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ON THE DUALITY OF BANACH FRAMES

Seyedeh Sara KARIMIZAD[1], Mohammad Sadegh ASGARI[2*]

A basic problem of interest in connection with the study of frames in Banach spaces is that of characterizing those Bessel sequences which can essentially be regarded as dual Banach frames. Dual Banach frames are Bessel sequences that have basis-like properties but which need not be bases. In this paper, we study this problem using the notion of dual and generalized dual for Bessel sequences with respect to a BK-space. We prove that duals and generalized duals of Banach frames are stable under small perturbations so that the perturbations results obtained in [5] is a special case of it. For generalized dual Banach frames constructed via perturbation theory, we provide a bound on the deviation from perfect reconstruction.

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1. Introduction

Frames with respect to a BK-space of scalar-valued sequences were extended firstly by Grochenig [8] in Banach spaces. Then ℓ^p-frames were introduced by Aldroubi et al. [1] and Christensen et al. [6] as a tool to obtain series expansions in shift-invariant spaces. Dual Banach frames are Bessel sequences that have basis-like properties but which need not be bases. In particular, they allow elements of a Banach space to be written as linear combinations of the Banach frame elements. Unfortunately, it is usually complicated to calculate a dual Banach frame explicitly. Hence we seek methods for constructing generalized duals. Approximate dual and pseudo-dual frames in Hilbert spaces are defined by Christensen in [7]. The main subject of this paper deals with the concepts of pseudo-dual and approximate dual Banach frames and examines their properties. We also investigate using of perturbation theory to construct pseudo-dual and approximate dual Banach frames.

The paper is organized as follows: in the rest of this Section, we will briefly recall the definitions and basic properties of Banach frames and bases that for more informations, we refer to [2, 3, 4, 9]. In Section 2, we discuss dual Banach

[1] Department of Mathematics, Faculty of Science, Islamic Azad University, Central Tehran Branch, Tehran, Iran, e-mail: s_ karimizad@yahoo.com
[2] Department of Mathematics, Faculty of Science, Islamic Azad University, Central Tehran Branch, Tehran, Iran, e-mail: moh.asgari@iauctb.ac.ir

frames and find some characterizations about them. In Section 3, we introduce the generalized duals for Banach frames and examine their properties and we show that these concepts are stable under small perturbations. In Section 4, we show that generalized dual Banach frames are stable under small perturbations of the Banach frame elements.

Throughout the paper X will be a separable Banach space and I is a countable index set that has been well-ordered. We shall denote by $\{I_n\}_{n=1}^{\infty}$ the family of subsets of the first n indices in I. If $|I| < \infty$ then $I_n = I$ for $n \geq |I|$.

Definition 1.1. Let X_d be a family of scalar-valued sequences indexed by I. Equip X_d with pointwise addition and scalar multiplication and let the coordinate functionals $\{\pi_i\}_{i\in I}$ on X_d be defined by

$$\pi_i(\{c_k\}_{k\in I}) = c_i, \quad \forall i \in I.$$

Then X_d with a norm $\|\cdot\|_{X_d}$ is called a BK-space, if $(X_d, \|\cdot\|_{X_d})$ is a Banach space and π_i is continuous operator from X_d to $\mathbb{C}$ for every $i \in I$. We call X_d solid if whenever $\{a_i\}_{i\in I}$ and $\{c_i\}_{i\in I}$ are sequences with $\{c_i\}_{i\in I} \in X_d$ and $|a_i| \leq |c_i|$, then it follows that $\{a_i\}_{i\in I} \in X_d$ and $\|\{a_i\}_{i\in I}\|_{X_d} \leq \|\{c_i\}_{i\in I}\|_{X_d}$. Moreover, the dual space X_d^* of X_d is also a BK-space of sequences $d = \{d_i\}_{i\in I}$ such that $d_i \in \mathbb{C}$ and

$$d(c) = \sum_{i\in I} c_i d_i \qquad \forall\, c \in X_d.$$

We note that $\ell^p(I)$ is a solid BK-space and if X_d is a solid BK-space such that there exists some $\{c_i\}_{i\in I} \in X_d$ with $c_i \neq 0$ for each $i \in I$, then every canonical unit vector $e_i = \{\delta_{ij}\}_{j\in I}$ is in X_d. We shall also require that the canonical unit vectors $\{e_i\}_{i\in I}$ form a Schauder basis for X_d. Moreover, if the series $\sum_{i\in I} c_i d_i$ is convergent for every $c \in X_d$ then $d \in X_d^*$ and if the above series converges for all $d \in X_d^*$, then $c \in X_d$.

Definition 1.2. A family $\{f_i\}_{i\in I} \subseteq X^*$ is a X_d-Bessel sequence for X if $\{f_i(x)\}_{i\in I} \in X_d$ for all $x \in X$; it is called a X_d-frame for X if it is a X_d-Bessel sequence and there exist $0 < A \leq B < \infty$ such that

$$A\|x\|_X \leq \|\{f_i(x)\}_{i\in I}\|_{X_d} \leq B\|x\|_X \qquad (1)$$

The constants A and B are called a lower and upper frame bound for X_d-frame. A X_d-frame $\{f_i\}_{i\in I}$ is called a Banach frame for X with respect to X_d if there exists a bounded linear operator $S_l: X_d \longrightarrow X$ such that $S_l(\{f_i(x)\}_{i\in I}) = x$ for all $x \in X$.

Definition 1.3. Let $\{f_i\}_{i\in I}$ be a X_d-frame for X. Then the X_d-frame condition implies that the coefficient mapping

$$U: X \to X_d, \qquad Ux = \{f_i(x)\}_{i\in I} \qquad \forall x \in X, \qquad (2)$$

is an isomorphism. The mapping U is called the *analysis* operator of $\{f_i\}_{i\in I}$. Also if $\{f_i\}_{i\in I}$ is a Banach frame for X with respect to X_d, then the extra condition in Definition 1.2 means that S_l is a left-inverse of analysis operator U, and thus US_l is a bounded linear projection of X_d onto the range $\mathcal{R}_U$. The mapping S_l is called the reconstruction operator of Banach frame and the optimal frame bounds are $\| S_l \|^{-1}$, $\|U\|$.

By replacing X_d by X_d^* and X by X^*, we can define a X_d^*-frame for X^* as an indexed set of elements from X as follows:

Definition 1.4. A sequence $\{x_i\}_{i\in I} \subseteq X$ is a X_d^*-Bessel sequence for X^* if $\{f(x_i)\}_{i\in I} \in X_d^*$ for all $f \in X^*$; it is called a X_d^*-frame for X^* with frame bounds $0 < A \leq B < \infty$ if $\{f(x_i)\}_{i\in I} \in X_d^*$ and

$$A\|f\|_{X^*} \leq \|\{f(x_i)\}_{i\in I}\|_{X_d^*} \leq B\|f\|_{X^*} \quad \text{for all } f \in X^*. \qquad (3)$$

The family $\{x_i\}_{i\in I}$ is called a Banach frame for X^* with respect to X_d^*, if it is a X_d^*-frame for X^* and there exists a bounded linear operator $S_r: X \longrightarrow X_d$ such that $S_r^*(\{f(x_i)\}_{i\in I}) = f$ for all $f \in X^*$.

Proposition 1.5. Let $\{x_i\}_{i\in I} \subseteq X$ and $\{f_i\}_{i\in I} \subseteq X^*$. Then

(i) $\{x_i\}_{i\in I}$ is a X_d^*-Bessel sequence for X^* with Bessel bound B, if and only if $\sum_{i\in I} c_i x_i$ converges in X for all $c \in X_d$ and $\|\sum_{i\in I} c_i\, x_i\|_X \leq B\|c\|_{X_d}$.

