CLASSIFICATION OF 3-DIMENSIONAL LEFT-INVARIANT STATISTICAL LIE GROUPS AND STATISTICAL WALLACH THEOREM

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We first characterize left-invariant statistical structures on Lie groups and determine the dimension of the affine space of all left-invariant statistical connections on an n-dimensional Lie group. Then, we classify all 2- and 3-dimensional left-invariant statistical Lie groups with the Cartan connection. As an application of this classification, we obtain a statistical Wallach theorem.

Keywords: Statistical structure, Codazzi pair, Left-invariant connection.

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1. Introduction

The study of natural geometric structures arising from families of probability distributions is known as information geometry. Indian scientist C. R. Rao introduced this geometry for defining the distance between statistical distributions that remains invariant under nonsingular parametrization transformations [14]. This potent branch of mathematics applies differential geometry methods to the realm of probability theory. The primary focus in this field is on statistical connections and statistical manifolds.

A manifold wherein each point corresponds to a probability distribution is referred to as a statistical manifold. More precisely, a statistical manifold is a triple (M, g, ∇) where (M, g) is a Riemannian manifold, (g, ∇) is a Codazzi pair and ∇ is a without torsion connection on M [13]. These geometric structures have been studied in differential geometry. However, statistical manifolds and dual affine connections were rediscovered in statistics to construct geometric theory for statistical inferences. Information geometry has numerous applications in various research fields such as physics, computer science and machine

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learning, as exemplified in [3, 12, 15].
In the second part of this article, we introduce the necessary definitions and preliminaries. In Section 3, we present the class of left-invariant statistical structures which forms a novel class of statistical structures with potential applications in the field. We provide an equivalent condition for the statistical nature of the pair (g, ∇) on a Lie group G by utilizing the one-to-one correspondence between left-invariant connections ∇ and bilinear maps μ on its Lie algebra $\mathfrak{g}$. In case $\mu(X,Y) = \frac{1}{2}[X,Y]$, we refer to the associated left-invariant connection to μ as Cartan connection. We have the following characterization.

Theorem 1.1. *Let G be a Lie group, g be a left-invariant Riemannian metric, and ∇ be the left-invariant torsion-free connection associated with a bilinear map μ on Lie algebra of G, denoted by $\mathfrak{g}$. Then, (G, g, ∇) is a statistical manifold if and only if*

$$\langle [X,Y], Z\rangle = \langle \mu(Y,Z), X\rangle - \langle Y, \mu(X,Z)\rangle, \quad \forall X,Y,Z \in \mathfrak{g}, \tag{1}$$

where $\langle\, , \,\rangle$ is the inner product induced by g on $\mathfrak{g}$.

Theorem 1.2. *Let G be an Abelian Lie group. For each left-invariant metric g on G, the triple (G, g, ∇) is a left-invariant statistical manifold with $\mu = 0$.*

Subsequently, using the condition (1), we classify all left-invariant statistical structures on 2- and 3-dimensional Lie groups. We demonstrate that a 2-dimensional left-invariant Lie group admits a statistical structure with Cartan connection only if it is Abelian; consequently, 2-dimensional non-Abelian Lie groups do not admit a statistical structure with the Cartan connection.

Theorem 1.3. *Let G be a 2-dimensional non-Abelian Lie group. There is no left-invariant metric g on G such that (G, g, ∇) is left-invariant statistical Lie group, where ∇ is the Cartan connection of G.*

Then, we shift our attention to 3-dimensional Lie groups. Referring to the notations in Table 1, we derive the following classification theorem, which succinctly states that the only 3-dimensional left-invariant Lie groups that possess a statistical structure with Cartan connection are $\mathbb{R}^3$, $SU(2)$ and G_I.

Theorem 1.4. *Let G be a simply connected 3-dimensional Lie group. Then G admits a left-invariant statistical structure with the Cartan connection if it falls into one of the following cases:*

i. $\mathbb{R}^3$.
ii. *The simple Lie group $SU(2)$ for $\gamma = \upsilon = \kappa$.*
iii. *The non-unimodular Lie group G_I.*

Finally, we establish statistical Wallach's Theorem in the context of statistical structures on Lie groups. Specifically, we prove that the only 3-dimensional simply connected Lie group that admits a statistical structure with the Cartan connection and positive sectional curvature is $SU(2)$.

Theorem 1.5. (**Statistical Wallach Theorem**) *The only 3-dimensional simply connected Lie group that admits a left-invariant positively curved statistical structure with Cartan connection is* $SU(2)$.

2. Preliminaries

Let M be a smooth n-dimensional manifold. A Riemannian metric g on M is expressed as a family of maps $(g_p)_{p\in M}$, such that for each $p \in M$, the map $g_p : T_pM \times T_pM \to \mathbb{R}$ is an inner product. If g is a Riemannian metric on M, then the pair (M, g) is called Riemannian manifold. A connection on a smooth manifold M is an $\mathbb{R}$-bilinear map $\nabla : \chi(M) \times \chi(M) \to \chi(M)$ satisfying the following properties:

$$\nabla_{fU}V = f\,\nabla_U V, \quad \nabla_U\, fV = (U \cdot f)\,V + f\,\nabla_U V, \tag{2}$$

for any smooth scalar function $f \in C^\infty(M)$ and any vector fields $U, V \in \chi(M)$. For any Riemannian manifold (M, g) there exists a unique connection ∇ on M satisfying the subsequent properties:

$$U \cdot g(V, W) = g(\nabla_U V, W) + g(V, \nabla_U W), \qquad [U, V] = \nabla_U V - \nabla_V U. \tag{3}$$

