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ON AN INTEGRAL AS AN INTERVAL FUNCTION

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Based on the total integrability we first define an integral of a real valued function f as an interval function associated to its antiderivative F. By introducing the concept of the residue of a function into the real analysis, the relationship between the integral so defined and the generalized Riemann integral is established.

Keywords: the fundamental theorem of calculus, a residue function

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1. Introduction

An antiderivative of a real-valued function f is just a function F whose derivative is f. The collection of functions $F + C$, where C is an arbitrary constant known as the constant of integration, is a nonunique inverse of the derivative f. Another way of stating this is that the set of all antiderivatives $F + C$ is an indefinite integral of f. In symbols, $\int f(x)\, dx = F + C$. So, the opposite process to differentiation is integration. The fundamental theorem of calculus, more precisely its the second part, allows definite integrals to be computed in terms of indefinite integrals. This part of the theorem states that if F is the antiderivative for f, then, under certain conditions, the definite integral of f over a compact interval $I \subset \mathbb{R}$ is equal to the difference between the values of an antiderivative F evaluated at the endpoints of the interval. In symbols, $\int_I f(x)\, dx = \Delta F(I)$. Here, ΔF is an associated interval function of F, such that $\Delta F(I) = F(v) - F(u)$ for any compact interval $I = [u, v]$, [6]. Obviously, if F is defined on I, then the sum of the changes in the value of F over I with any partition is equal to $\Delta F(I)$. Hence, an attempt has been made by *Sarić* [4, 5] to define an integral of f over I, as the sum of these changes in the value of F over I, for which the *Newton–Leibniz* formula (the second part of the fundamental theorem of calculus) to be valid unconditionally. The resulting integral is the so-called total *Kurzweil-Henstock* integral. Accordingly, instead of the set of functions $F + C$ we can use the associated interval function ΔF of F to be an integral of f. In symbols, $\int f(x)\, dx := \Delta F$. Therefore, the purpose of this note is to convert the fundamental theorem of calculus into the definition of integrability of f, as follows.

Definition 1.1. *Let f be a real valued function with antiderivative F and let $\mathfrak{I}$ be any collection of compact intervals I of the real line $\mathbb{R}$. If $\Delta F : \mathfrak{I} \to \mathbb{R}$ is the associated interval function of F, then f is integrable to $\Delta F(I)$ on $I \in \mathfrak{I}$. In symbols, $\int_I f(x)\, dx = \Delta F(I)$.*

When working with functions, which have a finite number of discontinuities on the compact interval $[a, b] \subset \mathbb{R}$, it does not really matter how these functions will be defined on the set E of discontinuities. Unless otherwise stated in what follows, we assume that the endpoints of $[a, b]$ do not belong to E. As this situation will arise frequently, we adopt the convention that such functions are equal to 0 at all points at which they have an infinite value ($\pm\infty$) or not be defined at all. Hence, we may define point functions $F_{ex} : [a, b] \mapsto \mathbb{R}$ and $f_{ex} : [a, b] \mapsto \mathbb{R}$ by extending F and its derivative f from $[a, b] \setminus E$ to E by $F_{ex}(x) = 0$

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and $f_{ex}(x) = 0$ for $x \in E$, so that

$$F_{ex}(x) = \begin{cases} F(x), \text{ if } x \in [a,b] \backslash E \\ 0, \text{ if } x \in E \end{cases} \quad \text{and}$$

$$f_{ex}(x) = \begin{cases} f(x), \text{ if } x \in [a,b] \backslash E \\ 0, \text{ if } x \in E \end{cases}. \tag{1}$$

If we denote any generalized *Riemann* integral of f_{ex} on I by $\mathcal{R} - \int_I f_{ex}(x)\,dx$, including the *Riemann* integral itself, then we will prove below the following result $\Delta F(I) = \int_I f(x)\,dx = \mathcal{R} - \int_I f_{ex}(x)\,dx + \Re$, where $\Re$ is the sum of residues of F on the set $E \subset I$ at whose points F is not differentiable. Clearly, if $\mathcal{R} - \int_I f_{ex}(x)\,dx$ does not exist, then $\mathcal{R} - \int_I f_{ex}(x)\,dx + \Re$ is reduced to the so-called indeterminate expression $\infty - \infty$ that actually have, in this situation, the real numerical value of $\Delta F(I)$.

