ON THE FIXED POINTS OF MULTIVALUED MAPPINGS IN $b-$METRIC SPACES AND THEIR APPLICATION TO LINEAR SYSTEMS

Fatemeh Lael[1], Naeem Saleem[2], Mujahid Abbas[3]

In this paper, we replied to an open problem related to a $b-$metric version of Banach's fixed point theorem. It was addressed with a partial answer by several authors, choosing a suitable contractive constant but we proved it in general, without adding any assumption in comparison with its classical one. Also, some new fixed point theorems for multivalued mappings in $b-$metric spaces are obtained. Furthermore, as applications, we showed the existence of a solution of an integral inclusion and a linear equation system. We provided two applications and examples to support our main results.

Keywords: fixed point, multivalued mapping, $b-$metric space, partial order

1. Introduction

Banach [4] proved a well known fixed point theorem called Banach fixed point theorem and it has various applications in different branches of sciences. Researchers around the globe extended and generalized it in several directions. These generalizations usually fall into two categories: first is to generalize the contractive condition and second to generalize the underlying space.

Several authors generalized metric spaces in various directions, in this regard, Bhaktin [3] extended the concept of a metric space by introducing $b-$metric spaces and proved Banach fixed point theorem in such spaces. Since then, several interesting fixed point results have been obtained in the setup of $b-$metric space [2, 3, 6, 8, 9, 10, 11, 24]

A $b-$metric space is a pair (X, d) where X is a nonempty set and $d : X \times X \to \mathbb{R}$ satisfies the following conditions:

(1): $0 \leq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,

(2): $d(x, y) = d(y, x)$,

(3): $d(x, z) \leq s[d(x, y) + d(y, z)]$, for all $x, y, z \in X$ and $s \geq 1$.

Obviously, for $s = 1$ every *$b-$metric space* is a *metric space* but the converse does not hold in general [25]. Note that a $b-$metric function $d : X \times X \to [0, \infty)$ is not necessarily *continuous* in each of its arguments [1]. However, if $b-$metric d is continuous in one variable, then it is continuous in other variables [27]. A sequence $\{x_n\}$ is Cauchy (convergent) in X if and only if $\{x_n\}$ is Cauchy (convergent) with respect to a $b-$metric. A $b-$metric space X is complete if every Cauchy sequence in X is convergent in X. Recall that, for any subset C of (X, d), a multivalued mapping f on a set C is denoted as $f : C \twoheadrightarrow X$ which assigns

[1]Department of Mathematics, Buein Zahra Technical University, Buein Zahra, Qazvin, Iran, e-mail: f_lael@dena.kntu.ac.ir

[2]Department of Mathematics, University of Management and Technology, 54770, Lahore, Pakistan, e-mail: naeem.saleem2@gmail.com

[3] Department of Mathematics, Government College University, 54770, Lahore, Pakistan, e-mail: abbas.mujahid@gmail.com

each element a in C a nonempty subset fa in X. An element $x \in C$ is said to be a fixed point of f if $x \in fx$.

In his article, we provided some examples (other than discussed in [1]) which show that a $b-$metric is *not* necessarily a metric (see also [25]).

Example 1.1. Suppose that (X, d) is a $b-$metric space with $s \geq 1$. Then (X, d^r) is a $b-$metric space for all $r \in \mathbb{R}^+$. Indeed, from the general form of Holder's inequality [23], for every $x, y, z \in X$ and $r \in \mathbb{R}^+$ with $1 + \frac{1}{r} \geq 1$, we obtain the following

$$d(x, y) \leq s(d(x, z) + d(z, y)) \leq (2s)(d^r(x, z) + d^r(z, y))^{\frac{1}{r}},$$

that is

$$d^r(x, y) \leq (2s)^r(d^r(x, z) + d^r(z, y)).$$

Hence d^r is a $b-$metric. As every metric d is a $b-$metric, d^r is a $b-$metric. However, d^r is not necessarily a metric. For instance, if $d(x, y) = |x-y|$ (a Euclidean metric), $d^2(x, y) = |x-y|^2$ is not a metric on $\mathbb{R}$.