(ii) $\{f_i\}_{i\in I}$ is a X_d-Bessel sequence for X with Bessel bound B, if and only if $\sum_{i\in I} d_i f_i$ converges in X^* for all $d \in X_d^*$ and $\|\sum_{i\in I} d_i\, f_i\|_{X^*} \leq B\|d\|_{X_d^*}$.

Proof. (i) Suppose that $\{x_i\}_{i\in I}$ is a X_d^*-frame for X^* with Bessel bound B. Let $c \in X_d$, then for every $m > n$ we have

$$\begin{aligned}\left\|\sum_{i\in I_m - I_n} c_i x_i\right\|_X &= \sup_{\substack{\|f\|_{X^*}\leq 1\\ f\in X^*}} \left|f\left(\sum_{i\in I_m - I_n} c_i x_i\right)\right| = \sup_{\substack{\|f\|_{X^*}\leq 1\\ f\in X^*}} \left|\sum_{i\in I_m - I_n} c_i f(x_i)\right| \\ &\leq \sup_{\substack{\|f\|_{X^*}\leq 1\\ f\in X^*}} \|\{f(x_i)\}_{i\in I}\|_{X_d^*} \left\|\sum_{i\in I_m - I_n} c_i e_i\right\|_{X_d} \\ &\leq B\left\|\sum_{i\in I_m - I_n} c_i e_i\right\|_{X_d}.\end{aligned}$$

Since $\{e_i\}_{i\in I}$ form a Schauder basis for X_d^*, hence $\left\|\sum_{i\in I_m - I_n} c_i e_i\right\|_{X_d}$ goes to zero as m, n tend to infinity. This shows that $\sum_{i\in I} c_i x_i$ converges in X and inequality holds. For the converse, assume that $\sum_{i\in I} c_i x_i$ converges in X for all $c \in X_d$ and the inequality satisfied. Then for every $f \in X^*$ we obtain

$$\sum_{i\in I} c_i f(x_i) = \lim_{n\to\infty} \sum_{i\in I_n} c_i f(x_i) = f\left(\lim_{n\to\infty} \sum_{i\in I_n} c_i x_i\right) = f\left(\sum_{i\in I} c_i x_i\right).$$

By our requirements on X_d and X_d^*, $\{f(x_i)\}_{i\in I} \in X_d^*$ and we have

$$\begin{aligned}|\{f(x_i)\}_{i\in I}(c)| &= \left|\sum_{i\in I} c_i\, f(x_i)\right| = \left|f\left(\sum_{i\in I} c_i\, x_i\right)\right| \\ &\leq \|f\|_{X^*}\left\|\sum_{i\in I} c_i\, x_i\right\|_X \leq B\|f\|_{X^*}\|c\|_{X_d},\end{aligned}$$

which implies that $\{x_i\}_{i\in I}$ is a X_d^*-Bessel sequence for X^* with Bessel bound B. The implication (ii) can be proved in a similar way.■

Definition 1.6. Let $\{x_i\}_{i\in I} \subseteq X$ be a X_d^*-Bessel sequence for X^*. Then Proposition 1.5 implies that the mapping

$$V: X_d \to X, \qquad Vc = \sum_{i\in I} c_i x_i \quad \forall c \in X_d, \qquad (4)$$

is a bounded operator. The mapping V is called the *synthesis* operator of $\{x_i\}_{i\in I}$. Also if $\{x_i\}_{i\in I}$ is a Banach frame for X^* with respect to X_d^*, then the extra condition in Definition 1.4 implies that S_r is a right-inverse of synthesis operator V, and thus $S_r V$ is a bounded linear projection of X_d onto the range $\mathcal{R}_{S_r}$. The mapping S_r is called the reconstruction operator of Banach frame and the optimal frame bounds are $\| S_r \|^{-1}$, $\|V\|$.

A family $\{f_i\}_{i\in I} \subseteq X^*$ is called total in X^* if $f_i(x) = 0$ for all $i \in I$, then $x = 0$. Similarly, $\{x_i\}_{i\in I}$ is total in X, if $f(x_i) = 0$, for all $i \in I$ then $f = 0$.

Definition 1.7. Let $\{x_i\}_{i\in I}$ be a sequence in X. Then

(i) $\{x_i\}_{i\in I}$ is a Schauder basis for X if for every $x \in X$ there exists an unique sequence of scalars $\{c_i\}_{i\in I}$ which is called the coordinates of x, such that $x = \sum_{i\in I} c_i x_i$.

(ii) $\{x_i\}_{i\in I}$ is a X_d-Riesz basis for X if it is a total set in X and there exist two positive constants $0 < A < B < \infty$ such that

$$A\|c\|_{X_d} \leq \|\sum_{i\in I} c_i x_i\|_X \leq B\|c\|_{X_d} \quad \forall\, c \in X_d. \tag{5}$$

The constants A and B are called the lower and upper Riesz bounds, respectively.

The following example shows that in a Hilbert space, there is a Banach frame with respect to a BK-space which is not a Banach frame with respect to $\ell^2(\mathbb{N})$.

Example 1.8. Suppose that $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for a separable Hilbert space $\mathcal{H}$. Consider the family $\{e_i + e_{i+1}\}_{i=1}^{\infty}$ which is a complete set in $\mathcal{H}$. By [2, Lemma 2.6] this family is a Banach frame with respect to the BK-space

$$X_d = \{\{< h, e_i + e_{i+1} >\}_{i=1}^{\infty} |\ h \in \mathcal{H}\},$$

but not a Banach frame for $\mathcal{H}$ with respect to $\ell^2(\mathbb{N})$.

Remark 1.9. Let $\{x_i\}_{i\in I}$ be a Schauder basis for X, then the linear functionals $\{f_i\}_{i\in I} \subseteq X^*$ defined by $f_i(\sum_{k\in I} c_k x_k) = c_i$ are called the coefficient functionals of $\{x_i\}_{i\in I}$. If we define the space X_d by

$$X_d = \{\{f_i(x)\}_{i\in I} \mid x \in X\},$$

and we equip X_d with the norm $\| \{f_i(x)\}_{i\in I} \|_{X_d} = \| x \|_X$, then X_d becomes a BK-space. The dual space of X_d and its norm as follows:

$$X_d^* = \{\{f(x_i)\}_{i\in I} |\ f \in X^*\}, \quad \text{and} \quad \|\{f(x_i)\}_{i\in I}\|_{X_d^*} = \|f\|_{X^*}.$$

This shows that $\{f_i\}_{i\in I}, \{x_i\}_{i\in I}$ are Banach frames for X, X^* with respect to X_d, X_d^*, respectively. Further since

$$\|\sum_{i\in I} f_i(x) x_i\|_X = \|x\|_X = \|\{f_i(x)\}_{i\in I}\|_{X_d},$$

thus $\{x_i\}_{i\in I}$ is also a X_d-Riesz basis for X.

Definition 1.10. Let $\{f_i\}_{i\in I} \subseteq X^*$, $\{x_i\}_{i\in I} \subseteq X$ be X_d-Bessel and X_d^*-Bessel sequences for X, X^* respectively. Then

(i) $\{f_i\}_{i\in I}$ is called a dual Banach frame for $\{x_i\}_{i\in I}$ in X with respect to X_d, if $x = \sum_{i\in I} f_i(x) x_i$ for all $x \in X$, with respect the norm topology on X.