2. Preliminaries

Given a compact interval $[a,b]$ in $\mathbb{R}$, let the collection $\mathcal{I}([a,b])$ be a family of all compact subintervals I of $[a,b]$. The *Lebesgue* measure in $\mathbb{R}$ is denoted by μ, however, for $I \subset \mathbb{R}$ we write $|I| = \Delta x(I)$ instead of $\mu(I)$. A partition $P[a,b]$ of a compact interval $[a,b] \in \mathbb{R}$ is a finite set (collection) of interval-point pairs $([a_i,b_i],x_i)_{i\le\nu}$, such that the subintervals $[a_i,b_i]$ are non-overlapping, $\cup_{i\le\nu}[a_i,b_i] = [a,b]$ and $x_i \in [a,b]$. The points $\{x_i\}_{i\le\nu}$ are the tags of $P[a,b]$, [1]. It is evident that there are many different ways to arrange the position of the tags x_i with respect to $[a_i,b_i]$. Each of these positions leads to one of a *Riemann* type definition of the generalized *Riemann* integral. If E is a subset of $[a,b]$, then the restriction of $P[a,b]$ to E is a finite collection of $([a_i,b_i],x_i) \in P[a,b]$, such that each pair of sets $[a_i,b_i]$ and E intersects in at least one point and all x_i are tagged in E. In symbols, $P[a,b]|_E = \{([a_i,b_i],x_i) \in P[a,b] \mid [a_i,b_i] \cap E \neq \emptyset \text{ and } x_i \in E\}$. Given $\delta : [a,b] \mapsto \mathbb{R}_+$, named a gauge, a point-interval pair $([a_i,b_i],x_i)$ is called *δ-fine* if $[a_i,b_i] \subseteq (x_i - \delta(x_i), x_i + \delta(x_i))$. Let $\mathcal{P}[a,b]$ be the family of all partitions $P[a,b]$ of $[a,b]$. If $E \subseteq [a,b]$, then for any position of the tags x_i with respect to $[a_i,b_i]$ the family of all *δ-fine* partitions $P[a,b]$ of $[a,b]$, such that $P[a,b]|_E \subset P[a,b]$, denoted by $\mathcal{P}_\delta[a,b]|_E$. In what follows we will use the following notations: $\sum_i \Delta F_{ex}([a_i,b_i]) = \Delta F(P[a,b]|_E)$ and $\sum_i f_{ex}(x_i)\,|[a_i,b_i]| = \delta F(P[a,b]|_E)$, whenever $([a_i,b_i],x_i) \in P[a,b]|_E$.

Definition 2.1. *Let $\varphi : \mathcal{I}[a,b] \mapsto \mathbb{R}$ be an arbitrary interval function and $E \subseteq [a,b]$. A point function $f : [a,b] \mapsto \mathbb{R}$ is the limit of ϕ on $[a,b] \backslash E$ if for every $\varepsilon > 0$ there exists a gauge $\delta : [a,b] \mapsto \mathbb{R}_+$, such that*

$$|\varphi([a_i,b_i]) - f(x_i)| < \varepsilon, \tag{2}$$

whenever $([a_i,b_i],x_i) \in P[a,b] \backslash P[a,b]|_E$ and $P[a,b] \in \mathcal{P}_\delta[a,b]|_E$.

Given a derivative-antiderivative pair (f and F), the derivative f is the limit of the interval function

$$\varphi(I) = \frac{\Delta F(I)}{\Delta x(I)} = \frac{1}{\Delta x(I)} \int_I f(x)\,dx, \tag{3}$$

where $\Delta F(I)$ is the associated interval function of F.

3. Main results

Let $F : [a,b] \mapsto \mathbb{R}$. It is an old result that F is continuous on $[a,b]$ if and only if the associated interval function ΔF of F converges to 0 at all points of $[a,b]$, [3]. Accordingly, we are now in a position to define the linear differential form on $[a,b]$.

For $F : [a,b] \mapsto \mathbb{R}$ let φ be defined by (3). Then, $dF = f dx$ as the limit of the interval function

$$\Delta F(I) = \int_I f(x)\,dx = \varphi(I)\Delta x(I) \tag{4}$$

on $[a,b]$ is a linear differential form on $[a,b]$

Clearly, if F is continuous on $[a,b]$ then $dF = fdx$ vanishes identically on $[a,b]$. In case F is differentiable to f everywhere on $[a,b]$ except for a set $E \subset [a,b]$ of *Lebesgue* measure zero, we can introduce into the analysis an interval-point function $\delta F : [a,b] \times \mathcal{I}([a,b]) \mapsto \mathbb{R}$ being the product of the point function f_{ex} defined by (1) and the interval function Δx, as follows