We recall the general form of Holder's inequality: Let $a_{ij} \geq 0$, $p_j > 0$, $(i = 1, 2, \dots, n,\ j = 1, 2, \dots, m)$ and let

$$\frac{1}{p_1} + \ldots + \frac{1}{p_m} \geq 1.$$

Then

$$\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \leq \prod_{j=1}^{m} (\sum_{i=1}^{n} a_{ij}^{p_j})^{\frac{1}{p_j}}.$$

Khan et.al. [12] introduced the notion of altering distance function as follows:

Definition 1.1. [12] *A mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an altering distance if the following conditions hold:*

(1): *φ is continuous and nondecreasing,*
(2): *$\varphi(t) = 0$ if and only if $t = 0$.*

Recently, Radenovíc et.al. [20] proved following Theorem which generalizes the results discussed in [7].

Theorem 1.1. [20] *Let (X, d) be a complete $b-$metric space with $s > 1$ and $T : X \to X$. Suppose that there exists an altering distance φ and constants $L \geq 0$, $\epsilon > 1$ such that for any $x, y \in X$,*

$$\varphi(s^\epsilon d(Tx, Ty)) \leq \varphi(S(x, y)) + L\varphi(I(x, y)), \tag{1}$$

holds, where

$$S(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\frac{1 + d(x, Tx)}{1 + d(x, y)}, \frac{d(x, Ty) + d(y, Tx)}{2s}\},$$

and

$$I(x, y) = \min\{d(x, Tx) + d(y, Ty), d(x, Ty), d(y, Tx)\},$$

then T has a fixed point.

In this article, we generalize and improve the fixed point results discussed in [20]. As an application of obtained results, we get a solution of integral inclusion and system of linear equations in $b-$metric spaces.

2. Main results

Throughout this section, B denotes a closed subset of a complete $b-$metric space $X = (X, d)$ and $f : B \twoheadrightarrow B$ is a closed multivalued mapping, T is a single valued mapping on (X, d) and φ is an altering distance function.

The following lemma and definitions are needed in the sequel.

Lemma 2.1. [17] *Consider a non-empty set $X = (X, d)$ be a $b-$metric space and $\{x_n\}$ is a sequence in X. Suppose that there exists some $k \in [0, 1)$ such that for every $n \in \mathbb{N}$,*

$$d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n)$$

holds. Then $\{x_n\}$ is a Cauchy sequence.

Definition 2.1. *A mapping f is called weakly Picard, if there exists a sequence $\{x_n\}$ with $x_0 \in B$, $x_1 \in fx_0$ and $x_{n+1} \in fx_n$ which converges to the fixed point of f.*

Definition 2.2. *A mapping f is continuous if for $x_n \to x$, and $y_n \to y$ such that $y_n \in fx_n$ implies that $y \in fx$.*

Definition 2.3. *Define*

$$d(a, fb) = \inf\{d(a, y) : y \in fb\}$$

and

$$d(a, C) = \inf\{d(a, c) : c \in C\}.$$

Moreover, Pompeiu-Hausdorff distance is defined as follows:

$$H_d(A, C) = \max\{\max_{a\in A} d(a, C), \max_{c\in C} d(A, c))\},$$

where A and C are subsets of B.

For fixed point results related with multivalued mappings employing the notion of a *Pompeiu-Hausdorff distance*, we refer to [5, 14, 16, 22, 26].

Now, we prove the main theorem of this article, which is a generalization of Theorem 1.1.

Theorem 2.1. *If there exists $z \in fx$ and $w \in fy$ such that*

$$\varphi(\lambda d(z, w)) \leq \varphi(S(x, y)) + L\varphi(I(x, y)), \tag{2}$$

holds, where constants $L \geq 0$, $\lambda > 1$, also

$$S(x, y) = \max\{d(x, y), d(x, fx), d(y, fy)\frac{1 + d(x, fx)}{1 + d(x, y)}, \frac{d(x, fy) + d(y, fx)}{2s}\}, \tag{3}$$

and

$$I(x, y) = \min\{d(x, fx) + d(y, fy), d(x, fy), d(y, fx)\}, \tag{4}$$

for all $x, y \in B$. Then f has a fixed point if and only if one of the following assumptions hold:

(i): *f is weakly Picard.*
(ii): *f is continuous.*
(iii): *d is continuous.*
(iv): *$\lambda > s$.*

Proof. Let $x_0 \in B$, there exists $x_1 \in fx_0$ such that we may find $x_2 \in fx_1$ and

$$\varphi(\lambda d(x_1, x_2)) \leq \varphi(S(x_0, x_1)) + L\varphi(I(x_0, x_1)). \tag{5}$$

From (4), $I(x_0, x_1) = 0$. Now, from (3), we have

$$\begin{aligned} S(x_0,x_1) &= \max\{d(x_0,x_1), d(x_0,fx_0), d(x_1,fx_1)\frac{1+d(x_0,fx_0)}{1+d(x_0,x_1)}, \\ &\quad \frac{d(x_0,fx_1)+d(x_1,fx_0)}{2s}\}, \\ &\leq \max\{d(x_0,x_1), d(x_0,x_1), d(x_1,x_2)\frac{1+d(x_0,x_1)}{1+d(x_0,x_1)}, \\ &\quad \frac{d(x_0,x_2)+d(x_1,x_1)}{2s}\}, \\ &\leq \max\{d(x_0,x_1), d(x_1,x_2), \frac{d(x_0,x_1)+d(x_1,x_2)}{2}\}, \\ &= \max\{d(x_1,x_2), d(x_0,x_1)\}. \end{aligned} \tag{6}$$

If $\max\{d(x_1,x_2), d(x_0,x_1)\} = d(x_1,x_2)$, then inequality (5) become

$$\varphi(\lambda d(x_1,x_2)) \leq \varphi(d(x_1,x_2)) + L\varphi(0).$$

As $\varphi(0) = 0$ and φ is nondecreasing, the above inequality can be written as

$$\lambda d(x_1,x_2) \leq d(x_1,x_2),$$

which is a contradiction because $\lambda > 1$. Hence $\max\{d(x_1,x_2), d(x_0,x_1)\} = d(x_0,x_1)$, then inequality (5) becomes

$$\varphi(\lambda d(x_1,x_2)) \leq \varphi(d(x_0,x_1)).$$

Thus we have

$$\lambda d(x_1,x_2) \leq d(x_0,x_1).$$

Continuing this way, we can construct a sequence $\{x_n\}$ such that $x_{n+1} \in fx_n$ and

$$\varphi(\lambda d(x_n,x_{n+1})) \leq \varphi(S(x_{n-1},x_n)) + L\varphi(I(x_{n-1},x_n)), \tag{7}$$

for every $n \geq 1$. After simple calculation, we have $I(x_{n-1},x_n) = 0$ and

$$\begin{aligned} S(x_{n-1},x_n) &= \max\{d(x_{n-1},x_n), d(x_{n-1},fx_{n-1}), d(x_n,fx_n)\frac{1+d(x_{n-1},fx_{n-1})}{1+d(x_{n-1},x_n)}, \\ &\quad \frac{d(x_{n-1},fx_n)+d(x_n,fx_{n-1})}{2s}\}, \\ &\leq \max\{d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1})\frac{1+d(x_{n-1},x_n)}{1+d(x_{n-1},x_n)}, \\ &\quad \frac{d(x_{n-1},x_{n+1})+d(x_n,x_n)}{2s}\}, \\ &\leq \max\{d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\}, \\ &= \max\{d(x_n,x_{n+1}), d(x_{n-1},x_n)\}. \end{aligned} \tag{8}$$

If $\max\{d(x_n,x_{n+1}), d(x_{n-1},x_n)\} = d(x_n,x_{n+1})$, then inequality (7) can be written as:

$$\varphi(\lambda d(x_n,x_{n+1})) \leq \varphi(d(x_n,x_{n+1})) + L\varphi(0).$$

As $\varphi(0) = 0$ and φ is nondecreasing, the above inequality can be written as

$$\lambda d(x_n,x_{n+1}) \leq d(x_n,x_{n+1}),$$

which is a contradiction.
Hence $\max\{d(x_n,x_{n+1}), d(x_{n-1},x_n)\} = d(x_{n-1},x_n)$. Now inequality (7) becomes

$$\varphi(\lambda d(x_n,x_{n+1})) \leq \varphi(d(x_{n-1},x_n)).$$

Thus we have

$$\lambda d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n),$$

for all $n \in \mathbb{N}$. It follows from Lemma (2.1) and completeness of B that there exists $x \in B$ such that $x_n \to x$ as $n \to \infty$.