(ii) $\{x_i\}_{i\in I}$ is called a dual Banach frame for $\{f_i\}_{i\in I}$ in X^* with respect to X_d^*, if $f = \sum_{i\in I} f(x_i) f_i$ for all $f \in X^*$, with respect the norm topology on X^*.

There is an equivalence assertion on duality of the X_d-Bessel and X_d^*-Bessel sequences.

Theorem 1.11. Let $\{f_i\}_{i\in I}, \{x_i\}_{i\in I}$ be X_d-Bessel and X_d^*-Bessel sequences for X, X^* respectively. Then the following conditions are equivalent:

(i) $\{f_i\}_{i\in I}$ is a dual Banach frame for $\{x_i\}_{i\in I}$ in X with respect to X_d.

(ii) $\{x_i\}_{i\in I}$ is a dual Banach frame for $\{f_i\}_{i\in I}$ in X^* with respect to X_d^*.
(iii) $f(x) = \sum_{i\in I} f_i(x)f(x_i)$ for all $x \in X, f \in X^*$.

Proof. To prove (i) ⇒ (ii), let $f \in X^*$ be arbitrary; then by Lemma 1.5(ii) the series $\sum_{i\in I} f(x_i)f_i$ is convergent in X^* and for every $x \in X$ we have

$$f(x) = f(\textstyle\sum_{i\in I} f_i(x)x_i) = \sum_{i\in I} f_i(x)f(x_i) = (\sum_{i\in I} f(x_i)f_i)(x)$$

This shows that $f = \sum_{i\in I} f(x_i)f_i$. The implications (ii) ⇒ (iii) and (iii) ⇒ (i) are obvious.■

The following result relates dual Banach frames to Banach frames.

Lemma 1.12. Every dual Banach frame is a Banach frame.

Proof. Let $\{f_i\}_{i\in I}$ be a dual Banach frame for $\{x_i\}_{i\in I}$ in X with respect to X_d and let U, V be the analysis and synthesis operators for $\{f_i\}_{i\in I}$ and $\{x_i\}_{i\in I}$ respectively. Then by Theorem 1.11 we have $U^*V^*(f) = f$ for all $f \in X^*$. These yields

$$\|f\|_{X^*} = \|U^*V^*(f)\|_{X^*} \leq \|U\|\|V^*(f)\|_{X_d^*} = \|U\|\|\{f(x_i)\}_{i\in I}\|_{X_d^*}.$$

The upper frame bound for $\{x_i\}_{i\in I}$ follows from $VU = I_X$. Similarly, we can show that $\{f_i\}_{i\in I}$ is also a Banach frame for X with respect to X_d.■

Definition 1.13. Let $\{f_i\}_{i\in I}$ and $\{x_i\}_{i\in I}$ be X_d-Bessel and X_d^*-Bessel sequences for X, X^* respectively. If one of the conditions in Theorem 1.11 is satisfied. Then the pair $(\{x_i\}_{i\in I}, \{f_i\}_{i\in I})$ is called a X_d-dual Banach frame for X.

The following result shows that every Banach frame have at least one dual

Theorem 1.14. Let $\{f_i\}_{i\in I}$ be a Banach frame for X with respect to X_d. Then there exists a X_d^*-Bessel sequence $\{\tilde{x}_i\}_{i\in I}$ for X^* such that $(\{\tilde{x}_i\}_{i\in I}, \{f_i\}_{i\in I})$ is a X_d-dual Banach frame for X.

Proof. Let U, S_l be the analysis and reconstruction operators for $\{f_i\}_{i\in I}$. Put $\tilde{x}_i = S_l(e_i)$ where $\{e_i\}_{i\in I}$ is the Schauder basis of the canonical unit vectors in X_d. We first show that $\{\tilde{x}_i\}_{i\in I}$ is a X_d^*-Bessel sequence for X^*. Given $c \in X_d$ and $m, n \in \mathbb{N}$ with $m > n$.

$$\left\|\sum_{i\in I_m - I_n} c_i\tilde{x}_i\right\|_X = \left\|\sum_{i\in I_m - I_n} c_iS_le_i\right\|_X = \left\|S_l\left(\sum_{i\in I_m - I_n} c_ie_i\right)\right\|_X$$
$$\leq \|S_l\|\left\|\sum_{i\in I_m - I_n} c_ie_i\right\|_{X_d}.$$

Since $c \in X_d$, $\sum_{i\in I} c_ie_i$ is convergent, this implies that $\left\{\sum_{i\in I_n} c_i\tilde{x}_i\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence and therefore it is convergent in X. Now Proposition 1.5 implies that $\{\tilde{x}_i\}_{i\in I}$ is a X_d^*-Bessel sequence for X^*. Moreover, for every $x \in X$ we have

$$x = S_lU(x) = S_l(\textstyle\sum_{i\in I} f_i(x)e_i) = \sum_{i\in I} f_i(x)S_le_i = \sum_{i\in I} f_i(x)\tilde{x}_i$$

This shows that $(\{\tilde{x}_i\}_{i\in I}, \{f_i\}_{i\in I})$ is a X_d-dual Banach frame for X.■

Via Theorem 1.11 the Banach frame $\{\tilde{x}_i\}_{i\in I}$ obtained in Theorem 1.14 is called the canonical dual Banach frame of $\{f_i\}_{i\in I}$ in X^* with respect to X_d^*. We also have a parallel result for Banach frames for X^* with respect to X_d^*.

Corollary 1.15. Let $\{x_i\}_{i\in I}$ be a Banach frame for X^* with respect to X_d^*. Then there exists a X_d-Bessel sequence $\{\tilde{f}_i\}_{i\in I}$ for X such that $\left(\{x_i\}_{i\in I}, \{\tilde{f}_i\}_{i\in I}\right)$ is a X_d-

dual Banach frame for X.

Proof. Let V and S_r be the synthesis and reconstruction operators for $\{x_i\}_{i\in I}$. Put $\tilde{f}_i = S_r^* e_i$, where $\{e_i\}_{i\in I}$ is the Schauder basis of the canonical unit vectors in X_d^*. Since for all $d \in X_d^*$ and $m,n \in \mathbb{N}$ with $m > n$ we have

$$\left\|\sum_{i\in I_m - I_n} d_i \tilde{f}_i\right\|_{X^*} = \left\|\sum_{i\in I_m - I_n} d_i S_r^* e_i\right\|_{X^*} = \left\|S_r^*\left(\sum_{i\in I_m - I_n} d_i e_i\right)\right\|_{X^*}$$
$$\leq \|S_r\|\left\|\sum_{i\in I_m - I_n} d_i e_i\right\|_{X_d^*}.$$

Therefore $\{\tilde{f}_i\}_{i\in I}$ is a X_d-Bessel sequence for X. Also for all $f \in X^*$ we have

$$f = S_r^* V^*(f) = S_r^*\left(\sum_{i\in I} f(x_i) e_i\right) = \sum_{i\in I} f(x_i) S_r^* e_i = \sum_{i\in I} f(x_i)\tilde{f}_i$$

From this the result follows.■

Similarly the Banach frame $\{\tilde{f}_i\}_{i\in I}$ obtained in Corollary 1.15 is called the canonical dual Banach frame of $\{x_i\}_{i\in I}$ in X with respect to X_d.

2. Characterizations and perturbations of Dual Banach frames

In this section we generalize some results of Christensen [4] to the situation of dual Banach frames. We give a characterization of dual Banach frames in terms of the synthesis and analysis operators without any knowledge of the frame bounds. We also show that every Banach frame has infinitely many dual Banach frames.

Theorem 2.1. Let $\{f_i\}_{i\in I}, \{x_i\}_{i\in I}$ be X_d-Bessel and X_d^*-Bessel sequences for X, X^* respectively. Then the following statements hold.