This unique connection is called the Levi-Civita connection of (M, g) and will be denoted by ∇^g.
For a group G with an identity element denoted, by e if G is a smooth manifold and the group operation is smooth, then G is called a Lie group. Consider a Lie group G endowed with a Riemannian metric g. If g is such that each left translation of G acts as isometry, then g is called left-invariant Riemannian metric. Similarly, g is called right-invariant if each right translation behaves as an isometry. When g is both left and right-invariant, it is termed the designation of being bi-invariant. A vector field X on a Lie group G is called left-invariant if it remains invariant under every left translation of G. The Lie algebra $\mathfrak{g}$ of a Lie group G is the tangent space T_eG, equipped with a Lie bracket operation defined by

$$[X, Y] = [X^L, Y^L](e), \qquad \forall X, Y \in \mathfrak{g}.$$

Here, X^L and Y^L represent left-invariant vector fields corresponding to X and Y, respectively. For each $X \in \mathfrak{g}$, the mapping $ad(X) : \mathfrak{g} \to \mathfrak{g}$ is defined by $ad(X)(Y) = [X, Y]$.
A one-to-one correspondence between left-invariant metrics on a Lie group G and inner products on Lie algebra $\mathfrak{g}$ of G can be established as follows [6, 8]. Consider an inner product $\langle , \rangle$ on $\mathfrak{g}$ and define the inner product $\langle X_a, Y_a\rangle_a$ on T_aG for all $X_a, Y_a \in T_aG$ and $a \in G$ as follow:

$$\langle X_a, Y_a\rangle_a = \langle (L_{a^{-1}})_{*_a}(X_a), (L_{a^{-1}})_{*_a}(Y_a)\rangle.$$

Let $\langle , \rangle$ be the inner product induced on $\mathfrak{g}$ by left-invariant Riemannian metric g. Then

(i) g is bi-invariant if and only if the following equation is satisfied:

$$\langle [X, Y], Z\rangle = \langle X, [Y, Z]\rangle. \tag{4}$$

(ii) If g is a bi-invariant metric, then the sectional curvature is obtained from the following relation:

$$\mathcal{K}(X, Y) = \frac{1}{4}\frac{\langle [X, Y], [X, Y]\rangle}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2}. \tag{5}$$

If, for each two left-invariant vector field U and V on G, the vector field $\nabla_U V$ is left-invariant, then ∇ is called a left-invariant connection on G. A bi-invariant connection is defined similarly, using both left and right vector fields. It is worth mentioning that Laquer classified bi-invariant affine connections on Lie groups in [10]. Let g be a left-invariant Riemannian metric on a Lie group G. In this case, the Riemannian connection of g is also a left-invariant connection. There exists a one-to-one correspondence between left-invariant connections on a Lie group G, and bilinear maps $\mu : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$. This correspondence is defined as follows [6, 8]. Given a left-invariant connection ∇ on G, we obtain the map $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined by

$$\mu(X, Y) = (\nabla_{X^L} Y^L)_e, \qquad \forall X, Y \in \mathfrak{g}. \tag{6}$$

Every bilinear map $\mu : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ can be decomposed as $\mu = \mu_A + \mu_S$, where μ_A is the anti-symmetric part of μ and μ_S is its symmetric part. The left-invariant connection associated with a bilinear map μ on $\mathfrak{g}$ is torsion-free if and only if

$$\mu(X, Y) - \mu(Y, X) = [X, Y], \quad \forall X, Y \in \mathfrak{g}. \tag{7}$$

Hence

$$\mu_A(X, Y) = \frac{1}{2}[X, Y], \quad \forall X, Y \in \mathfrak{g},$$

therefore,

$$\mu(X, Y) = \frac{1}{2}[X, Y] + \mu_S(X, Y), \quad \forall X, Y \in \mathfrak{g}. \tag{8}$$

Let ∇^g be the Levi-Civita connection of a left-invariant Riemannian metric g on a Lie group G and μ^g be the associated bilinear map to ∇^g. For all left-invariant vector fields X, Y and Z, the famous Koszul formula reads

$$2\langle Z, \nabla_X Y\rangle = \langle Z, [X, Y]\rangle + \langle [Z, X], Y\rangle + \langle [Z, Y], X\rangle. \tag{9}$$

Thus, the symmetric part of μ^g satisfies

$$2\langle Z, \mu^g_S(X, Y)\rangle = \langle [Z, X], Y\rangle + \langle [Z, Y], X\rangle. \tag{10}$$

Remark 2.1. *A symmetric bilinear map μ on a Lie algebra $\mathfrak{g}$ is torsion-free if and only if $\mathfrak{g}$ is Abelian. Thus, considering symmetric bilinear maps μ is very restrictive condition.*

Thus, we merely consider the anti-symmetric torsion-free bilinear map μ with $\mu_S = 0$. In this case, μ is called the Cartan connection of G which is the Levi-Civita connection of any bi-invariant metric on G [4]. The Riemann curvature of the Cartan connection is given by

$$R(X,Y)Z = -\frac{1}{4}[[X,Y],Z], \quad \forall X,Y,Z \in \mathfrak{g}.$$

A pair (g,∇) is called a Codazzi pair on a manifold M, if ∇ is a connection and g is a Riemannian metric such that the covariant derivative ∇g is a totally symmetric tensor field on M. If ∇ is torsion-free, then the pair (g,∇) is called a statistical structure, and the triple (M,g,∇) is called a statistical manifold [2].