The proof is obviously complete under assumption (i) and (ii).

Now assume (iii) holds. We know that for x_n, there exists $y_n \in fx$ such that

$$\varphi(\lambda d(y_n, x_n)) \leq \varphi(S(x_{n-1}, x)) + L\varphi(I(x_{n-1}, x)). \tag{9}$$

We can calculate

$$\begin{aligned} I(x_{n-1}, x) &= \min\{d(x_{n-1}, fx_{n-1}) + d(x, fx), d(x_{n-1}, fx), d(x, fx_{n-1})\}, \\ &\leq \min\{d(x_{n-1}, x_n) + d(x, fx), d(x_{n-1}, fx), d(x, x_n)\}, \end{aligned} \tag{10}$$

and

$$\begin{aligned} S(x_{n-1}, x) &= \max\{d(x_{n-1}, x), d(x_{n-1}, fx_{n-1}), d(x, fx)\frac{1 + d(x_{n-1}, fx_{n-1})}{1 + d(x_{n-1}, x)}, \\ &\quad \frac{d(x_{n-1}, fx) + d(x, fx_{n-1})}{2s}\}, \\ &\leq \max\{d(x_{n-1}, x), d(x_{n-1}, x_n), d(x, fx)\frac{1 + d(x_{n-1}, x_n)}{1 + d(x_{n-1}, x)}, \\ &\quad \frac{d(x_{n-1}, fx) + d(x, x_n)}{2s}\}. \end{aligned} \tag{11}$$

Since $x_n \to x$ as $n \to \infty$, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, x_{n-1}) = 0$. From inequality (10) and inequality (11) we have

$$\lim_{n\to\infty} I(x_{n-1}, x) = 0,$$

and

$$\limsup S(x_{n-1}, x) \leq d(x, fx),$$

So the inequality (9) yields

$$\varphi(\lambda \limsup d(y_n, x_n)) \leq \varphi(d(x, fx)). \tag{12}$$

Since $y_n \in fx$, we have

$$d(x, fx) \leq d(x, y_n),$$

for each $n \geq 1$. As φ is nondecreasing, inequality (12) can be written as

$$\lambda \limsup d(y_n, x_n) \leq d(x, fx) \leq d(x, y_n). \tag{13}$$

Now (iii) leads to $\lim_{n\to\infty} d(x_n, y_n) = d(x, y_n)$ and inequality (13) can be written as

$$\lambda \lim d(y_n, x) \leq \lim d(y_n, x),$$

Thus $\lim y_n = x$. Since fx is closed and $y_n \in fx$, we have $x \in fx$. Finally, if assumption (iv) is satisfied, by repeating the proof presented for an assumption (iii), inequality (13) is obtained. Therefore, it follows that

$$\lambda \limsup d(y_n, x_n) \leq d(x, y_n) \leq s(d(x, x_n) + d(x_n, y_n)).$$

Further,

$$(\lambda - s) \limsup d(y_n, x_n) \leq s \lim d(x, x_n).$$

Thus, $\lim d(x_n, y_n) = 0$, since $\lambda > s$. Also,

$$d(x, y_n) \leq s(d(x, x_n) + d(x_n, y_n)),$$

we have $\lim y_n = x$ and hence $x \in fx$. □

Theorem 1.1 is a particular case of Theorem 2.1 part (iv). As a single valued mapping T can be viewed as a multivalued mapping that takes $x \in X$ to a set $\{Tx\}$. If we consider $\lambda = s^{\epsilon}$ then equation (1) will be a particular case of equation (2). Therefore Theorem (2.1) part (iv) implies that T has a fixed point.

To support our main result, we provide the following example.