(i) Let $\{f_i\}_{i\in I}$ be a Banach frame for X with respect to X_d with the analysis operator U. Then the dual Banach frames for $\{f_i\}_{i\in I}$ in X^* with respect to X_d^* are precisely the families $\{x_i\}_{i\in I} = \{T_l e_i\}_{i\in I}$, where $T_l : X_d \to X$ is a bounded left-inverse of U.

(ii) Let $\{x_i\}_{i\in I}$ be a Banach frame for X^* with respect to X_d^* with the synthesis operator V. Then the dual Banach frames for $\{x_i\}_{i\in I}$ in X with respect to X_d are precisely the families $\{f_i\}_{i\in I} = \{T_r^* e_i\}_{i\in I}$ where $T_r : X \longrightarrow X_d$ is a bounded right-inverse of V.

Proof. The proof is identical to the proof of Theorem 1.14 and Corollary 1.15.■

The next result is analogous to [4, Lemma 5.7.3] for the situation of dual Banach frames.

Theorem 2.2. Let $\{f_i\}_{i\in I}, \{x_i\}_{i\in I}$ be Banach frames for X, X^* with respect to X_d, X_d^*, with the analysis and synthesis operators U, V respectively. Then the following holds:

(i) The bounded left-inverses of U are precisely the operators having the form $S_l + W(I_{X_d} - US_l)$, where $W: X_d \to X$ is a bounded operator and S_l denotes the reconstruction operator of $\{f_i\}_{i\in I}$.

(ii) The bounded right-inverses of V are precisely the operators having the form $S_r + (I_{X_d} - S_r V)W$, where $W: X \to X_d$ is a bounded operator and

S_r denotes the reconstruction operator of $\{x_i\}_{i\in I}$.

Proof. For the proof of (i), it is obvious that an operator of the given form is a left-inverse of U. On the other hand, if T_l is a left-inverse of U, then by taking $W = T_l$ we have $T_l = S_l + T_l(I_{X_d} - US_l)$. The argument for statement (ii) is similar.■

The next theorem is analogous to a well-known result in abstract frame theory [4, Theorem 5.7.4]. This theorem is a characterization of all dual Banach frames associated with a given Banach frame.

Theorem 2.3. Let $\{f_i\}_{i\in I}, \{x_i\}_{i\in I}$ be Banach frames for X, X^* with respect to X_d, X_d^* with the analysis and synthesis operators U, V, respectively. Then the following holds:

(i) The dual Banach frames of $\{f_i\}_{i\in I}$ in X^* with respect to X_d^* are precisely the families

$$\{z_k\}_{k\in I} = \{\tilde{x}_k + y_k - \textstyle\sum_{i\in I} f_i(\tilde{x}_k) y_i\}_{k\in I}$$

where $\{y_k\}_{k\in I}$ is a X_d^*-Bassel sequence for X^* and $\{\tilde{x}_i\}_{i\in I}$ denotes the canonical dual Banach frame of $\{f_i\}_{i\in I}$ in X^* with respect to X_d^*.

(ii) The dual Banach frames of $\{x_i\}_{i\in I}$ in X with respect to X_d are precisely the families

$$\{g_k\}_{k\in I} = \left\{\tilde{f}_k + h_k - \textstyle\sum_{i\in I} \tilde{f}_k(x_i) h_i\right\}_{k\in I}$$

where $\{h_k\}_{k\in I}$ is a X_d-Bessel sequence for X and $\{\tilde{f}_i\}_{i\in I}$ denotes the canonical dual Banach frame of $\{x_i\}_{i\in I}$ in X with respect to X_d.

Proof. (i) By Theorems 2.1, 2.2 we can characterize the dual Banach frames, $\{f_i\}_{i\in I}$in X^* with respect to X_d^* as families the form

$$\{z_k\}_{k\in I} = \left\{(S_l + W(I_{X_d} - US_l))e_k\right\}_{k\in I}$$

where $W: X_d \longrightarrow X$ is a bounded operator, or equivalently an operator of the form $W(c) = \sum_{i\in I} c_i y_i$ where $\{y_k\}_{k\in I}$ is a X_d^*-Bessel sequence for X^*. Inserting this expression for W we obtain

$$\begin{aligned}\{z_k\}_{k\in I} &= \{S_l e_k + W e_k - WUS_l e_k\}_{k\in I} \\ &= \{\tilde{x}_k + y_k - \textstyle\sum_{i\in I} f_i(\tilde{x}_k) y_i\}_{k\in I}.\end{aligned}$$

The proof for the statement (ii) is analogous.■

A nonzero operator $\Lambda \in B(X, Y)$ is called a left divisor of zero if there exists a nonzero operator $\Gamma \in B(Y, X)$ such that $\Lambda\Gamma = 0$; similarly, it is called a right divisor of zero if there exists a nonzero operator $\Gamma \in B(Y, X)$ such that $\Gamma\Lambda = 0$.

Theorem 2.4. Let $\{f_i\}_{i\in I}, \{x_i\}_{i\in I}$ be Banach frames for X, X^* with respect to X_d, X_d^*. Then the analysis and synthesis operators of them are right and left divisors of zero in $B(X, X_d)$, $B(X_d, X)$ respectively.

Proof. Suppose that U, V are the analysis and synthesis operators of $\{f_i\}_{i\in I}, \{x_i\}_{i\in I}$ respectively. Let

$\Lambda: X_d \longrightarrow X$ and $\Gamma: X \longrightarrow X_d$ by $\Lambda = I_X - US_l$ and $\Gamma = I_X - S_r V$, where S_l, S_r denote the reconstruction operators of $\{f_i\}_{i\in I}, \{x_i\}_{i\in I}$. Then we have $\Lambda U = 0$ and $V\Gamma = 0$.■

The following Theorem is another characterization of the dual Banach frames by the family of left and right divisors of zero.

Theorem 2.5. Let $\{f_i\}_{i\in I}, \{x_i\}_{i\in I}$ be Banach frames for X, X^* with respect to X_d, X_d^* with the analysis and synthesis operators U, V, respectively. Then the following holds:

(i) There exists a one to one correspondence between dual Banach frames of $\{f_i\}_{i\in I}$ in X^* with respect to X_d^* and the bounded operators $\Lambda: X_d \longrightarrow X$ such that $\Lambda U = 0$.

(ii) There exists a one to one correspondence between dual Banach frames of $\{x_i\}_{i\in I}$ in X with respect to X_d and the bounded operators $\Gamma: X \longrightarrow X_d$ such that $V\Gamma = 0$.

Proof. (i) Let $\{y_i\}_{i\in I}$ be a dual Banach frame of $\{f_i\}_{i\in I}$ in X^* with respect to X_d^* with the synthesis operator W. Define $\Lambda: X_d \longrightarrow X$ by $\Lambda = W - S_l$, where S_l denotes the reconstruction operator of $\{f_i\}_{i\in I}$. Clearly, Λ is a bounded operator and by using Theorem 1.11 we have $\Lambda U = WU - S_l U = 0$. For the opposite implication, suppose that Λ is a bounded operator from X_d in X such that $\Lambda U = 0$. Let $y_i = S_l e_i + \Lambda e_i,\ i \in I$, where $\{e_i\}_{i\in I}$ denotes the Schauder basis of the canonical unit vectors in X_d. As in the proof of Theorem 1.14, $\{y_i\}_{i\in I}$ is a X_d^*-Bessel sequence for X^* and for every $x \in X$ we have

$$\sum_{i\in I} f_i(x) y_i = \sum_{i\in I} f_i(x) S_l e_i + \sum_{i\in I} f_i(x) \Lambda e_i = S_l U x + \Lambda U x = x.$$

This shows that $\{y_i\}_{i\in I}$ is a dual Banach frame of $\{f_i\}_{i\in I}$ in X^* with respect to X_d^*.

