A GENERALIZED DISTANCE IN A CONE METRIC SPACE AND NEW COMMON FIXED POINT RESULTS

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In this paper we prove some common fixed point theorems by using the generalized distance in a cone metric space. Our theorems extend some recent results of Wang and Guo [Appl. Math. Lett. 24 (2011) 1735-1739] and Abbas and Jungck [J. Math. Anal. Appl. 341 (2008) 416-420].

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1. Introduction

Following Banach [5], if (X, d) is a complete metric space and f is a map of X satisfies $d(fx, fy) \leq \lambda d(x, y)$ for all $x, y \in X$ where $\lambda \in [0, 1)$, then f has a unique fixed point. Afterward, several fixed point theorems were considered by other authors [8, 11, 14, 17, 27]. The cone metric space was initiated in 2007 by Huang and Zhang [12] and several fixed and common fixed point results in cone metric spaces were introduced in [2, 3, 19, 20, 24, 26, 30, 31] and references were mentioned therein.

In 1996, Kada et al. [15] defined the concept of w-distance in complete metric spaces. Later, many authors proved some fixed point theorems in complete metric spaces (see [1, 18, 25]). Also, note that Saadati et al. [28] introduced a probabilistic version of the w-distance of Kada et al. [15] in a Menger probabilistic metric space. In the sequel, Cho et al. [7] and Wang and Guo [31] defined the concept of the c-distance in a cone metric space, which is a cone version of the w-distance of Kada et al. [15]. Recently, also, other authors proved some fixed and common fixed point theorems in cone metric spaces and tvs-cone metric spaces (see [6, 10, 13, 21, 22, 23, 29]).

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In this paper, we extend and develop the Banach's contraction theorem on c-distance in a cone metric space. Our results extend and improve some Wang and Guo's works and Abbas and Jungck's results.

Important note: It has been shown in [4] that, by renorming an ordered Banach space, every cone can be converted to a normal cone with constant $K = 1$, and every normal cone metric space can be renormed to be equivalent to a metric space. But, the c-distance is a generalization of the w-distance. Thus, our results are new. On the other hand, the results have not been proved under w-distance in metric spaces. Therefore, our fixed point theorems under c-distance are the most complete theorems in this field. Consequently, the obtained results extend, unify and generalize several well known comparable results in the existing literature (see [2, 19, 30, 31]).

2. **Preliminaries**

We begin with some important definitions.

Definition 2.1. ([9, 12]). Let E be a real Banach space and P a subset of E. Then P is called a cone if and only if
(a) P is closed, non-empty and $P \neq \{\theta\}$;
(b) $a, b \in \mathbf{R}, a, b \geq 0, x, y \in P$ implies $ax + by \in P$;
(c) if $x \in P$ and $-x \in P$, then $x = \theta$.

Given a cone $P \subset E$, a partial ordering $\preceq$ with respect to P is defined by

$$x \preceq y \Longleftrightarrow y - x \in P.$$

We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$. Also, we write $x \ll y$ if and only if $y - x \in intP$ (where $intP$ is the interior of P). If $intP \neq \emptyset$, the cone P is called solid. A cone P is called normal if there exists a number $K > 0$ such that, for all $x, y \in E$,

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq K\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P.

Definition 2.2. ([12]). Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies
$(d1)$ $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
$(d2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;
$(d3)$ $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.
Then, d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.3. ([12]). Let (X, d) be a cone metric space, $\{x_n\}$ a sequence in X and $x \in X$. Then
$(S1)$ $\{x_n\}$ converges to x if, for every $c \in E$ with $\theta \ll c$ there exists an $n_0 \in \mathbf{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$. We denote this by $\lim_{n\to\infty} d(x_n, x) = \theta$.

$(S2)$ $\{x_n\}$ is called a Cauchy sequence if, for every $c \in E$ with $\theta \ll c$ there exists an $n_0 \in \mathbf{N}$ such that $d(x_n, x_m) \ll c$ for all $m, n > n_0$. We denote this by $\lim_{n,m\to\infty} d(x_n, x_m) = \theta$.

