A COMPLETELY MONOTONIC FUNCTION RELATED TO THE q-TRIGAMMA FUNCTION

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In the paper, a function related to the q-trigamma function is proved to be completely monotonic. In order to prove this main result, two functions related to the logarithmic function are found to be completely monotonic.

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1. Introduction

The classical Euler gamma function $\Gamma(x)$ may be defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \,\mathrm{d} t = \frac{1}{x} \prod_{n=1}^\infty \left\{ \left(1 + \frac{1}{n}\right)^x \left(1 + \frac{x}{n}\right)^{-1} \right\} \tag{1.1}$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, is called the psi or digamma function, and the derivatives $\psi^{(i)}(x)$ for $i \in \mathbb{N}$, the set of all positive integers, are respectively called the polygamma functions. In particular, the functions $\psi'(x)$ and $\psi''(x)$ are called the trigamma and tetragamma functions.

The q-analogue $\Gamma_q(x)$ of the gamma function $\Gamma(x)$ may be defined for $x > 0$ by

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{i=0}^\infty \frac{1-q^{i+1}}{1-q^{i+x}} \tag{1.2}$$

when $0 < q < 1$, and by

$$\Gamma_q(x) = (q-1)^{1-x} q^{\binom{x}{2}} \prod_{i=0}^\infty \frac{1-q^{-(i+1)}}{1-q^{-(i+x)}} \tag{1.3}$$

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when $q > 1$. The q-psi function $\psi_q(x)$, the q-analogue of the psi function $\psi(x)$, may be defined by

$$\begin{aligned}\psi_q(x) = \frac{\Gamma_q'(x)}{\Gamma_q(x)} &= -\ln(1-q) + \ln q \sum_{k=0}^{\infty} \frac{q^{k+x}}{1-q^{k+x}} \\ &= -\ln(1-q) + \ln q \sum_{k=1}^{\infty} \frac{q^{kx}}{1-q^{k}}\end{aligned} \tag{1.4}$$

for $0 < q < 1$ and $x > 0$, and by

$$\psi_q(x) = -\ln(q-1) + \ln q\left(x - \frac{1}{2} - \sum_{n\ge 0} \frac{q^{-n-x}}{1-q^{-n-x}}\right) \tag{1.5}$$

for $q > 1$ and $x > 0$. The functions $\psi_q^{(k)}(x)$, the q-analogues of the polygamma functions $\psi^{(k)}(x)$, for $k \in \mathbb{N}$ are called the q-polygamma functions. For detailed information about the above formulas, see [2, 5, 6, 8, 11] and closely related references therein.

The above mentioned functions satisfy the following relations

$$\lim_{q\to 1^{\pm}} \Gamma_q(z) = \Gamma(z), \qquad \Gamma_q(x) = q^{\binom{x-1}{2}}\Gamma_{1/q}(x), \qquad \lim_{q\to 1^{\pm}} \psi_q(x) = \psi(x). \tag{1.6}$$

For more information, please refer to [2, pp. 493–496].

We recall from [7, Chapter XIII] and [16, Chapter IV] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and $(-1)^n f^{(n)}(x) \ge 0$ for $x \in I$ and $n \ge 0$. In [16, p. 161, Theorem 12b], it was stated that a necessary and sufficient condition that $f(x)$ should be completely monotonic for $0 < x < \infty$ is that $f(x) = \int_0^\infty e^{-xt}\,\mathrm{d}\alpha(t)$, where $\alpha(t)$ is non-decreasing and the integral converges for $0 < x < \infty$. In other words, a function is completely monotonic on $(0,\infty)$ if and only if it is a Laplace transform.

For $x > 0$, let

$$f(x) = \psi'(x) - \frac{1}{x} - \frac{1}{2x^2}. \tag{1.7}$$

For $x > 0$ and $0 < q < 1$, let

$$f_q(x) = \psi_q'(x) - \frac{(1-q)q^x}{1-q^x} - \frac{1}{2}\left[\frac{(1-q)q^x}{1-q^x}\right]^2. \tag{1.8}$$

It is clear that $\lim_{q\to 1^-} f_q(x) = f(x)$. So, we may regard $f_q(x)$ as the q-analogue of the function $f(x)$.