Example 2.1. Suppose that $f : \mathbb{R} \twoheadrightarrow \mathbb{R}$ is a multivalued mapping, defined as

$$fx = \{\frac{x}{2}, \frac{x}{3}, 1\},$$

and $(\mathbb{R}^2, d)$ is a $b-$metric space, where $d(x, y) = |x - y|^2$. Consider $\varphi(x) = x$, $L = \frac{1}{3}$ and $\lambda = 2$. For any $x, y \in \mathbb{R}$ and $z \in fx$. We have three cases:
Case 1: If $z = \frac{x}{2} \in fx$ then choose $w = \frac{y}{2}$.
Case 2: If $z = \frac{x}{3} \in fx$ then choose $w = \frac{y}{3}$.
Case 3: If $z = 1 \in fx$ then choose $w = z$.
Now, we have to show that the inequality (2) is satisfied.
Case 1 leads to

$$2|\frac{x}{2} - \frac{y}{2}|^2 = \frac{1}{2}|x - y|^2 \leq |x - y|^2,$$

therefore

$$2|z - w|^2 \leq |x - y|^2.$$

This implies that

$$\varphi(2|z - w|^2) \leq \varphi(|x - y|^2).$$

Case 2 and case 3, following on the same lines as in case 1. We have

$$|x - y|^2 \leq S(x, y).$$

So

$$\varphi(|x - y|^2) \leq \varphi(S(x, y)).$$

Since

$$\varphi(I(x, y)) \geq 0,$$

therefore

$$\varphi(|x - y|^2) \leq \varphi(S(x, y)) + \frac{1}{3}\varphi(I(x, y)).$$

Thus

$$\varphi(2|z - w|^2) \leq \varphi(S(x, y)) + L\varphi(I(x, y)).$$

Note that, 0 and 1 are the fixed points of f.

The following corollary is a $b-$metric version of Nadler's fixed point theorem.

Corollary 2.1. *Let B be a closed subset of a complete $b-$metric space X and $f : B \twoheadrightarrow B$ be a closed valued multivalued mapping. Also, there exists $z \in fx$ and $w \in fy$ such that*

$$d(z, w) \leq kd(x, y), \tag{14}$$

for each $x, y \in B$, then f has a fixed point.

Proof. Suppose $\lambda = \frac{1}{k} > 1$, where $k \neq 0$ (for $k = 0$, it is trivial). Since inequality (14) implies that the mapping f is continuous. Following the assumption (ii) of Theorem (2.1) implies that f has a fixed point. □

We know Nadler's fixed point theorem is a generalization of Banach fixed point theorem for multivalued mappings. So we have the following corollary which is an answer to an open problem and establishes a $b-$metric version of Banach contraction theorem.

Corollary 2.2. *Let B be a closed subset of a complete $b-$metric space X and consider a single valued mapping $T : X \to X$ satisfying*

$$d(Tx, Ty) \leq kd(x, y),$$

where $k \in [0, 1)$ and $x, y \in X$, then T has a fixed point.

The $b-$metric version of Banach fixed point theorem is already proved for $k \in (0, \frac{1}{s})$ but it was an open problem that whether T has a fixed point when $\frac{1}{s} \leq k < 1$. Indeed, we replied to this question in corollary 2.2 (for details, see 2.2).

Suppose that X is a $b-$metric space equipped with a partially order relation "$\preceq$" (see, [13, 21]). A multivalued mapping $f : X \twoheadrightarrow X$ is called *monotone* if for all $x \preceq y$, we have $u \preceq v$, for each $u \in fx$ and $v \in fy$ (see, [15, 19]).

Theorem 2.2. *Let (X, d) be a complete ordered $b-$metric space and f is a monotone multivalued mapping on X such that $x_0 \preceq fx_0$ for some $x_0 \in X$. Suppose, there exist constants $L \geq 0$ and $\lambda > s$ such that for $x, y \in X$ with $x \preceq y$ and $z \in fx$, there exists $w \in fy$ and $z \preceq w$ such that*

$$\varphi(\lambda d(z, w)) \leq \varphi(S(x, y)) + L\varphi(I(x, y)),$$

where $S(x, y)$ and $I(x, y)$ are given in (3) and (4). Then f has a fixed point.

Proof. The proof is closely modeled on the proof of Theorem 2.1 part (iv). □

Indeed, Theorem 2.2 is a generalization of Theorem 2.1 part (iv) in ordered $b-$metric space.