Lemma 2.4. ([12, 26]). Let (X, d) be a cone metric space and P be a normal cone with normal constant K. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y \in X$. Then the following hold:
(c_1) $\{x_n\}$ converges to x if and only if $d(x_n, x) \to \theta$ as $n \to \infty$.
(c_2) If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$.
(c_3) If $\{x_n\}$ converges to x, then $\{x_n\}$ is a Cauchy sequence.
(c_4) If $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.
(c_5) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$ as $n, m \to \infty$.

Lemma 2.5. ([7, 24]). Let E be a real Banach space with a cone P in E. Then, for all $u, v, w, c \in E$, the following hold:
(p_1) If $u \preceq v$ and $v \ll w$, then $u \ll w$.
(p_2) If $\theta \preceq u \ll c$ for each $c \in intP$, then $u = \theta$.
(p_3) If $u \preceq \lambda u$ where $u \in P$ and $0 < \lambda < 1$, then $u = \theta$.
(p_4) Let $x_n \to \theta$ in E, $\theta \preceq x_n$ and $\theta \ll c$. Then there exists positive integer n_0 such that $x_n \ll c$ for each $n > n_0$.
(p_5) If $\theta \preceq u \preceq v$ and k is a nonnegative real number, then $\theta \preceq ku \preceq kv$.
(p_6) If $\theta \preceq u_n \preceq v_n$ for all $n \in N$ and $u_n \to u$, $v_n \to v$ as $n \to \infty$, then $\theta \preceq u \preceq v$.

Definition 2.6. ([29, 31]). Let (X, d) be a cone metric space. A function $q : X \times X \to E$ is called a c-distance on X if the following are satisfied:
(q_1) $\theta \preceq q(x, y)$ for all $x, y \in X$;
(q_2) $q(x, z) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$;
(q_3) for all $n \geq 1$ and $x \in X$, if $q(x, y_n) \preceq u$ for some $u = u_x$, then $q(x, y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;
(q_4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Example 2.7. ([7, 29, 31]).
(1) Let (X, d) be a cone metric space and P be a normal cone. Put $q(x, y) = d(x, y)$ for all $x, y \in X$. Then q is a c-distance.
(2) Let $E = \mathbf{R}$, $P = \{x \in E : x \geq 0\}$ and $X = [0, \infty)$. Define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a c-distance.
(3) Let $E = C^1_{\mathbf{R}}[0, 1]$ with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and consider the cone $P = \{x \in E : x(t) \geq 0 \text{ on } [0, 1]\}$. Also, let $X = [0, \infty)$ and define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|\psi$ for all $x, y \in X$, where $\psi : [0, 1] \to R$ such that $\psi(t) = 2^t$. Then (X, d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = (x + y)\psi$ for all $x, y \in X$. Then q is c-distance.

(4) Let (X, d) be a cone metric space and P be a normal cone. Put $q(x, y) = d(w, y)$ for all $x, y \in X$, where $w \in X$ is a fixed point. Then q is a c-distance.

Remark 2.8. For c-distance q, $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ and $q(x, y) = q(y, x)$ does not necessarily hold for all $x, y \in X$.
Also, each w-distance q in a metric space (X, d) is a c-distance with $E = \mathbf{R}^+$ and $P = [0, \infty)$. But the converse does not hold. Thus, the c-distance is a generalization of the w-distance.

Lemma 2.9. Let (X, d) be a cone metric space and let q be a c-distance on X. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$. Suppose that $\{u_n\}$ and $\{v_n\}$ are two sequences in P converging to θ. Then the following hold:
(qp_1) If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq v_n$ for $n \in \mathbf{N}$, then $y = z$. Specifically, if $q(x, y) = \theta$ and $q(x, z) = \theta$, then $y = z$.
(qp_2) If $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq v_n$ for $n \in \mathbf{N}$, then $\{y_n\}$ converges to z.
(qp_3) If $q(x_n, x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.
(qp_4) If $q(y, x_n) \preceq u_n$ for $n \in \mathbf{N}$, then $\{x_n\}$ is a Cauchy sequence in X.

Proof. See [16, 29, 31]. □

3. **Fixed point results**

The following is the main theorem of this paper. We prove a common fixed point theorem by using c-distance. Our theorem extends the contractive condition from constant real numbers to some control functions (also, see [7, 16, 29]).