(ii) The proof is similar to (i).■

The following theorem is a perturbation result of dual Banach frames.

Theorem 2.6. Let $\{f_i\}_{i\in I}$ and $\{f'_i\}_{i\in I}$ be two Banach frames for X with respect to X_d with the canonical dual Banach frames $\{\tilde{x}_i\}_{i\in I}$ and $\{\widetilde{x'}_i\}_{i\in I}$respectively. Also let $\{y_i\}_{i\in I}$ be a fix alternate dual Banach frame of $\{f_i\}_{i\in I}$ in X^* with respect to X_d^* with the synthesis operator V and $\{f_i - f'_i\}_{i\in I}, \{\tilde{x}_i - \widetilde{x'}_i\}_{i\in I}$be two X_d-Bessel, X_d^*-Bessel sequences for X, X^* with sufficiently small Bessel bounds $\varepsilon > 0$. Then there exists an alternate dual Banach frame $\{y'_i\}_{i\in I}$ for $\{f'_i\}_{i\in I}$in X^* with respect to X_d^* such that $\{y_i - y'_i\}_{i\in I}$ is also a X_d^*-Bessel sequence in X^* and its bound is a multiple of ε.

Proof. Suppose that U, S_l and U', S'_l are the analysis and reconstruction operators of $\{f_i\}_{i\in I}$ and $\{f'_i\}_{i\in I}$ respectively. Then we have $\tilde{x}_i = S_l e_i$ and $\widetilde{x'}_i = S'_l e_i$ for all $i \in I$, where $\{e_i\}_{i\in I}$ denotes the Schauder basis of the canonical unit vectors in X_d. By using Theorem 2.5, there exists a bounded operator $\Lambda: X_d \longrightarrow X$ such that

$$\Lambda U = 0 \quad \text{and} \quad y_i = \tilde{x}_i + \Lambda e_i, \ \forall i \in I.$$

If we define $z_i = \widetilde{x'}_i + \Lambda e_i$ for all $i \in I$. Then it is easy to check that $\{z_i\}_{i\in I}$ is a X_d^*-Bessel sequence in X^* with the synthesis operator of $S'_l + \Lambda$. We claim that the bounded operator $\Gamma x = (S'_l + \Lambda)U'x = \sum_{i\in I} f'_i(x) z_i$ is invertible. In fact, for any $x \in X$, we have

$$\|x - \Gamma x\|_X = \left\|x - \sum_{i\in I} f'_i(x) z_i\right\|_X = \left\|\Lambda U' x\right\|_X = \left\|\Lambda U' x - \Lambda U x\right\|_X$$
$$\leq \|\Lambda\| \left\|U' x - U x\right\|_{X_d} \leq \varepsilon \|\Lambda\| \|x\|_X.$$

Therefore, if $\varepsilon\|\Lambda\| < 1$, then $\| I_X - \Gamma \| \leq \varepsilon \| \Lambda \| < 1$ and so Γ is invertible and we obtain $\left\|\Gamma^{-1}\right\| \leq \frac{1}{1-\|I_X - \Gamma\|} < \frac{1}{1-\varepsilon\|\Lambda\|}$. This implies that

$$\left\|I_X - \Gamma^{-1}\right\| = \left\|\Gamma^{-1}(\Gamma - I_X)\right\| \leq \left\|\Gamma^{-1}\right\| \|I_X - \Gamma\| \leq \frac{\varepsilon\|\Lambda\|}{1-\varepsilon\|\Lambda\|}.$$

Put $y'_i = \Gamma^{-1} z_i$ for all $i \in I$. It is trivial that $\{y'_i\}_{i\in I}$ is a X_d^*-Bessel sequence in X^* and we see from $\Gamma x = \sum_{i\in I} f'_i(x) z_i$ that $x = \sum_{i\in I} f'_i(x) y'_i$. Hence $\{y'_i\}_{i\in I}$ is a dual Banach frame for $\{f'_i\}_{i\in I}$ in X^* with respect to X_d^*. On the other hand, for every $c \in X_d$ we have

$$\left\|\sum_{i\in I} c_i (y_i - y'_i)\right\|_X = \left\|\sum_{i\in I} c_i (y_i - \Gamma^{-1} y_i + \Gamma^{-1} y_i - \Gamma^{-1} z_i)\right\|_X$$
$$\leq \left\|I_X - \Gamma^{-1}\right\| \|\sum_{i\in I} c_i y_i\|_X + \| \Gamma^{-1} \| \| \sum_{i\in I} c_i (y_i - z_i) \|_X$$
$$\leq \frac{\varepsilon\|\Lambda\|}{1-\varepsilon\|\Lambda\|} \|\sum_{i\in I} c_i y_i\|_X + \frac{1}{1-\varepsilon\|\Lambda\|} \left\|\sum_{i\in I} c_i \left(\tilde{x}_i - \widetilde{x'}_i\right)\right\|_X$$
$$\leq \frac{1+\|\Lambda\|\|V\|}{1-\varepsilon\|\Lambda\|} \| c \|_{X_d} \varepsilon.$$

This completes the proof.■

Corollary 2.7. Let $\{x_i\}_{i\in I}$ and $\{x'_i\}_{i\in I}$ be Banach frames for X^* with respect to X_d^* and let $\{f_i\}_{i\in I}$ be a dual Banach frame of $\{x_i\}_{i\in I}$ in X with respect to X_d. Let $\{f_i - f'_i\}_{i\in I}$ and $\{x_i - x'_i\}_{i\in I}$ be two X_d-Bessel X_d^*-Bessel sequences for X, X^* with sufficiently small Bessel bounds ε. Then there exists a dual Banach frame $\{f'_i\}_{i\in I}$ for $\{x'_i\}_{i\in I}$ in X with respect to X_d such that $\{f_i - f'_i\}_{i\in I}$ is a X_d-Bessel sequence in X and its bound is a multiple of ε.

Proof. The proof is similar to Theorem 2.6.■

3. Generalized dual Banach frames

In this section we generalize the concepts of pseudo-dual and approximate dual for Banach frames in Banach spaces and examines their properties.

Definition 3.1. Suppose that $\{f_i\}_{i\in I}, \{x_i\}_{i\in I}$ are X_d-Bessel and X_d^*–Bessel sequences for X, X^* with analysis and synthesis operators U, V respectively. Then

(i) $\{f_i\}_{i\in I}$ is called a pseudo-dual Banach frame for $\{x_i\}_{i\in I}$ in X with respect

to X_d, if the operator VU is a bijection on X.
(ii) $\{f_i\}_{i\in I}$ is an approximate dual Banach frame for $\{x_i\}_{i\in I}$ in X with respect to X_d, if $\|I_X - VU\| < 1$.

Note that if $\{f_i\}_{i\in I}$ is a pseudo-dual Banach frame for $\{x_i\}_{i\in I}$ in X with respect to X_d, then

$$x = \sum_{i\in I} f_i(x)(VU)^{-1}x_i \qquad \forall x \in X.$$

Thus $\{f_i\}_{i\in I}$ is a dual Banach frame for $\{(VU)^{-1}x_i\}_{i\in I}$ in X with respect to X_d. Therefore $\{f_i\}_{i\in I}$ is a Banach frame for X with respect to X_d. By symmetry $\{x_i\}_{i\in I}$ is also a Banach frame for X^* with respect to X_d^*. Obviously, every approximate dual Banach frame for X with respect to X_d is a pseudo-dual Banach frame.