3. Applications

A $b-$metric fixed point theorem can be used to provide sufficient conditions for finding a real continuous function u defined on $[a, b]$ such that

$$u(t) \in v(t) + \gamma \int_a^b G(t, s)g(s, u(s))ds, \quad t \in [a, b], \tag{15}$$

where γ is a constant, $g : [a, b] \times \mathbb{R} \twoheadrightarrow [a, b]$ is lower semicontinuous, $G : [a, b] \times [a, b] \to [0, \infty)$ and $v : [a, b] \to \mathbb{R}$ are given continuous functions. Let $X = C[a, b]$ be the set of all real continuous functions defined on $[a, b]$, $g_u : [a, b] \twoheadrightarrow [a, b]$ where $g_u(s) = g(s, u(s))$ and a $b-$metric on X defined as:

$$d(u, v) = \max_{a \leq t \leq b} \mid u(t) - v(t) \mid^2 .$$

Note that (X, d) is a complete $b-$metric space. Also, an integral inclusion problem (15) can be reformulated as: u is a solution of the problem (15) if and only if it is a fixed point of $f : X \twoheadrightarrow X$, where

$$fu = \{x \in X : x(t) \in v(t) + \gamma \int_a^b G(t, s)g(s, u(s))ds, \ t \in [a, b]\}.$$

Assume that:

1: $\mid \gamma \mid \leq 1$,

2: $\max\limits_{a \leq t \leq b} \int_a^b G^2(t, z)dz \leq \frac{1}{b-a}$,

3: for all $x, y \in X$ and $w_x(t) \in g_x(t)$, there exists $h_y(t) \in g_y(t)$ such that $|w_x(t) - h_y(t)|^2 \leq \frac{1}{2s} \mid x(t) - y(t) \mid^2, \ t \in [a, b]$,

then multivalued mapping f has a unique fixed point. Suppose that $x, y \in X$ and $w \in fx$, by definition, we have

$$w(t) \in v(t) + \gamma \int_a^b G(t, s)g(s, x(s))ds = v(t) + \gamma \int_a^b G(t, s)g_x(s)ds.$$

By Michael's selection theorem, (in [18] Theorem 1) it follows that there exists a continuous single valued mapping $w_x(s) \in g_x(s)$ such that $w(t) = v(t) + \gamma \int_a^b G(t,s)w_x(s)ds$. According to assumption 3, for $w_x(s) \in g_x(s)$, there exists an $h_y(s) \in g_y(s)$ such that

$$|w_x(s) - h_y(s)|^2 \leq \frac{1}{2s} \mid x(s) - y(s) \mid^2,$$

for all $s \in [a,b]$. We define

$$h(t) = v(t) + \gamma \int_a^b G(t,s)h_y(s)ds$$

that is

$$h(t) \in v(t) + \gamma \int_a^b G(t,s)g_y(s)ds.$$

Therefore $h \in fy$. Using the Cauchy-Schwarz inequality and conditions $1-3$, for $L = 0$, $\varphi(x) = \frac{x}{2s}$, we have

$$\begin{aligned}
\varphi(2sd(w,h)) &= \max_{a\leq t\leq b} |w(t) - h(t)|^2, \\
&= \max_{a\leq t\leq b} \mid v(t) + \gamma \int_a^b G(t,s)w_x(s)ds - (v(t) + \gamma \int_a^b G(t,s)h_y(s)ds) \mid^2, \\
&= \mid \gamma \mid^2 \max_{a\leq t\leq b} \mid \int_a^b G(t,s)(w_x(s) - h_y(s))ds \mid^2, \\
&\leq \mid \gamma \mid^2 \max_{a\leq t\leq b} \big\{ \int_a^b G^2(t,s)ds \int_a^b \mid w_x(s) - h_y(s) \mid^2 ds \big\}, \\
&= \mid \gamma \mid^2 \big\{ \max_{a\leq t\leq b} \int_a^b G^2(t,s)ds \big\} . \big\{ \int_a^b \mid w_x(s) - h_x(s) \mid^2 ds \big\}, \\
&\leq \frac{\mid \gamma \mid^2}{b-a} \big\{ \frac{1}{2s} \int_a^b \mid x(s) - y(s) \mid^2 ds \big\}, \\
&\leq \frac{\mid \gamma \mid^2}{2s(b-a)} \int_a^b \max_{a\leq s\leq b} \mid x(s) - y(s) \mid^2 ds, \\
&= \frac{\mid \gamma \mid^2}{2s} \max_{a\leq s\leq b} \mid x(s) - y(s) \mid^2, \\
&= \frac{\mid \gamma \mid^2}{2s} d(x,y) \leq \frac{1}{2s} d(x,y) \leq \frac{1}{2s} S(x,y), \\
&\leq \varphi(S(x,y)) + L\varphi(I(x,y)).
\end{aligned}$$