Theorem 3.1. Let (X, d) be a cone metric space, P be a normal cone with constant K and q be a c-distance. Also, let $f, g : X \to X$ be two mappings with $f(X) \subseteq g(X)$ and let $g(X)$ be a complete subspace of X. Suppose that there exist mappings $\alpha_i : X \to [0, 1)$ for $i = 1, 2, \cdots, 5$ such that the following conditions hold:
(i) $\alpha_i(fx) \le \alpha_i(gx)$ for all $x \in X$;
(ii) $(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5)(x) < 1$ for all $x \in X$;
(iii) $q(fx, fy) \preceq \alpha_1(gx)q(gx, gy)+\alpha_2(gx)q(gx, fx)+\alpha_3(gx)q(gy, fy)+\alpha_4(gx)q(gx, fy)+ \alpha_5(gx)q(gy, fx)$ for all $x, y \in X$;
(iv) $q(fy, fx) \preceq \alpha_1(gx)q(gy, gx)+\alpha_2(gx)q(fx, gx)+\alpha_3(gx)q(fy, gy)+\alpha_4(gx)q(fy, gx)+ \alpha_5(gx)q(fx, gy)$ for all $x, y \in X$;
If f and g satisfy

$$\inf\{\|q(fx, y)\| + \|q(gx, y)\| + \|q(gx, fx)\| : x \in X\} > 0$$

for all $y \in X$ with $y \ne fy$ or $y \ne gy$, then f and g have a common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. Since the range of g contains the range of f, there exists an $x_1 \in X$ such that $fx_0 = gx_1$. By induction, a sequence $\{x_n\}$ can be chosen such that $fx_n = gx_{n+1}$ for $n = 0, 1, 2, \cdots$. Now, set $x = x_{n-1}$ and $y = x_n$

in (iii). Thus, by (q_2), for $n \geq 1$, we get

$$\begin{aligned}
q(gx_n, gx_{n+1}) &= q(fx_{n-1}, fx_n) \\
&\preceq \alpha_1(gx_{n-1})q(gx_{n-1}, gx_n) + \alpha_2(gx_{n-1})q(gx_{n-1}, fx_{n-1}) + \alpha_3(gx_{n-1})q(gx_n, fx_n) \\
&\quad +\alpha_4(gx_{n-1})q(gx_{n-1}, fx_n) + \alpha_5(gx_{n-1})q(gx_n, fx_{n-1}) \\
&= \alpha_1(fx_{n-2})q(gx_{n-1}, gx_n) + \alpha_2(fx_{n-2})q(gx_{n-1}, gx_n) + \alpha_3(fx_{n-2})q(gx_n, gx_{n+1}) \\
&\quad +\alpha_4(fx_{n-2})q(gx_{n-1}, gx_{n+1}) + \alpha_5(fx_{n-2})q(gx_n, gx_n) \\
&\preceq \alpha_1(gx_{n-2})q(gx_{n-1}, gx_n) + \alpha_2(gx_{n-2})q(gx_{n-1}, gx_n) + \alpha_3(gx_{n-2})q(gx_n, gx_{n+1}) \\
&\quad +\alpha_4(gx_{n-2})q(gx_{n-1}, gx_{n+1}) + \alpha_5(gx_{n-2})q(gx_n, gx_n) \\
&\;\vdots \\
&\preceq \alpha_1(gx_0)q(gx_{n-1}, gx_n) + \alpha_2(gx_0)q(gx_{n-1}, gx_n) + \alpha_3(gx_0)q(gx_n, gx_{n+1}) \\
&\quad +\alpha_4(gx_0)q(gx_{n-1}, gx_{n+1}) + \alpha_5(gx_0)q(gx_n, gx_n) \\
&\preceq \alpha_1(gx_0)q(gx_{n-1}, gx_n) + \alpha_2(gx_0)q(gx_{n-1}, gx_n) + \alpha_3(gx_0)q(gx_n, gx_{n+1}) \\
&\quad +\alpha_4(gx_0)[q(gx_{n-1}, gx_n) + q(gx_n, gx_{n+1})] \\
&\quad +\alpha_5(gx_0)[q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_n)]. \qquad (1)
\end{aligned}$$