Example 2.1. *Consider Riemannian space* $(\mathbb{R}^2, g)$ *where* $g = dx^2 + dy^2$. *Let* $\{e_1, e_2\}$ *be an orthonormal basis and the connection* ∇ *be defined as follows:*

$$\nabla_{e_1}e_1 = e_2, \ \nabla_{e_2}e_2 = 0, \ \nabla_{e_1}e_2 = \nabla_{e_2}e_1 = e_1.$$

Then the triple $(\mathbb{R}^2, g, \nabla)$ *is a non-trivial statistical manifold.*

Let G be a Lie group. Then a statistical structure (g,∇) on G is said to be left-invariant if both g and ∇ are left invariant. In this case, the triple (G,g,∇) is called a left-invariant statistical Lie group. Clearly, for any left-invariant Riemannian metric g on a Lie group G, the triple (G,g,∇^g) is a left-invariant statistical Lie group. Our research shows that very little information is available on statistical Lie groups and very limited research has been done in this area. A connection ∇^* on a manifold M is referred to as the dual of ∇ with respect to a Riemannian metric g on M if the following equation holds for all vector fields $U,V,W \in \chi(M)$:

$$U \cdot g(V,W) = g(\nabla_U V, W) + g(V, \nabla^*_U W). \tag{11}$$

If (g,∇) is a Codazzi pair on a manifold M, then the pair (g,∇^*) constitutes a Codazzi pair on M too and we have the following relationship

$$2\nabla^g = \nabla + \nabla^*. \tag{12}$$

If a Riemannian metric g and a connection ∇ on a Lie group G are left-invariant, then it can be easily verified that ∇^* is also a left-invariant connection on G.

Example 2.2. *Let* $M = \{f(x,\theta) : \theta = (\mu,\sigma) \in \mathbb{R}^2, \ \sigma > 0\}$, *where*

$$f(x,\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} exp(-\frac{|x-\mu|^2}{2\sigma^2}).$$

The Fisher information matrix is given by

$$[g_{ij}(\theta)] = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{bmatrix},$$

where $\theta = (\theta_1, \theta_2) = (\mu, \sigma)$ *and*

$$g_{ij}(\theta) = \int_{-\infty}^{+\infty} f(x,\theta) \frac{\partial lnf(x,\theta)}{\partial\theta_i} \frac{\partial lnf(x,\theta)}{\partial\theta_j} dx.$$

Let $g = \frac{1}{\sigma^2} d\theta_1^2 + \frac{2}{\sigma^2} d\theta_2^2$. *Then* (M, g) *is isometric with* $(\mathbb{H}, h)$*, where*

$$\mathbb{H} = \{(x, y) \in \mathbb{R}^2 |\ y > 0\},$$

and

$$h = \frac{dx^2 + dy^2}{y^2}.$$

Hence, (M, g) *is a left-invariant statistical manifold. Since* $\mathbb{H}$ *has the group structure,* (M, g) *can be regarded as a left-invariant statistical Lie group. Actually, this example served as our inspiration to define and study left-invariant statistical Lie groups.*

3. **Left-Invariant Statistical Lie Groups**

This section delves into the studying and characterizing left-invariant statistical structures on Lie groups. Let (M, g) be a Riemannian manifold. We explore that $\mathcal{A}_{(M,g)}$, the set of all connections ∇ on M such that (M, g, ∇) is a statistical manifold, is an affine space whose associated vector space is $\mathcal{D}^1_2(M)^{ss}$ which is the set of all $(1, 2)$-tensor fields $D : \chi(M) \times \chi(M) \to \chi(M)$ satisfying:

$$D(X, Y) = D(Y, X), \quad g(D(X, Z), Y) = g(X, D(Y, Z)). \tag{13}$$

It is obvious that ∇^g is a special and noteworthy element of $\mathcal{A}_{(M,g)}$. By considering ∇^g as the origin of $\mathcal{A}_{(M,g)}$, one can see that every $\nabla \in \mathcal{A}_{(M,g)}$ can be expressed as $\nabla = \nabla^g + D$, for some $D \in \mathcal{D}^1_2(M)^{ss}$.
Now, let G be a Lie group and g be a left-invariant Riemannian metric on G. In general, suppose that (G, g, ∇) is a left-invariant statistical Lie group. For each $\alpha \in \mathbb{R}$, let us set

$$\nabla^\alpha = (1 - \alpha)\nabla^g + \alpha\nabla.$$

∇^α is called the α-connection [5]. Since $\mathcal{A}_{(G,g)}$ is an affine space, it follows that the triple (G, g, ∇^α) is also a left-invariant statistical Lie group.
A natural question is: **How large is** $\mathcal{A}_{(G,g)}$? To provide an answer to this question, we suppose that ∇ is an arbitrary left-invariant connection which belongs to $\mathcal{A}_{(G,g)}$ and let μ be the bilinear map associated with ∇. Then

$$\mu(X, Y) = \mu^g(X, Y) + D(X, Y) = \frac{1}{2}[X, Y] + \mu^g_S(X, Y) + D(X, Y), \tag{14}$$

where μ^g is the bilinear mapping associated with the Levi-Civita connection of g and $D : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ belongs to $\mathcal{D}^1_2(\mathfrak{g})^{ss}$. The description of elements of $\mathcal{A}_{(G,g)}$ given by (14) tells us that $\mathcal{A}_{(G,g)}$ is the same size as the vector space $\mathcal{D}^1_2(\mathfrak{g})^{ss}$. Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal base for $\mathfrak{g}$ with respect to the inner product $\langle , \rangle$ on $\mathfrak{g}$. For an arbitrary $D \in \mathcal{D}^1_2(\mathfrak{g})^{ss}$, let us define $D_{ijk} = \langle D(e_i, e_j), e_k\rangle$.