In recent years, the complete monotonicity of the function (1.7) was proved, generalized, and applied in [1, 3, 5, 6, 8, 11], [9, Theorem 1.1], [12, Theorem 1.3], [13, pp. 1977–1978], [14, Theorem 2]. For more information on this topic, please refer to related texts in the survey articles [10, 15] and closely related references therein.

The goal of this paper is to prove the complete monotonicity of $f_q(x)$ for $0 < q < 1$ on $(0,\infty)$. Our main result may be stated as the following theorem.

Theorem 1.1. *For $0 < q < 1$, the function $f_q(x)$ defined by* (1.8) *is completely monotonic on* $(0,\infty)$.

2. Lemmas

To prove our main result, we need the following lemmas.

Lemma 2.1. *For $i \in \mathbb{N}$ and $q \in (0,1)$, we have*

$$\psi_q^{(i)}(x) = (\ln q)^{i+1} \sum_{k=1}^{\infty} \frac{k^i q^{kx}}{1-q^k}. \tag{2.1}$$

Proof. This follows from the definition of $\psi_q(x)$ by (1.4), direct differentiation, and the induction. □

Lemma 2.2. *For $q \in (0,1)$ and $x \in (0,\infty)$, we have*

$$\sum_{k=1}^{\infty} k q^{(k+1)x} = \frac{q^{2x}}{(1-q^x)^2}. \tag{2.2}$$

Proof. This can be deduced from the series expansion

$$\frac{1}{(1-x)^2} = \sum_{i=0}^{\infty} (i+1) x^i \tag{2.3}$$

for $x \in (0,1)$ and replacement of x by q^x in (2.3). □

Lemma 2.3. *For $0 < q < 1$ and $x \in (0,\infty)$, we have*

$$\psi_q'(x) - \psi_q'(x+1) = (\ln q)^2 \sum_{k=1}^{\infty} k q^{kx}. \tag{2.4}$$

Proof. By Lemma 2.1 for $i = 1$, we have

$$\begin{aligned}\psi_q'(x) - \psi_q'(x+1) &= (\ln q)^2 \sum_{k=1}^{\infty} \frac{k q^{kx}}{1-q^k} - (\ln q)^2 \sum_{k=1}^{\infty} \frac{k q^k q^{kx}}{1-q^k} \\ &= (\ln q)^2 \sum_{k=1}^{\infty} \frac{k q^{kx}(1-q^k)}{1-q^k} = (\ln q)^2 \sum_{k=1}^{\infty} k q^{kx}.\end{aligned}$$

Lemma 2.3 is thus proved. □

Remark 2.1. In [4, p. 1245, Theorem 4.4], the identity

$$\psi_q^{(k-1)}(x+1) - \psi_q^{(k-1)}(x) = -\frac{\mathrm{d}^{k-1}}{\mathrm{d} x^{k-1}} \left(\frac{q^x}{1-q^x} \right) \ln q \tag{2.5}$$

for $x \in (0,\infty)$ and $k \in \mathbb{N}$ was deduced. It is not difficult to see that the identity (2.4) is a special case of (2.5).

Lemma 2.4. *For $0 < q < 1$ and $i \in \mathbb{N}$, the limit $\lim_{x\to\infty} [f_q(x)]^{(i-1)} = 0$ is valid, where $f_q(x)$ is defined by (1.8).*

Proof. It is apparent that $\lim_{x\to\infty} f_q(x) = 0$.

Differentiating and making use of (2.1) and (2.2) result in

$$[f_q(x)]^{(i)} = \psi_q^{(i+1)}(x) - \left[\frac{(1-q)q^x}{1-q^x} \right]^{(i)} - \left[\frac{(1-q)^2 q^{2x}}{2(1-q^x)^2} \right]^{(i)}$$

$$= \psi_q^{(i+1)}(x) - (1-q)\left[\sum_{\ell=0}^{\infty} q^{x(\ell+1)}\right]^{(i)} - \frac{(1-q)^2}{2}\left[\sum_{\ell=0}^{\infty}(\ell+1)q^{x(\ell+2)}\right]^{(i)}$$