Similarly, set $x = x_{n-1}$ and $y = x_n$ in (iv). Thus, by (q_2), for $n \geq 1$, we get

$$\begin{aligned}
q(gx_{n+1}, gx_n) &= q(fx_n, fx_{n-1}) \\
&\preceq \alpha_1(gx_n)q(gx_n, gx_{n-1}) + \alpha_2(gx_n)q(gx_n, gx_{n-1}) + \alpha_3(gx_n)q(gx_{n+1}, gx_n) \\
&\quad +\alpha_4(gx_n)[q(gx_{n+1}, gx_n) + q(gx_n, gx_{n-1})] \\
&\quad +\alpha_5(gx_n)[q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_n)] \\
&\preceq \alpha_1(gx_0)q(gx_n, gx_{n-1}) + \alpha_2(gx_0)q(gx_n, gx_{n-1}) + \alpha_3(gx_0)q(gx_{n+1}, gx_n) \\
&\quad +\alpha_4(gx_0)[q(gx_{n+1}, gx_n) + q(gx_n, gx_{n-1})] \\
&\quad +\alpha_5(gx_0)[q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_n)]. \qquad (2)
\end{aligned}$$

Adding up (1) and (2), we have

$$\begin{aligned}
q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_n) &\preceq (\alpha_1(gx_0) + \alpha_2(gx_0) + \alpha_4(gx_0))[q(gx_{n-1}, gx_n) + q(gx_n, gx_{n-1})] \\
&\quad +(\alpha_3(gx_0) + \alpha_4(gx_0) + 2\alpha_5(gx_0))[q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_n)]. \qquad (3)
\end{aligned}$$

Now, set $v_n = q(gx_n, gx_{n+1}) + q(gx_{n+1}, x_n)$ in (3). Thus, we have

$$v_n \preceq (\alpha_1(gx_0) + \alpha_2(gx_0) + \alpha_4(gx_0))v_{n-1} + (\alpha_3(gx_0) + \alpha_4(gx_0) + 2\alpha_5(gx_0))v_n.$$

So, $v_n \preceq hv_{n-1}$ for all $n \geq 1$ with

$$h = \frac{\alpha_1(gx_0) + \alpha_2(gx_0) + \alpha_4(gx_0)}{1 - (\alpha_3(gx_0) + \alpha_4(gx_0) + 2\alpha_5(gx_0))} < 1,$$

since $(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5)(x) < 1$ for all $x \in X$. Repeating this process, we get $v_n \preceq h^n v_0$ for $n = 0, 1, 2, \cdots$. Thus,

$$q(gx_n, gx_{n+1}) \preceq v_n \preceq h^n \Big(q(gx_0, gx_1) + q(gx_1, gx_0)\Big) \qquad (4)$$

for all $n = 0, 1, 2, \cdots$. Now, for positive integer m and n with $m > n \geq 1$, it follows from (4) and $h < 1$ that

$$\begin{aligned} q(gx_n, gx_m) &\preceq q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_{n+2}) + \cdots + q(gx_{m-1}, gx_m) \\ &\preceq (h^n + h^{n+1} + \cdots + h^{m-1})\Big(q(gx_0, gx_1) + q(gx_1, gx_0)\Big) \\ &\preceq \frac{h^n}{1-h}\Big(q(gx_0, gx_1) + q(gx_1, gx_0)\Big). \end{aligned} \tag{5}$$

Lemma 2.9 implies that $\{gx_n\}$ is a Cauchy sequence in X. Since $g(X)$ is a complete subspace of X, there exists a point $x' \in g(X)$ such that $gx_n \to x'$ as $n \to \infty$. By (5) and (q_3)

$$q(gx_n, x') \preceq \frac{h^n}{1-h}\Big(q(gx_0, gx_1) + q(gx_1, gx_0)\Big), \qquad n = 0, 1, 2, \cdots .$$