Then, conditions proposed in (13) are equivalent to $D_{ijk} = D_{jik}$ and $D_{ijk} = D_{jki}$ for all indices i, j and k. It is a straightforward combinatorial problem to compute the number of all D_{ijk}s satisfying the two conditions. Indeed, finding the dimension of the vector space $\mathcal{D}^1_2(\mathfrak{g})^{ss}$, we obtain the following.

Proposition 3.1. *$\mathcal{A}_{(G,g)}$ is a $\frac{n(n+1)(n+2)}{6}$-dimensional affine space, where n denotes the dimension of the Lie group G.*

Remark 3.1. *Note that Hirica and et al. presented the same result as Proposition 3.1 in Proposition 4 in* [9] *with a different approach.*

Theorem 3.1. *Let G be a Lie group, g a left-invariant Riemannian metric, and ∇ a torsion-free left-invariant connection associated with a bilinear map μ on G. Then, the triple (G, g, ∇) is a left-invariant statistical Lie group if and only if*

$$\langle [X, Y], Z\rangle = \langle \mu(Y, Z), X\rangle - \langle Y, \mu(X, Z)\rangle, \qquad \forall X, Y, Z \in \mathfrak{g}, \tag{15}$$

where $\langle\, ,\, \rangle$ is the inner product induced by g on $\mathfrak{g}$.

Proof. Let (G, g, ∇) be a left-invariant statistical Lie group and X, Y and $Z \in \mathfrak{g}$. Since the pair (g, ∇) is a Codazzi pair, we have

$$(\nabla_{X^L} g)(Y^L, Z^L) = (\nabla_{Y^L} g)(X^L, Z^L), \tag{16}$$

where X^L, Y^L and Z^L represent left-invariant vector fields corresponding to X, Y and Z, respectively. By definition, (16) can be expressed as follows

$$\begin{aligned} X^L \cdot g(Y^L, Z^L) \quad &- \quad g(\nabla_{X^L} Y^L, Z^L) - g(Y^L, \nabla_{X^L} Z^L) \\ &= \quad Y^L \cdot g(X^L, Z^L) - g(\nabla_{Y^L} X^L, Z^L) - g(X^L, \nabla_{Y^L} Z^L). \end{aligned} \tag{17}$$

Since g and vector fields X^L, Y^L and Z^L are left-invariant, it follows from (17)

$$g(\nabla_{X^L} Y^L, Z^L) + g(Y^L, \nabla_{X^L} Z^L) = g(\nabla_{Y^L} X^L, Z^L) + g(X^L, \nabla_{Y^L} Z^L), \tag{18}$$

and

$$g(\nabla_{X^L} Y^L - \nabla_{Y^L} X^L, Z^L) = g(X^L, \nabla_{Y^L} Z^L) - g(Y^L, \nabla_{X^L} Z^L). \tag{19}$$

Since ∇ is torsion-free, (19) gives us

$$g([X^L, Y^L], Z^L) = g(X^L, \nabla_{Y^L} Z^L) - g(Y^L, \nabla_{X^L} Z^L). \tag{20}$$

Evaluating (20) at the point $e \in G$, we have (15).
Conversely, suppose that (15) holds and assume that g is the left-invariant Riemannian metric associated with $\langle\, ,\, \rangle$ and ∇ is the left-invariant torsion-free connection associated with μ. Substituting $\mu = \mu^g + D$ into (15) and taking into account (G, g, ∇^g) is a left-invariant statistical Lie group, we infer that D satisfies

$$\langle D(X, Z), Y\rangle = \langle X, D(Y, Z)\rangle, \;\; \forall X, Y, Z \in \mathfrak{g}. \tag{21}$$

Since ∇ and ∇^g are torsion-free, it follows that D is symmetric. Thus, D satisfies conditions (13), and consequently, (G, g, ∇) is a left-invariant statistical Lie group. This completes the proof. □

For computational purposes, we are going to express (14) in a local coordinates. Let us consider an orthonormal base $\{e_1, e_2, ..., e_n\}$ for $\mathfrak{g}$ with respect to the inner product $\langle , \rangle$ on $\mathfrak{g}$. Since $[e_i, e_j] \in \mathfrak{g}$, there exists a set of real numbers c_{ijk} $(i, j, k = 1, 2, ..., n)$ such that $[e_i, e_j] = \sum_k c_{ijk} e_k$. In [11], J. Milnor proved that if g is a left-invariant Riemannian metric on G and ∇^g is the Levi-Civita connection of g, then

$$\mu^g(e_i, e_j) = \nabla^g_{e_i} e_j = \frac{1}{2} \sum_k (c_{ijk} - c_{jki} + c_{kij}) e_k.$$

Since μ^g is a map from $\mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}$, we have $\mu^g(e_i, e_j) = \sum_k \mu^g_{ijk} e_k$. With this assumption, we have the following result.