$$= \psi_q^{(i+1)}(x) - (1-q)(\ln q)^i \sum_{\ell=0}^{\infty}(\ell+1)^i q^{x(\ell+1)}$$

$$- \frac{(1-q)^2}{2}(\ln q)^i \sum_{\ell=0}^{\infty}(\ell+1)(\ell+2)^i q^{x(\ell+2)} \to 0$$

as $x \to \infty$ for $0 < q < 1$. The proof of Lemma 2.4 is complete. □

Lemma 2.5. *The function* $h(t) = (\ln t)^2 + t(t-1)(t-2)\ln t + \frac{1}{2}(t-1)^3$ *is completely monotonic on* $(0,1]$.

Proof. A straightforward computation yields

$$\begin{aligned}
\left[(\ln t)^2\right]^{(i)} &= \frac{(-1)^{i-1}2(i-1)!\ln t}{t^i} + \sum_{k=1}^{i-1}\frac{(-1)^i i!}{k(i-k)}\frac{1}{t^i} \\
&= \frac{(-1)^{i-1}2(i-1)!\ln t}{t^i} + \frac{(-1)^i 2(i-1)!}{t^i}\sum_{k=1}^{i-1}\frac{1}{k} \\
&= \frac{(-1)^i 2(i-1)!}{t^i}\left[\sum_{k=1}^{i-1}\frac{1}{k} - \ln t\right], \\
[t(t-1)(t-2)\ln t]' &= t^2 - 3t + 2 + \left(3t^2 - 6t + 2\right)\ln t, \\
[t(t-1)(t-2)\ln t]'' &= 6(t-1)\ln t + 5t + \frac{2}{t} - 9, \\
[t(t-1)(t-2)\ln t]^{(3)} &= 11 - \frac{2}{t^2} - \frac{6}{t} + 6\ln t, \\
[t(t-1)(t-2)\ln t]^{(i+3)} &= \frac{(-1)^{i+1}2(i+1)!}{t^{i+2}} + \frac{(-1)^{i+1}6i!}{t^{i+1}} + \frac{(-1)^{i-1}6(i-1)!}{t^i} \\
&= \frac{(-1)^{i+1}2(i-1)![i(i+1) + 3it + 3t^2]}{t^{i+2}}.
\end{aligned}$$

Accordingly,

$$\begin{aligned}
h'(t) &= \frac{5t^2}{2} - 6t + \frac{7}{2} + \left(3t^2 - 6t + \frac{2}{t} + 2\right)\ln t, \\
h''(t) &= \frac{8t^3 - 12t^2 + 2t + 2 + 2\left(3t^3 - 3t^2 - 1\right)\ln t}{t^2}, \\
h^{(3)}(t) &= \frac{14t^3 - 6t^2 - 2t - 6 + \left(6t^3 + 4\right)\ln t}{t^3}, \\
h^{(i+3)}(t) &= \frac{(-1)^{i+1}2(i+2)!}{t^{i+3}}\left[\sum_{k=1}^{i+2}\frac{1}{k} - \ln t\right] \\
&\quad + \frac{(-1)^{i+1}2(i-1)![i(i+1) + 3it + 3t^2]}{t^{i+2}}
\end{aligned}$$

$$= \frac{(-1)^{i+1}2(i+2)!}{t^{i+3}}\left[\sum_{k=1}^{i+2}\frac{1}{k} - \ln t + \frac{i(i+1)t + 3it^2 + 3t^3}{i(i+1)(i+2)}\right]$$

for $i \in \mathbb{N}$. It is clear that

$$(-1)^{i+3}h^{(i+3)}(t) = \frac{2(i+2)!}{t^{i+3}}\left[\sum_{k=1}^{i+2}\frac{1}{k} - \ln t + \frac{i(i+1)t + 3it^2 + 3t^3}{i(i+1)(i+2)}\right] > 0 \tag{2.6}$$

on the interval $(0,1]$ for $i \in \mathbb{N}$. This implies that $h^{(3)}(t)$ is strictly increasing on $(0,1]$. From $h^{(3)}(1) = h''(1) = h'(1) = h(1) = 0$, we obtain $h^{(3)}(t) \le 0$, $h''(t) \ge 0$, $h'(t) \le 0$, and $h(t) \ge 0$ on $(0,1]$. In conclusion, the function $h(t)$ is completely monotonic on $(0,1]$. Lemma 2.5 is proved. □

Lemma 2.6. *The function* $p(t) = (\ln t)^2 + (t-2)(t-1)^2$ *is completely monotonic on* $(0,1]$.