The next result follows immediately from the definition. We leave the proof to interested readers.

Corollary 3.2. Let $\{f_i\}_{i\in I}$ and $\{x_i\}_{i\in I}$ be X_d-Bessel and X_d^*-Bessel sequences for X, X^* with analysis and synthesis operators U and V respectively. Then the following statements are equivalent:

(i) $\{f_i\}_{i\in I}$ is a pseudo-dual Banach frame for $\{x_i\}_{i\in I}$ in X with respect to X_d.
(ii) $x = \sum_{i\in I} f_i((VU)^{-1}x)x_i = \sum_{i\in I} f_i(x)(VU)^{-1}x_i \quad \forall x \in X$.
(iii) $f = \sum_{i\in I} f((VU)^{-1}x_i)f_i = \sum_{i\in I} f(x_i)(U^*V^*)^{-1}f_i \quad \forall f \in X^*$.
(iv) For every $x \in X$ and $f \in X^*$ we have

$$f(x) = \sum_{i\in I} f((VU)^{-1}x_i)f_i(x) = \sum_{i\in I} f(x_i)f_i((VU)^{-1}x).$$

Theorem 3.3. Let $\{f_i\}_{i\in I}$ be an approximate dual Banach frame for $\{x_i\}_{i\in I}$ in X with respect to X_d with the analysis and synthesis operators U, V, respectively. Then the following holds:

(i) $\{(U^*V^*)^{-1}f_i\}_{i\in I}$ is a dual Banach frame for $\{x_i\}_{i\in I}$ in X with respect to X_d and $(U^*V^*)^{-1}f_i = f_i + \sum_{n=1}^{\infty}(I_{X^*} - U^*V^*)^n f_i$
(ii) For fixed $n \in \mathbb{N}$, consider the partial sum

$$f_{ni} = f_i + \sum_{j=1}^{n}(I_{X^*} - U^*V^*)^j f_i.$$

Then $\{f_{ni}\}_{i\in I}$ is an approximate dual Banach frame of $\{x_i\}_{i\in I}$ in X with respect to X_d. Denoting its associated analysis operator by U_n, we have

$$\| I_{X^*} - U_n^*V^* \| \leq \| I_{X^*} - U^*V^* \|^{n+1} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Proof. (i) If $\{f_i\}_{i\in I}$ is an approximate dual Banach frame for $\{x_i\}_{i\in I}$ then the operator VU is a bijection on X and for all $x \in X$ we have

$$x = (VU)(VU)^{-1}x = \sum_{i\in I} f_i((VU)^{-1}x)x_i = \sum_{i\in I}(U^*V^*)^{-1}(f_i)x_i.$$

This shows that $\{(U^*V^*)^{-1}f_i\}_{i\in I}$ is a dual Banach frame for $\{x_i\}_{i\in I}$ in X with respect to X_d. Moreover, the inverse of VU can be written as follows:

$$(VU)^{-1} = (I_X - (I_X - VU))^{-1} = I_X + \sum_{n=1}^{\infty}(I_X - VU)^n$$

From this the result in (i) follows.
(ii) For any $x \in X$ we have

$$\begin{aligned}(I_X - VU_n)x &= x - \sum_{i\in I} f_{ni}(x)x_i \\ &= x - \sum_{i\in I}\sum_{j=0}^{n}(I_{X^*} - U^*V^*)^j(f_i)(x)x_i \\ &= x - \sum_{j=0}^{n}\big(I_X - (I_X - VU)\big)(I_X - VU)^j x \\ &= (I_X - VU)^{n+1}x\end{aligned}$$

Thus

$$\| I_X - VU_n \| = \| (I_X - VU)^{n+1} \| \leq \| I_X - VU \|^{n+1} \to 0 \quad \text{as} \quad n \to \infty. \blacksquare$$

The following result shows that the image of a dual Banach frame under a bounded invertible operator is a pseudo-dual Banach frame.

Theorem 3.4. Suppose that $\{f_i\}_{i\in I}$ and $\{x_i\}_{i\in I}$ are X_d-Bessel and X_d^*–Bessel sequences for $X,\ X^*$ respectively, and let $\{\alpha_j\}_{j=1}^{N}$ be a finite sequence of complex numbers such that $\sum_{j=1}^{N}\alpha_j \neq 0$. Then the following holds:

(i) If for all $1 \leq j \leq N$, $\{f_{ij}\}_{i\in I} \subseteq X^*$ is a dual Banach frame for $\{x_i\}_{i\in I}$ in X with respect to X_d and $\Lambda: X \longrightarrow X$ is an invertible operator, then the sequence $\{g_i\}_{i\in I} \subseteq X^*$ defined by $g_i = \sum_{j=1}^{N}\alpha_j\Lambda^*(f_{ij}), (i \in I)$ is a pseudo-dual Banach frame for $\{x_i\}_{i\in I}$ in X with respect to X_d.

(ii) If for all $1 \leq j \leq N$, $\{f_i\}_{i\in I}$ is a dual Banach frame for $\{x_{ij}\}_{i\in I}$ in X with respect to X_d and $\Lambda: X \longrightarrow X$ is an invertible operator, then $\{f_i\}_{i\in I}$ is also a pseudo-dual Banach frame for the sequence $\{y_i\}_{i\in I} \subseteq X$ defined by $y_i = \sum_{j=1}^{N}\alpha_j\Lambda x_{ij}$ in X with respect to X_d.

Proof. (i) Let U and V be the analysis and synthesis operators of $\{g_i\}_{i\in I}$ and $\{x_i\}_{i\in I}$ respectively. For every $x \in X$ we have

$$\begin{aligned}VU(x) &= \sum_{i\in I} g_i(x)x_i = \sum_{i\in I}\sum_{j=1}^{N}\alpha_j\Lambda^*(f_{ij})(x)x_i \\ &= \sum_{j=1}^{N}\alpha_j\sum_{i\in I} f_{ij}(\Lambda x)\,x_i = \Big(\sum_{j=1}^{N}\alpha_j\Big)\Lambda x,\end{aligned}$$

hence VU is invertible. From this the result follows.

(ii) The proof is similar to (i).■

Theorem 3.5. Let $\{x_i\}_{i\in I},\ \{y_i\}_{i\in I}$ be X_d-Riesz bases for X with the canonical dual Banach frames $\{\tilde{f}_i\}_{i\in I}, \{\tilde{g}_i\}_{i\in I}$ respectively. Then $\{\tilde{f}_i\}_{i\in I}$ is a pseudo-dual Banach frame for $\{y_i\}_{i\in I}$ in X with respect to X_d.

Proof. If V_1 and V_2 are the synthesis operators of $\{x_i\}_{i\in I}, \{y_i\}_{i\in I}$, then V_1, V_2 are invertible and V_1^{-1}, V_2^{-1} are the analysis operators of $\{\tilde{f}_i\}_{i\in I}, \{\tilde{g}_i\}_{i\in I}$ respectively. Thus $V_2V_1^{-1}$ is a bijection on X and for every $x \in X$ we have

$$x = \sum_{i\in I} \tilde{f}_i(x)V_1V_2^{-1}y_i = \sum_{i\in I} \tilde{f}_i(V_1V_2^{-1}x)y_i.$$

From this the claim follows immediately.■

Theorem 3.6. Suppose that $\{f_i\}_{i\in I}$ and $\{x_i\}_{i\in I}$ are X_d-Bessel and X_d^*–Bessel sequences for X, X^* respectively, and let Λ, Γ be two invertible operators on X. Then $\{f_i\}_{i\in I}$ is a pseudo-dual Banach frame for $\{x_i\}_{i\in I}$ in X with respect to X_d if and only if $\{\Lambda^* f_i\}_{i\in I}$ is a pseudo-dual Banach frame for $\{\Lambda x_i\}_{i\in I}$ in X with respect to X_d.