Since P is a normal cone with normal constant K, we have

$$\|q(gx_n, x')\| \leq K\Big(\frac{h^n}{1-h}\Big)\|q(gx_0, gx_1) + q(gx_1, gx_0)\|, \qquad n = 0, 1, 2, \cdots , \tag{6}$$

and

$$\|q(gx_n, gx_m)\| \leq K\Big(\frac{h^n}{1-h}\Big)\|q(gx_0, gx_1) + q(gx_1, gx_0)\|, \tag{7}$$

for all $m > n \geq 1$. If $fx' \neq x'$ or $gx' \neq x'$, then, by the hypothesis, (6) and (7) with $m = n + 1$, we have

$$\begin{aligned} 0 &< \inf\{\|q(fx, x')\| + \|q(gx, x')\| + \|q(gx, fx)\| : x \in X\} \\ &\leq \inf\{\|q(fx_n, x')\| + \|q(gx_n, x')\| + \|q(gx_n, fx_n)\| : n \geq 1\} \\ &= \inf\{\|q(gx_{n+1}, x')\| + \|q(gx_n, x')\| + \|q(gx_n, gx_{n+1})\| : n \geq 1\} \\ &\leq \inf\Big\{K(\frac{h^{n+1}}{1-h})\|q(gx_0, gx_1) + q(gx_1, gx_0)\| + K(\frac{h^n}{1-h})\|q(gx_0, gx_1) + q(gx_1, gx_0)\| \\ &\quad +K(\frac{h^n}{1-h})\|q(gx_0, gx_1) + q(gx_1, gx_0)\| : n \geq 1\Big\} = 0. \end{aligned}$$

which is a contradiction. Hence $x' = fx' = gx'$. This completes the proof. □

The following corollaries are obtained from Theorem 3.1.

Corollary 3.2. Let (X, d) be a cone metric space, P be a normal cone with constant K and q be a c-distance on X. Suppose that the mappings $f, g : X \to X$ satisfy the following two contractive conditions:

$$\begin{aligned} q(fx, fy) &\preceq \alpha_1 q(gx, gy) + \alpha_2 q(gx, fx) + \alpha_3 q(gy, fy) \\ &\quad +\alpha_4 q(gx, fy) + \alpha_5 q(gy, fx), \\ q(fy, fx) &\preceq \alpha_1 q(gy, gx) + \alpha_2 q(fx, gx) + \alpha_3 q(fy, gy) \\ &\quad +\alpha_4 q(fy, gx) + \alpha_5 q(fx, gy) \end{aligned}$$

for all $x, y \in X$, where α_i for $i = 1, 2, \cdots , 5$ are nonnegative constants such that

$$\alpha_1 + \alpha_2 + \alpha_3 + 2(\alpha_4 + \alpha_5) < 1.$$

If the range of g contains the range of f, $g(X)$ is a complete subspace of X, f and g satisfy

$$\inf\{\|q(fx,y)\|+\|q(gx,y)\|+\|q(gx,fx)\| : x\in X\}>0$$

for all $y\in X$ with $y\neq fy$ or $y\neq gy$, then f and g have a common fixed point in X.

Proof. We can prove this result by applying Theorem 3.1 with $\alpha_i(x)=\alpha_i$ for $i=1,2,\cdots,5$. □

Corollary 3.3. Let (X,d) be a cone metric space, P a normal cone with constant K and q be a c-distance. Also, let $f,g:X\to X$ be two mappings with $f(X)\subseteq g(X)$ and $g(X)$ be a complete subspace of X. Suppose that there exist mappings $\alpha_i : X\to[0,1)$ for $i=1,\cdots,4$ such that the following conditions hold:
(i) $\alpha_i(fx)\leq\alpha_i(gx)$ for all $x\in X$;
(ii) $(\alpha_1+\alpha_2+\alpha_3+2\alpha_4)(x)<1$ for all $x\in X$;
(iii) $q(fx,fy)\preceq\alpha_1(gx)q(gx,gy)+\alpha_2(gx)q(gx,fx)+\alpha_3(gx)q(gy,fy)+\alpha_4(gx)q(gx,fy)$ for all $x,y\in X$;
If f and g satisfy

$$\inf\{\|q(fx,y)\|+\|q(gx,y)\|+\|q(gx,fx)\| : x\in X\}>0$$

for all $y\in X$ with $y\neq fy$ or $y\neq gy$, then f and g have a common fixed point in X.