$$\delta F(I,x) = f_{ex}(x)\,\Delta x(I). \tag{5}$$

As we can see, there is a difference between the interval-point function $\delta F(I,x)$ and the interval function $\Delta F(I)$, as well as between their limits on E. However, by *Definition 2.1*, since f is the limit of φ on $[a,b]\backslash E$, given $\varepsilon > 0$ there exists a gauge $\delta : [a,b] \mapsto \mathbb{R}_+$, such that

$$|\delta F([a_i,b_i],x_i) - \Delta F([a_i,b_i])| < \varepsilon \Delta x([a_i,b_i]), \tag{6}$$

whenever $([a_i,b_i],x_i) \in P[a,b]\backslash P[a,b]|_E$ and $P[a,b] \in \mathcal{P}_\delta[a,b]|_E$. So, in this emphasized case fdx is the limit of both δF and ΔF on $[a,b]\backslash E$.

Remember, there are many different ways to arrange the position of the tags x_i with respect to $[a_i,b_i]$, each of which leads to one type of the generalized *Riemann* integral defined by the following definition.

Definition 3.1. *For $[a,b] \in \mathbb{R}$ let $E \subset [a,b]$ be a set of Lebesgue measure zero at whose points a real valued function f is not defined. A point function $f_{ex} : [a,b] \mapsto R$ is generalized Riemann integrable to a real point $\mathcal{F}$ on $[a,b]$ if for every $\varepsilon > 0$ there exists a gauge $\delta : [a,b] \mapsto \mathbb{R}_+$, such that*

$$|\delta F(P[a,b]) - \mathcal{F}| < \varepsilon, \tag{7}$$

whenever $P[a,b] \in \mathcal{P}_\delta[a,b]|_E$. In symbols, $\mathcal{F} := \mathcal{R} - \int_a^b f(x)\,dx$.

If $x_i \in [a_i,b_i]$ and the gauge $\delta(x)$ has a positive infimum on $[a,b]$, then the previous definition becomes that of the ordinary *Riemann* integral.

The following two definitions introduce the concept of the residue of a function into the real analysis.

Definition 3.2. *For $[a,b] \in \mathbb{R}$ let $E \subset [a,b]$ be a set of Lebesgue measure zero at whose points a real valued function F is not defined. The function F is said to be basically summable (BS_δ) on E to a real number $\Re$, if for every $\varepsilon > 0$ there exists a gauge $\delta : [a,b] \mapsto \mathbb{R}_+$, such that $|\Delta F(P[a,b]|_E) - \Re| < \varepsilon$, whenever $P[a,b] \in \mathcal{P}_\delta[a,b]|_E$. If in addition E can be written as a countable union of sets on each of which F is BS_δ, then F is said to be BSG_δ on E. In symbols, $\Re := \sum_{x\in E} f(x)\,dx$.*

If F is absolutely continuous on $[a,b]$, that means it has negligible variation on E, then $\Re$ is equal to zero, [1].

Definition 3.3. *The linear differential form $dF = fdx$ is a residue function of F. In symbols, $\mathfrak{R} := dF$.*

Obviously, the residue function of F being basically summable (BS_δ) on $E \subset [a,b]$ to a real number $\Re$ is a null function on $[a,b]$ (*A function $F : [a,b] \mapsto \mathbb{R}$ is said to be a null function on $[a,b]$, if the set $\{x \in [a,b] \mid F(x) \neq 0\}$ is a set of Lebesgue measure zero*, see *2.4 Definition* in [1]) and

$$\Re = \sum_{x\in E} \mathfrak{R}(x). \tag{8}$$

On the other hand, for any compact interval $[a,b] \in \mathbb{R}$ the infinite sum $\sum_{x\in[a,b]} \mathfrak{R}(x)$ is in fact the integral of f on $[a,b]$ since the antiderivative F of f is by *Definition 3.2* basically summable (BS_δ) on $[a,b]$ to $\Delta F([a,b])$, so that

$$\Delta F([a,b]) = \sum_{x\in[a,b]} \mathfrak{R}(x). \tag{9}$$

In case when F has a certain number of discontinuities within $[a,b]$, gathered together into the set $E \subset [a,b]$, at which its derivative f can take values $\pm\infty$ or not be defined at all, the sum $\sum_{x\in[a,b]\backslash E} \mathfrak{R}(x)$ reduces to the sum $\sum_{x\in[a,b]} f_{ex}(x)\,dx = \mathcal{R} - \int_a^b f_{ex}(x)\,dx$, since $f_{ex}dx$

is the limit of δF on $[a,b]$. Hence, if we split the sum $\sum_{x\in[a,b]} \mathfrak{R}(x)$ into two sums of $\mathfrak{R}(x)$ over two separate sets $[a,b]\setminus E$ and E, then we finally obtain that

$$\int_a^b f(x)\,dx = \mathcal{R} - \int_a^b f_{ex}(x)\,dx + \mathfrak{R}. \tag{10}$$

In what follows we shall formulate the result (10) as a theorem and prove it explicitly.