Proof. Direct differentiation gives $p'(t) = 5 - 8t + 3t^2 + \frac{2\ln t}{t}$,

$$p''(t) = \frac{2}{t^2} + 6t - 8 - \frac{2\ln t}{t^2}, \quad p^{(3)}(t) = 6 - \frac{6}{t^3} + \frac{4\ln t}{t^3},$$

and

$$p^{(i+3)}(t) = \left[(\ln t)^2\right]^{(i+3)} = \frac{(-1)^{i+1}2(i+2)!}{t^{i+3}}\left[\sum_{k=1}^{i+2}\frac{1}{k} - \ln t\right]$$

for $i \in \mathbb{N}$. For $t \in (0,1]$, it is obvious that

$$(-1)^{i+3}p^{(i+3)}(t) = \frac{2(i+2)!}{t^{i+3}}\left[\sum_{k=1}^{i+2}\frac{1}{k} - \ln t\right] > 0, \quad i \in \mathbb{N}.$$

This implies that $p^{(3)}(t)$ is strictly increasing on $(0,1]$. From $p^{(3)}(1) = p''(1) = p'(1) = p(1) = 0$, it is derived that $p^{(3)}(t) \le 0$, $p''(t) \ge 0$, $p'(t) \le 0$, and $p(t) \ge 0$ on $(0,1]$. In a word, the function $p(t)$ is completely monotonic on $(0,1]$. The proof of Lemma 2.6 is complete. □

3. **Proof of Theorem 1.1**

Now it is time to supply a proof of Theorem 1.1.

Direct calculation and utilization of Lemmas 2.2 and 2.3 yield

$$\begin{aligned}
f_q(x) - f_q(x+1) &= \psi_q'(x) - \psi_q'(x+1) - \frac{(1-q)q^x}{1-q^x} - \frac{1}{2}\left[\frac{(1-q)q^x}{1-q^x}\right]^2 \\
&\quad + \frac{(1-q)q^{x+1}}{1-q^{x+1}} + \frac{1}{2}\left[\frac{(1-q)q^{x+1}}{1-q^{x+1}}\right]^2 \\
&= (\ln q)^2\sum_{k=1}^{\infty}kq^{kx} + (1-q)q^x\left(\frac{q}{1-q^{x+1}} - \frac{1}{1-q^x}\right) \\
&\quad + \frac{1}{2}(1-q)^2q^{2x}\left[\frac{q^2}{(1-q^{x+1})^2} - \frac{1}{(1-q^x)^2}\right] \\
&= (\ln q)^2\sum_{k=1}^{\infty}kq^{kx} + (1-q)q^x\left[\sum_{k=0}^{\infty}q^{k(x+1)+1} - \sum_{k=0}^{\infty}q^{kx}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}(1-q)^2q^{2x}\left[\sum_{k=0}^{\infty}(k+1)q^{k(x+1)+2}-\sum_{k=0}^{\infty}(k+1)q^{kx}\right]\\
&=(\ln q)^2\sum_{k=1}^{\infty}kq^{kx}+(1-q)\sum_{k=0}^{\infty}\bigl(q^{k+1}-1\bigr)q^{(k+1)x}\\
&\quad+\frac{1}{2}(1-q)^2\sum_{k=0}^{\infty}(k+1)\bigl(q^{k+2}-1\bigr)q^{(k+2)x}\\
&=\sum_{k=1}^{\infty}\biggl\{\biggl[\frac{1}{2}(1-q)k+1\biggr](1-q)\bigl(q^{k+1}-1\bigr)\\
&\quad+(\ln q)^2(k+1)\biggr\}q^{(k+1)x}+\bigl[(\ln q)^2-(1-q)^2\bigr]q^x.
\end{aligned}
$$