Hence all the conditions of Theorem 2.1 part (iv) are satisfied, which implies that f has a unique fixed point $u \in X$ such that the integral inclusion (15) has a solution that belongs to $C[a,b]$.

Now, we are going to provide an application of the Banach's fixed point theorem in $b-$metric spaces to establish the existence of the unique solution of linear system of equations:

Consider we have the following system of linear equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1,$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2,$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n,$$

which has a unique solution under certain conditions. Then, we have to define

$$\gamma_{ij} = \begin{cases} a_{ij}+1 & i=j, \\ a_{ij} & i \neq j. \end{cases}$$

and a $b-$metric as:

$$d(x,y) = \max_j (x_j - y_j)^2,$$

for all $x, y \in \mathbb{R}^n$. Also, the self-mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ is defined as:

$$Tx = (A+I)x - b,$$

where A is an $n \times n$ matrix with a_{ij} arrays and I is an identical matrix, $b = [b_1, \ldots, b_n]^\top$. Now, we have to show that the self-mapping T satisfies the Banach's contraction principle in $b-$metric spaces. Using Cauchy-Schwarz inequality, we have

$$\begin{aligned} d(Tx,Ty) &= \max_{1\le i\le n} (\sum_{j=1}^n \gamma_{ij}(x_j - y_j))^2, \\ &\le \max_{1\le i\le n} (\sum_{j=1}^n \gamma_{ij}^2 \ \sum_{j=1}^n (x_j - y_j)^2), \\ &\le n^2 \max_{i,j} \gamma_{ij}^2 \ \max_j (x_j - y_j)^2, \\ &\le n^2 \max_{i,j} \gamma_{ij}^2 \ d(x,y). \end{aligned}$$

Corollary 2.2 implies that if $n^2 \max_{i,j} \gamma_{ij}^2 < 1$ then T has a fixed point. So, the linear system has a unique solution.

4. Conclusion

In this article, we defined $\varphi-$multivalued contractive mapping and obtained fixed point results in "b-metric space". As a consequence of our main result, we obtained Nadler's theorem in "b-metric space"and Banach fixed point theorem by relaxing the assumptions on contractive constant $k \in [\frac{1}{s}, 1)$ in contraction theorems proved for $b-$metric spaces. In this way, we addressed an open problem by showing that the results hold even contractive constant k lies in $[\frac{1}{s}, 1)$, where $s \ge 1$. We also presented an application to a particular form of integral inclusions and to the system of a linear equation to support the results thus obtained.

REFERENCES

[1] *A. Aghajani, M. Abbas and J. R. Roshan*, Common fixed point of generalized weak contractive mappings in partially ordered G_b-metric spaces, Filomat, **28**(2014), No. 6, 1087-1101.

[2] *M.U. Ali, T. Kamran and M. Postolache*, Solution of Volterra integral inclusion in b-metric spaces via new fixed point theorem, Nonlinear Anal. Model. Control, **22** (2017), No. 1, 17-30.

[3] *J. A. Bakhtin*, The contraction principle in quasi metric spaces, Funct. Anal. Unianowsk Gos. Ped. Inst., **30** (1989), 26-37.

[4] *S. Banach*, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math., **3** (1922), No. 1, 133-181.

[5] *M. Boriceanu*, Strict fixed point theorems for multivalued operators in $b-$metric spaces, Int. J. Mod. Math., **4** (2009), No. 2, 285-301.

[6] *S. Czerwik*, Contraction mappings in $b-$metric spaces, Acta Math. Inform. Univ. Ostraviensis, **1** (1993), 5-11.