Theorem 3.1. *For a compact interval $[a,b] \in \mathbb{R}$ let $E \subset [a,b]$ be a set of Lebesgue measure zero at whose points a real valued function F defined and differentiable on $[a,b]\setminus E$ and its derivative f can take values $\pm\infty$ or not be defined at all. If F is basically summable (BS_δ) on E to $\mathfrak{R}$, then f_{ex} is generalized Riemann integrable on $[a,b]$ and*

$$\int_a^b f(x)\,dx = \mathcal{R} - \int_a^b f_{ex}(x)\,dx + \mathfrak{R}. \tag{11}$$

Proof. Let F_{ex} and f_{ex} be defined by (1). Since F is BS_δ on E to $\mathfrak{R}$ it follows from *Definition 3.2.* that for every $\varepsilon > 0$ there exists a gauge $\delta : [a,b] \mapsto \mathbb{R}_+$, such that $|\Delta F(P[a,b]|_E) - \mathfrak{R}| < \varepsilon$, whenever $P[a,b] \in \mathcal{P}_\delta[a,b]|_E$. In addition, $f_{ex}(x) \equiv 0$ on E and $\Delta F(P[a,b]) = \Delta F([a,b])$ whenever $P[a,b] \in \mathcal{P}[a,b]$. Hence, by the result (6), for every $\varepsilon > 0$ there exists a gauge $\delta : [a,b] \mapsto \mathbb{R}_+$, such that

$$\begin{aligned} &|\delta F(P[a,b]) - [\Delta F([a,b]) - \mathfrak{R}]| \leq \\ &\leq |\delta F(P[a,b]\setminus P[a,b]|_E) - \Delta F(P[a,b]\setminus P[a,b]|_E)| + \\ &+ |\Delta F(P[a,b]|_E) - \mathfrak{R}| < \varepsilon(|[a,b]| + 1), \end{aligned}$$

whenever $P[a,b] \in \mathcal{P}_\delta[a,b]|_E$. So, by *Definition 3.1*, f_{ex} is generalized *Riemann* integrable on $[a,b]$ and $\mathcal{R} - \int_a^b f_{ex}(x)\,dx = \Delta F([a,b]) - \mathfrak{R}$, that is $\int_a^b f(x)\,dx = \mathcal{R} - \int_a^b f_{ex}(x)\,dx + \mathfrak{R}$. □

This result provides an extension of *Cauchy's* result from the calculus of residues in $\mathbb{R}$ (compare with results in [4]).

4. Example

Let $C : [0,1] \mapsto \mathbb{R}$ be the *Cantor* function, [2]. Its derivative c is a null function on $[0,1]$ that is not defined on the *Cantor* set $\mathcal{C}$. Since the generalized *Riemann* integral of $c_{ex} : [0,1] \mapsto 0$ is equal to zero on $[0,1]$ it follows from (10) that $\mathfrak{R} = \int_0^1 c(x)\,dx - \mathcal{R} - \int_0^1 c_{ex}(x)\,dx = \Delta C([0,1]) - 0 = 1$. So, the sum of the changes in the value of C over $\mathcal{C}$ is reduced to the so-called indeterminate expression $\infty \cdot 0$ (the residue function $\mathfrak{R}$ of C vanishes identically on $[0,1]$ because C is continuous on $[0,1]$), that actually have, in this situation, the real numerical value of 1 (it means that C is not absolutely continuous and has no negligible variation on $\mathcal{C}$). Let's prove it once more. For the *Cantor* function with the total length of 2 on $[0,1]$ the total length of all line segments contained within $[0,1]\setminus\mathcal{C}$, on each of which C is constant, is as follows, $\frac{1}{2}\sum_{n=1}^{+\infty}(\frac{2}{3})^n = \frac{1}{2}(3-1) = 1$. Hence, the sum of the changes in the value of C over $\mathcal{C}$ is equal to $2 - 1$.

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