Theorem 3.2. *Let (G, g, ∇) be a left-invariant statistical Lie group and let μ be the bilinear mapping corresponding to ∇. If ∇^g is the Levi-Civita connection of g and μ^g is the bilinear mapping corresponding to ∇^g, then*

$$\mu(e_i, e_j) = \frac{1}{2} \sum_k \{(\mu^g_{jki} - \mu^g_{ikj}) - (\mu^g_{kij} - \mu^g_{jik}) + (\mu^g_{ijk} - \mu^g_{kji}) + 2D_{ijk}\} e_k, \quad (22)$$

where $D_{ijk} = \langle D(e_i, e_j), e_k \rangle$.

Proof. By equation (14), we have

$$\sum_k \mu_{ijk} e_k = \mu(e_i, e_j) = \mu^g(e_i, e_j) + D(e_i, e_j) = \sum_k (\mu^g_{ijk} + D_{ijk}) e_k.$$

Since μ^g is the Levi-Civita connection of g and $\{e_1, e_2, ..., e_n\}$ is an orthonormal basis for $\mathfrak{g}$, we have

$$\mu^g(e_i, e_j) = \frac{1}{2} \sum_k (c_{ijk} - c_{jki} + c_{kij}) e_k. \quad (23)$$

Since the pair (g, ∇^g) is a statistical structure on the Lie group G, we have

$$\begin{aligned} c_{ijk} &= \langle [e_i, e_j], e_k \rangle \\ &= \langle \mu^g(e_j, e_k), e_i \rangle - \langle e_j, \mu^g(e_i, e_k) \rangle \\ &= \mu^g_{jki} - \mu^g_{ikj}. \end{aligned}$$

Similarly, we have

$$c_{kij} = \mu^g_{ijk} - \mu^g_{kji},$$

and

$$c_{jki} = \mu^g_{kij} - \mu^g_{jik}.$$

Hence

$$\begin{aligned} \mu(e_i, e_j) &= \frac{1}{2} \sum_k (c_{ijk} - c_{jki} + c_{kij} + 2D_{ijk}) e_k \\ &= \frac{1}{2} \sum_k \{(\mu^g_{jki} - \mu^g_{ikj}) - (\mu^g_{kij} - \mu^g_{jik}) + (\mu^g_{ijk} - \mu^g_{kji}) + 2D_{ijk}\} e_k. \end{aligned}$$

□

Example 3.1. [5] *Consider the left-invariant statistical Lie group* (G, g, ∇), *where* $G = \{(x, y) \in \mathbb{R}^2 |\ y > 0\}$ *and* $g = \frac{dx^2 + 2dy^2}{\frac{2}{y}}$. *Let* $\{e_1, e_2\}$ *be an orthonormal basis for* $\mathfrak{g}$ *such that the Levi-Civita connection of* g *in this basis is given as follows:*

$$\nabla^g_{e_1} e_1 = \frac{1}{\sqrt{2}} e_2,\ \nabla^g_{e_1} e_2 = -\frac{1}{\sqrt{2}} e_1,\ \nabla^g_{e_2} e_1 = \nabla^g_{e_2} e_2 = 0.$$

If $D : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *is defined by*

$$D(e_1, e_1) = -\frac{\sqrt{2}}{2} e_2,\ D(e_1, e_2) = D(e_2, e_1) = -\frac{\sqrt{2}}{2} e_1,\ D(e_2, e_2) = -\sqrt{2} e_2,$$

then we have

$$\mu^g_{212} = \mu^g_{122} = \mu^g_{111} = \mu^g_{211} = \mu^g_{221} = \mu^g_{222} = 0,\ \mu^g_{121} = -\frac{1}{\sqrt{2}},\ \mu^g_{112} = \frac{1}{\sqrt{2}},$$

and

$$D_{212} = D_{111} = D_{221} = D_{122} = 0,\ D_{112} = D_{121} = D_{211} = -\frac{\sqrt{2}}{2},\ D_{222} = -\sqrt{2}.$$

Hence

$$\begin{aligned}
\nabla_{e_1} e_1 &= \mu(e_1, e_1) \\
&= \frac{1}{2} \sum_k \{(\mu^g_{1k1} - \mu^g_{1k1}) - (\mu^g_{k11} - \mu^g_{11k}) + (\mu^g_{11k} - \mu^g_{k11}) + 2D_{11k}\} e_k \\
&= D_{111} e_1 + \{(\mu^g_{112} - \mu^g_{211}) + D_{112}\} e_2 = 0.
\end{aligned}$$

$$\begin{aligned}
\nabla_{e_1} e_2 &= \mu(e_1, e_2) \\
&= \frac{1}{2} \sum_k \{(\mu^g_{2k1} - \mu^g_{1k2}) - (\mu^g_{k12} - \mu^g_{21k}) + (\mu^g_{12k} - \mu^g_{k21}) + 2D_{12k}\} e_k \\
&= \{(\mu^g_{211} - \mu^g_{112}) + D_{121}\} e_1 + D_{122} e_2 = -\sqrt{2} e_1.
\end{aligned}$$

Similarly, we have $\nabla_{e_2} e_1 = -\frac{1}{\sqrt{2}} e_1$ *and* $\nabla_{e_2} e_2 = -\frac{2}{\sqrt{2}} e_2$.