Proof. Suppose that U, U_Γ and V, V_Λ are the analysis and synthesis operators of $\{f_i\}_{i\in I}$, $\{\Gamma^*(f_i)\}_{i\in I}$ and $\{x_i\}_{i\in I}$, $\{\Lambda x_i\}_{i\in I}$ respectively. This claim follows immediately from the fact that for each $x \in X$ we have

$$V_\Lambda U_\Gamma(x) = \sum_{i\in I} \Gamma^*(f_i)(x)\Lambda x_i = \Lambda(\sum_{i\in I} f_i(\Gamma x)x_i) = \Lambda VU\Gamma x.$$

This finishes the proof.■

Theorem 3.7. Let $\{f_i\}_{i\in I}$, $\{x_i\}_{i\in I}$ be two X_d-Bessel and X_d^*-Bessel sequences for X, $X^*$and let Λ, Γ be invertible operators on X. If $\{f_i\}_{i\in I}$ is a dual Banach frame for $\{x_i\}_{i\in I}$ in X with respect to X_d then $\{\Gamma^*(f_i)\}_{i\in I}$ is a pseudo-dual Banach frame for $\{\Lambda x_i\}_{i\in I}$ in X with respect to X_d.

Proof. The hypotheses imply that $x = \sum_{i\in I} f_i(x)x_i$ therefore

$$\sum_{i\in I} \Gamma^*(f_i)(x)\Lambda x_i = \Lambda(\sum_{i\in I} f_i(\Gamma x)x_i) = \Lambda\Gamma x.$$

From this the result follows at once.■

Next we give a method for constructing a family of pseudo-dual Banach frames from a given Banach frame.

Theorem 3.8. Let $\{f_i\}_{i\in I}$ be a pseudo-dual Banach frame for $\{x_i\}_{i\in I}$ in X with respect to X_d with the analysis and synthesis operators U, V respectively. Let α, β be two complex numbers such that $\alpha + \beta = 1$. Then the sequence $\{g_i\}_{i\in I}$ defined by $g_i = \alpha f_i + \beta(VU)^*(\tilde{f}_i)$, is a pseudo-dual Banach frame for $\{x_i\}_{i\in I}$ in X with respect to X_d, where $\{\tilde{f}_i\}_{i\in I}$ is the canonical dual Banach frame of $\{x_i\}_{i\in I}$.

Proof. For every $x \in X$ we have

$$\begin{aligned}\sum_{i\in I} g_i(x)x_i &= \alpha \sum_{i\in I} f_i(x)x_i + \beta \sum_{i\in I}(VU)^*(\tilde{f}_i)(x)x_i \\ &= \alpha VUx + \beta \sum_{i\in I} \tilde{f}_i(VUx)x_i \\ &= (\alpha + \beta)VUx = VUx. \ ■\end{aligned}$$

4. Perturbation results of generalized dual Banach frames

In this section we show that generalized dual Banach frames are stable under small perturbations of the Banach frame elements so that the perturbation results obtained in [5] is a special case of it.

Theorem 4.1. Let $\{f_i\}_{i\in i}$ be a Banach frame for X with respect to X_d with the analysis and reconstruction operators $U,\ S_l$. Assume that $\{g_i\}_{i\in I} \subseteq X^*$ and there exist $\lambda, \mu \geq 0$ such that

(i) $2(\lambda \parallel U \parallel + \mu)\|S_l\| < 1$.

(ii) $\parallel \{f_i(x) - g_i(x)\}_{i\in I} \parallel_{X_d} \leq \lambda \parallel \{f_i(x)\}_{i\in I} \parallel_{X_d} + \mu \parallel x \parallel_X$,

for all $x \in X$. Then $\{g_i\}_{i\in I}$ is a Banach frame for X with respect to X_d and $\{g_i\}_{i\in I}$, $\{f_i\}_{i\in I}$ are approximate dual Banach frames of $\{\tilde{x}_i\}_{i\in I}$, $\{\tilde{y}_i\}_{i\in I}$ in X with respect to X_d respectively, where $\{\tilde{x}_i\}_{i\in I}$, $\{\tilde{y}_i\}_{i\in I}$ are the canonical dual Banach frame of $\{f_i\}_{i\in I}$, $\{g_i\}_{i\in I}$ in X^* with respect to X_d^*.

Proof. If U' is the analysis operator of $\{g_i\}_{i\in I}$, then by the hypotheses we have

$$\parallel U'x \parallel_{X_d} \leq \parallel U'x - Ux \parallel_{X_d} + \parallel Ux \parallel_{X_d} \leq ((\lambda + 1) \parallel U \parallel + \mu) \parallel x \parallel_X.$$

for all $x \in X$. This establishes the upper frame bound for $\{g_i\}_{i\in I}$. On the other hand, from $S_l U = I_X$ we have $\parallel I_X - S_l U' \parallel \leq \|S_l\|\|U - U'\| < 1$ and this implies that $\{g_i\}_{i\in I}$ is an approximate dual Banach frame of $\{\tilde{x}_i\}_{i\in I}$ and so

$$\|(S_l U')^{-1}\| \leq \frac{1}{1-(\lambda\|U\|+\mu)\|S_l\|}.$$

If we set $S' = (S_l U')^{-1} S_l$, then $S'U' = I_X$ which implies that $\{g_i\}_{i\in I}$ is a Banach frame for X with respect to X_d and

$$\|S'\| \leq \frac{\|S_l\|}{1-(\lambda\|U\|+\mu)\|S_l\|}.$$

Finally, from $\tilde{y}_i = S'(e_i)$ we obtain

$$\|I_X - S'U\| = \|S'U' - S'U\| \leq \|S'\|\|U' - U\| \leq \frac{(\lambda\|U\|+\mu)\|S_l\|}{1-(\lambda\|U\|+\mu)\|S_l\|} < 1.$$

This concludes the proof.

Theorem 4.2. Let $\{f_i\}_{i\in I}$ be a dual Banach frame of $\{x_i\}_{i\in I}$ in X with respect to X_d with the analysis and synthesis operators $U,\ V$. Assume that $\{g_i\}_{i\in I}$ is a sequence in X^* and there exist $\lambda,\ \mu \geq 0$ such that

(i) $2(\lambda \parallel U \parallel + \mu) \parallel V \parallel < 1$.

(ii) $\parallel \{f_i(x) - g_i(x)\}_{i\in I} \parallel_{X_d} \leq \lambda \parallel \{f_i(x)\}_{i\in I} \parallel_{X_d} + \mu \parallel x \parallel_X \quad \forall x \in X$.

Then $\{g_i\}_{i\in I}$ is an approximate dual Banach frame for $\{x_i\}_{i\in I}$ in X with respect to X_d.

Proof. The proof is similar to that of Theorem 4.1.■

Theorem 4.3. Let $\{f_i\}_{i\in I}$ be a dual Banach frame of $\{x_i\}_{i\in I}$ in X with respect to X_d with the analysis and synthesis operators $U,\ V$. Assume that $\{y_i\}_{i\in I}$ is a sequence in X and there exist $\lambda, \mu \geq 0$ such that

(i) $(\lambda + \mu\|U\|)(1 + \|V\|\|U\|) < 1$.

