace the vector $\bar{\boldsymbol{U}}_{n,1}$ with $\boldsymbol{X}_N(0)\,\boldsymbol{M}^T\,\boldsymbol{D}^T$. Next, we replace the first and last rows of the augmented matrix $[\boldsymbol{U}_n;\boldsymbol{F}_n]$ by the vectors $[\bar{\boldsymbol{U}}_{n,0};b_0^{n+1}]$ and $[\bar{\boldsymbol{U}}_{n,1};b_1^{n+1}]$. Thus, the following modified linear system of equations is obtained

$$[\bar{\boldsymbol{U}}_n;\bar{\boldsymbol{F}}_n]=\begin{bmatrix} \bar{u}_{0,0} & \bar{u}_{0,1} & \bar{u}_{0,2} & \bar{u}_{0,3} & \cdots & \bar{u}_{0,N} & ; & b_0^{n+1} \\ u_{1,0} & u_{1,1} & u_{1,2} & u_{1,3} & \cdots & u_{1,N} & ; & f_n(x_1) \\ u_{2,0} & u_{2,1} & u_{2,2} & u_{2,3} & \cdots & u_{2,N} & ; & f_n(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & ; & \vdots \\ u_{N-1,0} & u_{N-1,1} & u_{N-1,2} & u_{N-1,3} & \cdots & u_{N-1,N} & ; & f_n(x_{N-1}) \\ \bar{u}_{1,0} & \bar{u}_{1,1} & \bar{u}_{1,2} & \bar{u}_{1,3} & \cdots & \bar{u}_{1,N} & ; & b_1^{n+1} \end{bmatrix}. \tag{23}$$

Now, by solving the above linear system we are able to obtain the unknown Bessel coefficients in (16).

5. Numerical experiments

Our aim is to demonstrate the advantages of the proposed combined Taylor and Bessel-collocation method by simulating a test problem for the nonlinear initial and boundary value problem (1)-(3). The numerical models and calculations are verified through a comparison with existing computational schemes and experimental measurements. We use MATLAB 2017a for simulation analysis.

We consider $w_0(x)=2x$ and $\lambda=1$ in (3) to get following exact solution [2, 14, 5, 11]

$$w(x,t)=\frac{2x}{1+t}.$$

We first consider $\Delta t=1$, $T=10$, and then looking for the approximate solutions in the form $\mathcal{W}_{n+1,N}(x)=\sum_{l=0}^{N}a_{l,n}\,\mathbb{B}_l(x)$ for $n=0,1,\ldots,10$. Using the proposed method with $N=1$, the following approximative solutions for $0\leqslant x\leqslant 1$ are obtained

$\mathcal{W}_{1,1}(x)=1.0\,x$, $\quad\mathcal{W}_{2,1}(x)=0.66666666666666666667\,x$,
$\mathcal{W}_{3,1}(x)=0.5\,x$, $\quad\mathcal{W}_{4,1}(x)=0.4\,x$,
$\mathcal{W}_{5,1}(x)=0.33333333333333333333\,x$, $\quad\mathcal{W}_{6,1}(x)=0.28571428571428571429\,x$,
$\mathcal{W}_{7,1}(x)=0.25\,x$, $\quad\mathcal{W}_{8,1}(x)=0.22222222222222222222\,x$,
$\mathcal{W}_{9,1}(x)=0.2\,x$, $\quad\mathcal{W}_{10,1}(x)=0.18181818181818181818\,x$,

which are well matched to the exact solutions at $t=1,2,\ldots,10$ respectively. Graphs of approximation solutions $\mathcal{W}_{n+1,2}(x)$ for $n=0,1,\ldots,9$ using $N=2$ are visualized in Fig. 1. In order to validate our results, we make a comparison in the next experiments. Table 1 shows the numerical solutions obtained by the proposed scheme evaluated at $t=0.1$ and various $x\in[0,1]$. Note the same results are obtained with either $N=1,2$ or $N=3$. The corresponding absolute errors are also reported in Table 1. Furthermore, analogue results of the previously well-established methods are given in Table 1. These include the Haar wavelet quasilinearization method (HWQM) [2], the cubic trigonometric B-spline (CTBS) collocation method [5], the quintic B-spline (QBS) [11] collocation finite element approach, and a collocation method based on bi-variate (generalized) fractional-order of the Chebyshev functions