Let $g_q(t)=\frac{1}{2}(1-q)[(1-q)(t+1)+2]\bigl(q^t-1\bigr)+(\ln q)^2t$ for $0<q<1$ and $t\in(0,\infty)$. Then

$$
\begin{aligned}
g_q'(t)&=(\ln q)^2+\frac{1}{2}(\ln q)(q-1)q^t[q(t-1)+q-3]+\frac{1}{2}(q^t-1)(1-q)^2,\\
g_q''(t)&=\frac{1}{2}(q-1)q^t(\ln q)[(q-1)t\ln q+2q+q\ln q-3\ln q-2]\\
&\triangleq\frac{1}{2}(q-1)q^t(\ln q)\varphi(t,q),\\
\varphi(1,q)&=2[q-1+(q-2)\ln q],\quad \frac{\mathrm{d}\varphi(1,q)}{\mathrm{d}q}=2\biggl[2\biggl(1-\frac{1}{q}\biggr)+\ln q\biggr]<0.
\end{aligned}
$$

Since $\varphi(1,q)$ is decreasing with respect to $q\in(0,1)$ and $\varphi(1,1)=0$, so $\varphi(1,q)>0$ for $q\in(0,1)$. It is obvious that $\varphi(t,q)$ is increasing with respect to t, so $\varphi(t,q)>0$ for $(t,q)\in[1,\infty)\times(0,1)$. Hence, the second derivative $g_q''(t)$ is positive for $(t,q)\in[1,\infty)\times(0,1)$ and $g_q'(t)$ is increasing with respect to $t\in[1,\infty)$. From Lemma 2.5, we have $g_q'(1)=(\ln q)^2+q\bigl(q^2-3q+2\bigr)\ln q+\frac{1}{2}(q-1)^3>0$, hence $g_q'(t)>0$ for $(t,q)\in(1,\infty)\times(0,1)$, equivalently, the function $g_q(t)$ for $0<q<1$ is increasing with respect to $t\in[1,\infty)$. By virtue of Lemma 2.6, we have $g_q(1)=(q-2)(q-1)^2+(\ln q)^2>0$ for $q\in(0,1)$. Thus, the function $g_q(t)$ is positive for $(t,q)\in[1,\infty)\times(0,1)$. Consequently,

$$
\begin{aligned}
[f_q(x)-f_q(x+1)]^{(i-1)}&=(\ln q)^{i-1}\biggl\{\bigl[(\ln q)^2-(1-q)^2\bigr]q^x\\
&\quad+\sum_{k=1}^{\infty}(k+1)^{i-1}\bigl[g_q(k+1)+(1-q)^2\bigl(1-q^{k+1}\bigr)\bigr]q^{(k+1)x}\biggr\}
\end{aligned}
$$

for $i\in\mathbb{N}$. This means that

$$
\begin{aligned}
(-1)^{i-1}[f_q(x)-f_q(x+1)]^{(i-1)}&=(-1)^{i-1}(\ln q)^{i-1}\biggl\{\bigl[(\ln q)^2-(1-q)^2\bigr]q^x\\
&\quad+\sum_{k=1}^{\infty}(k+1)^{i-1}\bigl[g_q(k+1)+(1-q)^2\bigl(1-q^{k+1}\bigr)\bigr]q^{(k+1)x}\biggr\}>0
\end{aligned}
$$

which can be rearranged as $(-1)^{i-1}[f_q(x)]^{(i-1)} > (-1)^{i-1}[f_q(x+1)]^{(i-1)}$. By induction and Lemma 2.4, it follows that

$$(-1)^{i-1}[f_q(x)]^{(i-1)} > (-1)^{i-1}[f_q(x+1)]^{(i-1)} > (-1)^{i-1}[f_q(x+2)]^{(i-1)} \\ > \cdots > (-1)^{i-1}[f_q(x+k)]^{(i-1)} \ge (-1)^{i-1}\lim_{k\to\infty}[f_q(x+k)]^{(i-1)} = 0$$

for $(i,k) \in \mathbb{N}^2$. So the function $f_q(x)$ for $0 < q < 1$ is completely monotonic on $(0,\infty)$. The proof of Theorem 1.1 is complete.

4. Conclusions

The main result in this paper generalizes a conclusion applied extensively in many literature on the gamma function and polygamma functions.

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