Corollary 3.1. *With the above assumptions, if* $\mathcal{K}$ *is the sectional curvature of* μ, *then*

$$\begin{aligned}
\mathcal{K}(e_i, e_j) &= \frac{1}{2} \sum_k \{(\mu^g_{jjk} - \mu^g_{kjj}) + D_{jjk}\} \sum_t \{(\mu^g_{ktt} - \mu^g_{ttk}) + D_{tkt}\} \\
&- \frac{1}{4} \sum_k \{(\mu^g_{jki} - \mu^g_{ikj}) - (\mu^g_{kij} - \mu^g_{jik}) + (\mu^g_{ijk} - \mu^g_{kji}) + 2D_{ijk}\} \\
&\times \sum_t \{(\mu^g_{ktj} - \mu^g_{jtk}) - (\mu^g_{tjk} - \mu^g_{kjt}) + (\mu^g_{jkt} - \mu^g_{tkj}) + 2D_{jkt}\} \\
&- \frac{1}{2} \sum_k (\mu^g_{jki} - \mu^g_{ikj}) \sum_t \{(\mu^g_{jtk} - \mu^g_{ktj}) - (\mu^g_{tkj} - \mu^g_{jkt}) + (\mu^g_{kjt} - \mu^g_{tjk}) + 2D_{kjt}\}.
\end{aligned}$$

Proof. We know that

$$\begin{aligned}\mathcal{K}(e_i,e_j) &= \langle R(e_i,e_j)e_j,e_i\rangle\\ &= \langle \nabla_{e_i}\nabla_{e_j}e_j,e_i\rangle - \langle \nabla_{e_j}\nabla_{e_i}e_j,e_i\rangle - \langle \nabla_{[e_i,e_j]}e_j,e_i\rangle\\ &= \langle \mu(e_i,\mu(e_j,e_j)),e_i\rangle - \langle \mu(e_j,\mu(e_i,e_j)),e_i\rangle - \langle \mu([e_i,e_j],e_j),e_i\rangle.\end{aligned}$$

Now, by using equation (22) and performing a straightforward calculation, the result is obtained. □

Example 3.2. *With the notations of Example 3.1, we have*

$$\begin{aligned}\mathcal{K}(e_1,e_2) &= \frac{1}{2}\sum_k\{(\mu^g_{22k}-\mu^g_{k22})+D_{22k}\}\sum_t\{(\mu^g_{ktt}-\mu^g_{ttk})+D_{tkt}\}\\ &- \frac{1}{4}\sum_k\{(\mu^g_{2k1}-\mu^g_{1k2})-(\mu^g_{k12}-\mu^g_{21k})+(\mu^g_{12k}-\mu^g_{k21})+2D_{12k}\}\\ &\times \sum_t\{(\mu^g_{kt2}-\mu^g_{2tk})-(\mu^g_{t2k}-\mu^g_{k2t})+(\mu^g_{2kt}-\mu^g_{tk2})+2D_{2kt}\}\\ &- \frac{1}{2}\sum_k(\mu^g_{2k1}-\mu^g_{1k2})\sum_t\{(\mu^g_{2tk}-\mu^g_{kt2})-(\mu^g_{tk2}-\mu^g_{2kt})+(\mu^g_{k2t}-\mu^g_{t2k})+2D_{k2t}\}\\ &= \frac{1}{2}\{(\mu^g_{221}-\mu^g_{122})+D_{221}\}\{(\mu^g_{122}-\mu^g_{221})+D_{212}\}+\frac{1}{2}D_{222}\{(\mu^g_{211}-\mu^g_{112})+D_{121}\}\\ &- \{(\mu^g_{211}-\mu^g_{112})+D_{121}\}\{D_{211}+(\mu^g_{122}-\mu^g_{221})+D_{212}\}\\ &- \{(\mu^g_{221}-\mu^g_{122})+D_{122}\}\{(\mu^g_{221}-\mu^g_{122})+D_{221}+D_{222}\}\\ &- \{(\mu^g_{211}-\mu^g_{112})\}\{(\mu^g_{211}-\mu^g_{112})+D_{121}+D_{122}\}\\ &- \{(\mu^g_{221}-\mu^g_{122})\}\{(\mu^g_{221}-\mu^g_{122})+D_{221}+D_{222}\}\\ &= -2+\sqrt{2}.\end{aligned}$$

Proposition 3.2. *Let G be an Abelian Lie group. If $\mu\equiv 0$, then for each left-invariant metric g on G, the triple (G,g,∇) is a left-invariant statistical Lie group.*

Proof. Let G be an Abelian Lie group. Then $\mathfrak{g}$ is Abelian, and hence $[X,Y]=0$, for all $X,Y\in\mathfrak{g}$. Now, if $\mu\equiv 0$, then the equation (15) holds trivially. □

Up to isomorphism, there exists a unique 2-dimensional non-Abelian Lie algebra. Indeed, any 2-dimensional non-Abelian Lie algebra $\mathfrak{g}$ has a basis $\{X,Y\}$ with the bracket given by $[X,Y]=X$.