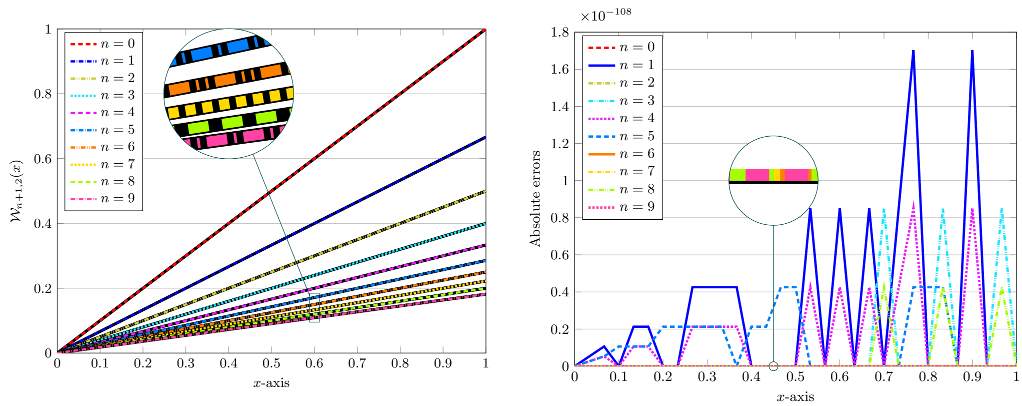

FIGURE 1. Comparison of numerical model results with the exact solutions (left) and the corresponding absolute errors at $t = s\Delta t, s = 1, 2, \ldots, 10$ (right) for $\Delta t = 1, N = 2$, and $T = 10$.

(B-GFCF) [14]. It can be observed that our numerical results are in excellent alignment with the corresponding exact solutions while needing significantly less computational efforts compared to other existing methods. We next consider $T = 10$,

TABLE 1. Results of the absolute errors for $N = 1, 2, 3$ and various $x \in [0, 1]$ at time $t = 0.1$.

	Present ($N = 1, 2, 3$)		HWQA [2]	CTBS [5]	QBS [11]	B-GFCF [14]
$\frac{x}{128}$	Numerical	A.E.	256 nodes	128 nodes	128 nodes	225 nodes
1	0.01420454545454545454545	0	1.0_{-11}	5.087110_{-8}	6.384110_{-6}	7.75_{-16}
3	0.04261363636363636363636	0	5.0_{-11}	5.841630_{-8}	1.400953_{-6}	2.93_{-15}
5	0.07102272727272727272727	0	7.0_{-11}	7.164139_{-8}	5.637072_{-7}	1.45_{-15}
7	0.09943181818181818181818	0	1.0_{-10}	6.602419_{-8}	4.695398_{-6}	1.10_{-14}
9	0.1278409090909090909091	0	1.0_{-10}	6.333948_{-8}	5.169489_{-6}	1.82_{-14}
59	0.8380681818181818181818	0	1.1_{-09}	5.930189_{-8}	3.997275_{-6}	9.20_{-14}
61	0.8664772727272727272727	0	2.0_{-10}	5.862473_{-8}	3.415639_{-6}	8.73_{-14}
63	0.8948863636363636363636	0	4.0_{-10}	5.790172_{-8}	2.199316_{-6}	7.89_{-14}
65	0.9232954545454545454545	0	1.0_{-09}	5.713122_{-8}	1.356332_{-6}	6.86_{-14}
67	0.9517045454545454545455	0	1.0_{-09}	5.531186_{-8}	4.494994_{-6}	5.86_{-14}
69	0.9801136363636363636364	0	2.0_{-09}	5.544210_{-8}	4.728788_{-6}	5.09_{-14}
119	1.690340909090909090909	0	4.0_{-09}	1.291484_{-8}	1.341862_{-7}	1.26_{-13}
121	1.718750000000000000000	0	4.0_{-09}	1.020638_{-8}	5.446600_{-9}	1.20_{-13}
123	1.747159090909090909091	0	4.0_{-09}	7.406883_{-9}	1.386507_{-6}	1.31_{-13}
125	1.775568181818181818182	0	4.0_{-09}	4.514845_{-9}	2.113729_{-6}	1.52_{-13}
127	1.803977272727272727273	0	4.0_{-09}	1.528773_{-9}	1.522962_{-6}	1.51_{-13}

$\Delta t = 0.05$, and $N = 2$ for computations obtained over long-time periods. The snapshots of numerical solutions at different time instants $t = s\Delta t$, $s = 1, 2, \ldots, 200$ are shown in Fig. 2. The corresponding approximated solutions for $t = \Delta t$ and $t = T$

are obtained as follows

$$\mathcal{W}_{1,2}(x) = 1.90476190476190476 19\,x,$$
$$\mathcal{W}_{200,2}(x) = -8.5159\times 10^{-108}\,x^2 + 0.181818181818181818\,x + 1.4193\times 10^{-109}.$$