(ii) $\|\sum_{i\in I} c_i(x_i - y_i)\|_X \leq \lambda\|\sum_{i\in I} c_i x_i\|_X + \mu\|c\|_{X_d}$, for all $c \in X_d$. Then $\{f_i\}_{i\in I}$ is an approximate dual Banach frame for $\{y_i\}_{i\in I}$ in X with respect to X_d. Furthermore, there exists a X_d-Bessel sequence $\{g_i\}_{i\in I}$ for X such that it is an approximate dual Banach frame for $\{x_i\}_{i\in I}$ in X with respect to X_d.

Proof. The hypotheses imply that $\{y_i\}_{i\in I}$ is a X_d^*-Bessel sequences for X^*. Let V' be the synthesis operator of $\{y_i\}_{i\in I}$. For all $x \in X$ we have

$$\begin{aligned}\| x - V'Ux \|_X &= \| \textstyle\sum_{i\in I} f_i(x)(x_i - y_i) \| \\ &\leq \lambda \| x \|_X + \mu \| Ux \|_{X_d} \\ &\leq (\lambda + \mu \| U \|) \| x \|_X \\ &< (1+\| V \|\| U \|)(\lambda + \mu \| U \|) \| x \|_X.\end{aligned}$$

From this we obtain $\|I_X - V'U\| < \lambda + \mu\|U\| < 1$, which implies that $\{f_i\}_{i\in I}$ is an approximate dual Banach frame for $\{y_i\}_{i\in I}$ in X with respect to X_d and

$$\|(V'U)^{-1}\| \leq \frac{1}{1-(\lambda+\mu\|U\|)}.$$

Therefore if we define $g_i = ((V'U)^{-1})^*(f_i)$, for all $i \in I$, then we have

$$x = V'U(V'U)^{-1}x = \sum_{i\in I} g_i(x)y_i,$$

Which shows that $\{g_i\}_{i\in I}$ is a dual Banach frame for $\{y_i\}_{i\in I}$ in X with respect to X_d with the analysis operator $U' = U(V'U)^{-1}$. We also have

$$\begin{aligned}\|I_X - VU'\| &\leq \|V\|\|U - U'\| \\ &\leq \|V\|\|U\|\|I_X - (V'U)^{-1}\| \\ &\leq \|V\|\|U\|\|(V'U)^{-1}\|\|I_X - V'U\| \\ &\leq \|V\|\|U\| \frac{\lambda+\mu\|U\|}{1-(\lambda+\mu\|U\|)} < 1.\end{aligned}$$

Therefore $\{g_i\}_{i\in I}$ is an approximate dual Banach frame for $\{x_i\}_{i\in I}$ in X with respect to X_d.■

Corollary 4.4. Let $\{f_i\}_{i\in I}$ be a dual Banach frame of $\{x_i\}_{i\in I}$ in X with respect to X_d with the analysis and synthesis operators U, V. Assume that $\{y_i\}_{i\in I}$ is a sequence in X such that $\{\|x_i - y_i\|\}_{i\in I} \in X_d^*$ and

$$(1 + \|V\|\|U\|)\|U\|\|\{\|x_i - y_i\|\}_{i\in I}\|_{X_d^*} < 1.$$

Then $\{f_i\}_{i\in I}$ is an approximate dual Banach frame for $\{y_i\}_{i\in I}$ in X with respect to X_d and there exists a X_d-Bessel sequence $\{g_i\}_{i\in I}$ for X such that it is an approximate dual Banach frame for $\{x_i\}_{i\in I}$ in X with respect to X_d.

Proof. Since for every $c \in X_d$ we have

$$\|\sum_{i\in I} c_i(x_i - y_i)\|_X \leq \|\{\|x_i - y_i\|\}_{i\in I}\|_{X_d^*}\|c\|_{X_d},$$

therefore by $\lambda = 0$ and $\mu = \|\{\|x_i - y_i\|\}_{i\in I}\|_{X_d^*}$, the result follows from Theorem 4.3.■

Corollary 4.5. Let $\{f_i\}_{i\in I}$ be a dual Banach frame of $\{x_i\}_{i\in I}$ in X with respect to X_d with the analysis and synthesis operators U, V. Assume that exists a family $\{\Lambda_j\}_{j\in J}$ of bounded operators on X and a family $\{\alpha_{ij}\}_{i\in I, j\in J}$ of scalars so that

$$y_i = x_i - \sum_{j\in J} \alpha_{ij}\Lambda_j x_i \qquad \forall i \in I.$$

If $\alpha_j = \sup_{i\in I} |\alpha_{ij}| < \infty$, for all $j \in J$ and

$$(1 + \|V\|\|U\|)\|V\|\|U\| \sum_{j\in J} \alpha_j \|\Lambda_j\| < 1,$$

then $\{f_i\}_{i\in I}$ is an approximate dual Banach frame for $\{y_i\}_{i\in I}$ in X with respect to X_d and there exists a X_d-Bessel sequence $\{g_i\}_{i\in I}$ for X such that it is an approximate dual Banach frame for $\{x_i\}_{i\in I}$ in X with respect to X_d.

Proof. For all $j \in J$ and $c \in X_d$ we have

$$\begin{aligned}\left\|\sum_{i\in I} c_i\alpha_{ij}x_i\right\|_X &= \sup_{\|f\|_{X_d^*}=1} \left|\sum_{i\in I} c_i\alpha_{ij}f(x_i)\right| \\ &\le \sup_{\|f\|_{X_d^*}=1} \|V^*(f)\|_{X_d^*} \left\|\{\alpha_{ij}c_i\}_{i\in I}\right\|_{X_d} \\ &\le \alpha_j\|V\|\|c\|_{X_d}.\end{aligned}$$

This yields

$$\begin{aligned}\left\|\sum_{i\in I} c_i(x_i - y_i)\right\|_X &= \left\|\sum_{i\in I}\sum_{j\in J} c_i\alpha_{ij}\Lambda_j x_i\right\|_X \\ &= \left\|\sum_{j\in J} \Lambda_j\left(\sum_{i\in I} c_i\alpha_{ij}x_i\right)\right\|_X \\ &\le \sum_{j\in J}\|\Lambda_j\| \left\|\sum_{i\in I} c_i\alpha_{ij}x_i\right\|_X \\ &\le \|c\|_{X_d}\|V\| \sum_{j\in J} \alpha_j\|\Lambda_j\|.\end{aligned}$$

Now with $\lambda = 0$ and $\mu = \|V\| \sum_{j\in J} \alpha_j\|\Lambda_j\|$, the result follows from Theorem 4.3.■

Theorem 4.6. Let $\{f_i\}_{i\in I}$, $\{g_i\}_{i\in I}$ be two pseudo-dual Banach frames for $\{x_i\}_{i\in I}$ in X with respect to X_d with the analysis and synthesis operators U_1, U_2 and V, respectively. If $\| (VU_1)^{-1} \|\| V \|\| U_2 \| < 1$, then the sequence $\{f_i + g_i\}_{i\in I}$ is a pseudo-dual Banach frame for $\{x_i\}_{i\in I}$ in X with respect to X_d.

Proof. Since $\| (VU_1)^{-1} \|\| V \|\| U_2 \| < 1$ the operator $I_X + (VU_1)^{-1}VU_2$ is invertible, which implies that $V(U_1 + U_2)$ is invertible. We have

$$\sum_{i\in I}(f_i + g_i)(x)x_i = \sum_{i\in I} f_i(x)x_i + \sum_{i\in I} g_i(x)x_i = V(U_1 + U_2)x,$$

for all $x \in X$. Therefore $\{f_i + g_i\}_{i\in I}$ is a pseudo-dual Banach frame for $\{x_i\}_{i\in I}$.■

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