Theorem 3.3. *Let G be a 2-dimensional non-Abelian Lie group and ∇ be the Cartan connection on G. Then, there is no left-invariant metric g on G such that (G,g,∇) is a left-invariant statistical Lie group.*

Proof. Let G be a 2-dimensional non-Abelian Lie group and suppose that $\langle,\rangle$ is an inner product on $\mathfrak{g}$ such that (G,g,∇) is a left-invariant Lie group, where g is the left-invariant Riemannian metric on G induced by $\langle,\rangle$ and ∇ is the

Cartan connection on G. Suppose that $\{X, Y\}$ is an orthonormal basis with respect to $\langle , \rangle$ for $\mathfrak{g}$ such that $[X, Y] = X$. For every vector $Z = \lambda X + \eta Y \in \mathfrak{g}$ with non-zero $\lambda \in \mathbb{R}$, we have

$$\langle [X, Y], Z \rangle = \lambda, \quad \langle \mu(Y, Z), X \rangle - \langle Y, \mu(X, Z) \rangle = -\frac{1}{2}\lambda. \tag{24}$$

By (15), we must have $\lambda = 0$ which is a contradiction. This completes the proof. □

Corollary 3.2. *Let G be a 2-dimensional Lie group and ∇ be the Cartan connection on G. Then, G admits a left-invariant statistical structure (g, ∇) if and only if G is an Abelian group.*

4. Classification of Left-Invariant 3-Dimensional Lie Groups

Now, we deal with 3-dimensional Lie groups. In the context of 3-dimensional Lie groups, the work of Ha-Bumlee in [7], focused on 3-dimensional Lie algebras and the classification of Left-invariant Riemannian metrics on simply connected 3-dimensional Lie groups. Considering a basis $\{x, y, z\}$ for a 3-dimensional Lie algebra, it can be shown to be isometric isomorphic to one of the presented Lie algebras endowed with the given inner product in Table 1 (for more details see Table 1). In this section, building upon Ha-Bumlee's findings and utilizing (15), we classify all left-invariant 3-dimensional Lie groups endowed with the Cartan connection. This classification is essential in delineating the scope of the class of left-invariant statistical Lie groups.

Theorem 4.1. *Let G be a simply connected 3-dimensional Lie group and ∇ be the Cartan connection on G. Then, G admits a left-invariant statistical structure (g, ∇) if and only if it falls into one of the following cases:*

i. $\mathbb{R}^3$.
ii. *The simple Lie group $SU(2)$ for $\gamma = \upsilon = \kappa$.*
iii. *The non-unimodular Lie group G_I.*

Proof. It is sufficient to examine the conditions under which each of the cases in Table 1 satisfies the equation (15). It is obvious that in Case 1, $G = \mathbb{R}^3$ admits a left-invariant statistical structure.
As for Case 7, based on the information provided in Table 1, we have

$$\langle\, [e_1, e_2], e_3 \,\rangle = \upsilon, \tag{25}$$

$$\langle\, \mu(e_2, e_3), e_1 \,\rangle - \langle\, e_2, \mu(e_1, e_3) \,\rangle = \langle\, \frac{1}{2}\,[e_2, e_3], e_1 \,\rangle - \langle\, e_2, \frac{1}{2}\,[e_1, e_3] \,\rangle = \frac{1}{2}\,\gamma + \frac{1}{2}\,\kappa. \tag{26}$$

Similarly, we have

$$\langle\, [e_3, e_1], e_2 \,\rangle = \kappa, \quad \langle\, \mu(e_1, e_2), e_3 \,\rangle - \langle\, e_1, \mu(e_3, e_2) \,\rangle = \frac{1}{2}\,\upsilon + \frac{1}{2}\,\gamma. \tag{27}$$

TABLE 1. Euclidean 3-dimensional Lie algebras.