[7] *R. George, K. P. Reshma and A. Padmavati*, Fixed point theorems for cyclic contractions in $b-$metric spaces, J. Nonlinear Funct. Anal., **2015** (2015), No. 5, 45-56.

[8] *N. Hussain, D. Doric, Z. Kadelbarg and S.Radenović*, Suzuki-type fixed point results in metric type spaces, Fixed Point Theor. Appl., **2012** (2012), No. 126, 1-12.
[9] *M. Jovanović, Z.Kadelburg and S. Radenović*, Common fixed point results in metric-type spaces, Fixed Point Theor. Appl., **2010** (2010), No. 1, 1-16.
[10] *T. Kamran, M. Postolache, M.U. Ali and Q. Kiran*, Feng and Liu type F-contraction in $b-$metric spaces with application to integral equations, J. Math. Anal., **7** (2016), No. 5, 18-27.
[11] *M. A. Khamsi and N. Hussain*, KKM mappings in metric type spaces, Nonlinear Anal., **73** (2010), 3123-3129.
[12] *M. S. Khan, M. Swaleh and S. Sessa*, Fixed point theorems by altering distances between the points, Bull. Austral. Math. Soc., **30** (1984), No. 1, 1-9.
[13] *P. Kumrod and W. Sintunavarat*, A new contractive condition approach to φ-fixed point results in metric spaces and its applications, J. Comput. Appl. Maths., **311** (2016), 194-204.
[14] *H. Kumar Pathak, R. P. Agarwal and Y. Je Cho*, Coincidence and fixed points for multi-valued mappings and its application to nonconvex integral inclusions, J. Comput. Appl. Maths., **283**(2015), 201-217.
[15] *F. Lael and Z. Heidarpour*, Fixed point theorems for a class of generalized nonexpansive mappings, Fixed Point Theory Appl., **2016** (2016), No. 82, 1-7.
[16] *N. Mehmood, A. Azam and L.D.R. Kočinac*, Multivalued fixed point results in cone metric spaces, Topol. Appl., **179** (2015), 156-170.
[17] *R. Miculescu and A. Mihail*, New fixed point theorems for set-valued contractions in $b-$metric spaces, J. Fixed Point Theory Appl., **19** (2017), 2153-2163.
[18] *E. Michael*, A selection theorem, Proceedings of the American Mathematical Society, **17**(1966), No. 6, 1404-1406.
[19] *G. Petrusel*, Fixed point results for multivalued contractions on ordered gauge spaces, Cent.l Eur. J. Math., **7** (2009), No. 3, 520-528.
[20] *S. Radenović, T. Došenović, T. A. Lampert and Z. Golubović*, A note on some recent fixed point results for cyclic contractions in $b-$metric spaces and an application to integral equations, Appl. Maths. Comput., **273**(2016), 155-164.
[21] *A. N. Sadigh and S. Ghods*, Coupled coincidence point in ordered cone metric spaces with examples in game theory, Int. J. Nonlinear Anal. Appl., **7** (2016), No. 1, 183-194.
[22] *N. Saleem, I. Habib and M. D. Sen*, Some New Results on Coincidence Points for Multivalued Suzuki-Type Mappings in Fairly Complete Spaces, Computation, **8** (2020), No. 17, 1-23.
[23] *W. Shan He*, Generalization of a sharp Holder's inequality and its application, J. Math. Anal. Appl., **332** (2007), No. 1, 741-750.
[24] *W. Shatanawi, A. Pitea and R. Lazovic*, Contraction conditions using comparison functions on $b-$metric spaces, Fixed Point Theory Appl., **2014** (2014), No. 135, 1-11.
[25] *S. L. Singh and B. Prasad*, Some coincidence theorems and stability of iterative procedures, Comput. Math. Appl., **55** (2008), 2512-2520.
[26] *W. Sintunavarat and P. Kumam*, Common fixed point theorem for cyclic generalized multi-valued contraction mappings, Appl. Math. Lett., **25** (2012), No. 11, 1849-1855.
[27] *T. Van An, L. Quoc Tuyen and N. Van Dung*, Stone-type theorem on $b-$metric spaces and applications, Topol. Appl., **185** (2015), 50-64.