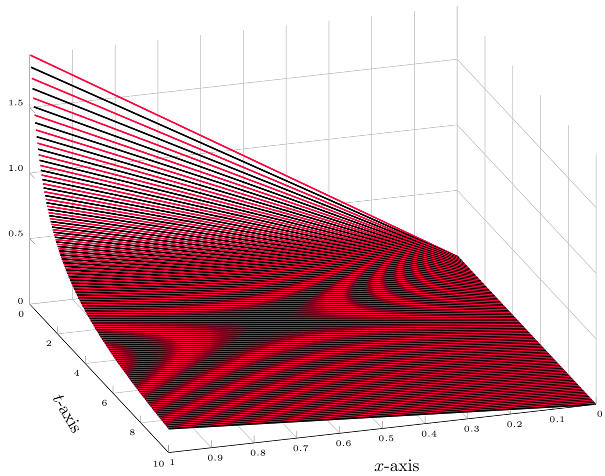

FIGURE 2. Numerical solutions at different time instants $t = s\Delta t, s = 1, 2, \dots, 200$ with $\Delta t = 0.05$, $T = 10$, and $N = 2$.

In Table 2, we calculate the maximum absolute errors denotes by L_∞ as well as the L_2 error norms evaluated at the final time $t = T$ via

$$L_\infty := \max_{0\leqslant x\leqslant 1} |w(x,T) - \mathcal{W}_{M+1,N}(x)|,\ \ L_2 := \Big(\frac{\int_0^1 [w(x,T) - \mathcal{W}_{M+1,N}(x)]^2 dx}{N+1}\Big)^{\frac{1}{2}}.$$

We use different $N = 1, 2, \dots, 5$ in Table 2. Also various final times $T = 50, 100, 500$, and $T = 1000$ are utilized with the step size $\Delta t = 1$. As seen in Table 2, the achievement of excellent approximations to the exact solutions are possible using only a few terms of Bessel polynomials.

TABLE 2. Comparison of L_∞ and L_2 error norms for various $N = 1, 2, \dots, 5$ and $\Delta t = 1$ on large time domain evaluated at the final times $t = T$, with $T = 50, 100, 500, 1000$.

	$T = 50$		$T = 100$		$T = 500$		$T = 1000$	
N	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2
1	0	0	5.322_{-110}	0	6.653_{-111}	0	0	0
2	5.322_{-110}	0	5.322_{-110}	0	6.653_{-111}	0	0	0
3	1.064_{-109}	0	5.322_{-110}	0	6.653_{-111}	0	0	0
4	2.129_{-109}	0	1.064_{-109}	0	1.331_{-110}	0	9.980_{-111}	0
5	1.064_{-109}	0	1.064_{-109}	0	1.331_{-110}	0	6.653_{-111}	0

Finally, the results presented in Table 2 indicate that the corresponding approximated solutions for $N = 2$, $\Delta t = 1$, and $T = 50, 100, 500, 1000$ are

$$\mathcal{W}_{50,2}(x) = 1.06448996000203768 \times 10^{-109}\, x^2 + 0.0392156862745098039 22\, x,$$
$$\mathcal{W}_{100,2}(x) = -5.3224498000101883999 \times 10^{-110}\, x^2 + 0.0198019801980198019 8\, x,$$
$$\mathcal{W}_{500,2}(x) = 0.00399201596806387225 55\, x,$$
$$\mathcal{W}_{1000,2}(x) = 0.00199800199800199800 2\, x.$$

The exact solutions for these values of T are respectively

$$w(x,50) = \frac{2x}{51}, \quad w(x,100) = \frac{2x}{101}, \quad w(x,500) = \frac{2x}{501}, \quad w(x,1000) = \frac{2x}{1001}.$$

The exact solutions indicate that our developed combined approach gives highly convincing accuracy level to simulate the HS equation on large time domains especially under a large time step.

6. **Conclusions**

A practical matrix approach based on novel Bessel polynomials was presented to solve the Hunter-Saxton equation. For the time discretization, the popular Taylor expansion method with second-order accuracy was used. Hence, a collocation approach based on Bessel polynomials is applied to approximate the space variable in each time level. With the aid of the matrix representations of these polynomials and the collocation points, the scheme transformed the model problem into a system of algebraic linear equations. The efficiency of the proposed technique has been assessed by means of numerical experiments. Comparisons with available well-established numerical simulations and experimental measurements have also been made. Based on the experiments, it was found that the numerical approximations were in an excellent agreement, which demonstrated the reliable efficiency and the great potential of the presented technique for the Hunter-Saxton equation even under a large time step on long-time computations.

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