Case	Algebra structure	Associated simply connected Lie group	Left-invariant Riemannian metric
1	$[e_1,e_2]=0$ $[e_1,e_3]=0$ $[e_2,e_3]=0$	$\mathbb{R}^3$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
2	$[e_1,e_2]=e_3$ $[e_1,e_3]=0$ $[e_2,e_3]=0$	The Heisenberg group Nil	$\begin{pmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{pmatrix}$ $\gamma>0$
3	$[e_1,e_2]=0$ $[e_1,e_3]=-e_1$ $[e_2,e_3]=e_2$	The solvable Lie group Sol	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & \upsilon \end{pmatrix}$ $\upsilon>0$
4	$[e_1,e_2]=0$ $[e_1,e_3]=-e_1$ $[e_2,e_3]=e_2$	The solvable Lie group Sol	$\begin{pmatrix} 1 & 1 & 0 \\ 1 & \kappa & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $\kappa>1$ $\upsilon>0$
5	$[e_1,e_2]=0$ $[e_1,e_3]=e_2$ $[e_2,e_3]=-e_1$	The solvable Lie group $\tilde{E}_0(2)$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $0<\kappa\leq 1$ $\upsilon>0$
6	$[e_1,e_2]=2e_3$ $[e_1,e_3]=-2e_2$ $[e_2,e_3]=-2e_1$	The simple Lie group $P\tilde{S}L(2,\mathbb{R})$	$\begin{pmatrix} \gamma & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $\kappa\geq\upsilon>0$ $\gamma>0$
7	$[e_1,e_2]=e_3$ $[e_1,e_3]=-e_2$ $[e_2,e_3]=e_1$	The simple Lie group $SU(2)$	$\begin{pmatrix} \gamma & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $\gamma\geq\kappa\geq\upsilon$ $\upsilon>0$
8	$[e_1,e_2]=0$ $[e_1,e_3]=-e_1$ $[e_2,e_3]=-e_2$	The non-unimodular Lie group G_I	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $\upsilon>0$
9	$[e_1,e_2]=0$ $[e_1,e_3]=-e_2$ $[e_2,e_3]=ce_1-2e_2$	The non-unimodular Lie group G_c	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $0<\kappa\leq\lvert c\rvert$ $\upsilon>0$
10	$[e_1,e_2]=0$ $[e_1,e_3]=-e_2$ $[e_2,e_3]=ce_1-2e_2$	The non-unimodular Lie group G_c	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $\kappa>0$ $\upsilon>0$ $c=0$
11	$[e_1,e_2]=0$ $[e_1,e_3]=-e_2$ $[e_2,e_3]=ce_1-2e_2$	The non-unimodular Lie group G_c	$\begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $\upsilon>0$ $c=0$
12	$[e_1,e_2]=0$ $[e_1,e_3]=-e_2$ $[e_2,e_3]=ce_1-2e_2$	The non-unimodular Lie group G_c	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $0<\kappa\leq 1$ $c=1$ $\upsilon>0$
13	$[e_1,e_2]=0$ $[e_1,e_3]=-e_2$ $[e_2,e_3]=ce_1-2e_2$	The non-unimodular Lie group G_c	$\begin{pmatrix} 1 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $0<\kappa\leq 1$ $c=1$ $\upsilon>0$ $0<\gamma<1$
14	$[e_1,e_2]=0$ $[e_1,e_3]=-e_2$ $[e_2,e_3]=ce_1-2e_2$	The non-unimodular Lie group G_c	$\begin{pmatrix} 1 & 1 & 0 \\ 1 & \kappa & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $0<\kappa\leq c$ $c>0$ $\upsilon>0$
15	$[e_1,e_2]=0$ $[e_1,e_3]=-e_2$ $[e_2,e_3]=ce_1-2e_2$	The non-unimodular Lie group G_c	$\begin{pmatrix} 1 & \kappa & 0 \\ \kappa & 1 & 0 \\ 0 & 0 & \upsilon \end{pmatrix}$ $0<\kappa\leq 1$ $\upsilon>0$ $\gamma\pi=\sqrt{1-c}$

Also, we have

$$\langle\, [e_2,e_3],e_1 \,\rangle=\gamma, \quad \langle\, \mu(e_3,e_1),e_2 \,\rangle-\langle\, e_3,\mu(e_2,e_1) \,\rangle=\frac{1}{2}\,\kappa+\frac{1}{2}\,\upsilon. \tag{28}$$

By equating these three relations, we conclude that $G = SU(2)$ admits a left-invariant statistical structure (g, ∇) provided that $\gamma = \kappa = \upsilon$.
The Case 8, the non-unimodular Lie group G_I, can be proven with a similar argument presented for Case 7. It is a direct computation, to show that other cases given in Table 1 do not satisfy (15). Thus, we get the proof. □

The celebrated Wallach theorem states that the only 3-dimensional simply connected Lie group admitting a left-invariant Riemannain metric whose sectional curvature is positive, is $SU(2)$ [11]. Among Cases 1, 7 and 8 in Table 1, only Case 7 is a simply connected compact Lie group. This observation leads us to a Wallach type theorem for left-invariant statistical Lie groups.

Theorem 4.2. *(***Statistical Wallach Theorem***) The only 3-dimensional simply connected Lie group with the Cartan connection that admits a left-invariant statistical structure with positive sectional curvature is $SU(2)$.*

Proof. A straightforward calculation demonstrates that the left-invariant metric on $SU(2)$ given in Table 1 with $\gamma = \kappa = \upsilon$ satisfies the following relation:

$$\langle [e_1, e_2], e_3 \rangle = \langle e_1, [e_2, e_3] \rangle.$$

Therefore, this left-invariant metric on $SU(2)$ is a bi-invariant metric on $SU(2)$ and its sectional curvature in the direction of the plane generated by the vectors e_1 and e_2 is obtained from the following relation:

$$\mathcal{K}(e_1, e_2) = \frac{1}{4} \frac{\langle [e_1, e_2], [e_1, e_2] \rangle}{\langle e_1, e_1 \rangle \langle e_2, e_2 \rangle - \langle e_1, e_2 \rangle^2} = \frac{1}{4} \frac{\upsilon^2}{\gamma\kappa} > 0. \tag{29}$$

Similarly, we have $\mathcal{K}(e_1, e_3) = \frac{1}{4}\frac{\kappa^2}{\gamma\upsilon}$ and $\mathcal{K}(e_2, e_3) = \frac{1}{4}\frac{\gamma^2}{\kappa\upsilon}$. Thus, the left-invariant Riemannain metric on $SU(2)$ defined in Table 1 with $\gamma = \kappa = \upsilon$ has positive constant sectional curvature $\frac{1}{4}$. Moreover, $(SU(2), g, \nabla)$ is a left -invariant statistical Lie group. This completes the proof. □

5. Conclusions

We first characterized left-invariant statistical structures on Lie groups and determined the dimension of the affine space of all left-invariant statistical connections on an n-dimensional Lie group. Then, we classified all 2- and 3-dimensional left-invariant statistical Lie groups with the Cartan connection. As an application of this classification, we obtained a statistical Wallach theorem.

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