Proof. In Theorem 3.1, consider $\alpha_5(x)=0$. Note that we only need to accuracy the relation (iii) in Corollary 3.3. □

Corollary 3.4. Let (X,d) be a cone metric space, P be a normal cone with constant K and q be a c-distance on X. Suppose that the mappings $f,g:X\to X$ satisfy the following condition:

$$q(fx,fy)\quad\preceq\quad\alpha_1q(gx,gy)+\alpha_2q(gx,fx)+\alpha_3q(gy,fy)+\alpha_4q(gx,fy)$$

for all $x,y\in X$, where α_i for $i=1,\cdots,4$ are nonnegative constants such that

$$\alpha_1+\alpha_2+\alpha_3+2\alpha_4<1.$$

If the range of g contains the range of f, $g(X)$ is a complete subspace of X, f and g satisfy

$$\inf\{\|q(fx,y)\|+\|q(gx,y)\|+\|q(gx,fx)\| : x\in X\}>0$$

for all $y\in X$ with $y\neq fy$ or $y\neq gy$, then f and g have a common fixed point in X.

Proof. We can prove this result by applying Corollary 3.3 with $\alpha_i(x)=\alpha_i$ for $i=1,2,3,4$. □

In Theorem 3.1, if $g=i_X$ is the identity map on X, then we get the following theorem of Hardy-Rogers type on c-distance in a cone metric space.

Theorem 3.5. Let (X,d) be a complete cone metric space and P be a normal cone with constant K. Also, let q be a c-distance and $f:X\to X$ be a mapping. Suppose that there exist mappings $\alpha_i : X\to[0,1)$ such that the following conditions hold:

(i) $\alpha_i(fx) \leq \alpha_i(x)$ for all $x \in X$;
(ii) $(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5)(x) < 1$ for all $x \in X$;
(iii) $q(fx, fy) \preceq \alpha_1(x)q(x, y) + \alpha_2(x)q(x, fx) + \alpha_3(x)q(y, fy) + \alpha_4(x)q(x, fy) + \alpha_5(x)q(y, fx)$ for all $x, y \in X$;
(iv) $q(fy, fx) \preceq \alpha_1(x)q(y, x) + \alpha_2(x)q(fx, x) + \alpha_3(x)q(fy, y) + \alpha_4(x)q(fy, x) + \alpha_5(x)q(fx, y)$ for all $x, y \in X$;
If f satisfies

$$\inf\{\|q(fx, y)\| + \|q(x, y)\| + \|q(x, fx)\| : x \in X\} > 0$$

for all $y \in X$ with $y \neq fy$, then f has a fixed point in X.

Corollary 3.6. Let (X, d) be a cone metric space, P be a normal cone with constant K and q be a c-distance on X. Suppose that the mapping $f : X \to X$ satisfies the following two contractive conditions:

$$\begin{aligned} q(fx, fy) &\preceq \alpha_1 q(x, y) + \alpha_2 q(x, fx) + \alpha_3 q(y, fy) \\ &\quad + \alpha_4 q(x, fy) + \alpha_5 q(y, fx), \\ q(fy, fx) &\preceq \alpha_1 q(y, x) + \alpha_2 q(fx, x) + \alpha_3 q(fy, y) \\ &\quad + \alpha_4 q(fy, x) + \alpha_5 q(fx, y) \end{aligned}$$

for all $x, y \in X$, where α_i for $i = 1, 2, \cdots, 5$ are nonnegative constants such that

$$\alpha_1 + \alpha_2 + \alpha_3 + 2(\alpha_4 + \alpha_5) < 1.$$

If f satisfies

$$\inf\{\|q(fx, y)\| + \|q(x, y)\| + \|q(x, fx)\| : x \in X\} > 0$$

for all $y \in X$ with $y \neq fy$, then f has a fixed point in X.

Proof. We can prove this result by applying Theorem 3.5 with $\alpha_i(x) = \alpha_i$ for $i = 1, 2, \cdots, 5$. □

Corollary 3.7. Let (X, d) be a complete cone metric space and P be a normal cone with constant K. Also, let q be a c-distance and $f : X \to X$ be a mapping. Suppose that there exist mappings $\alpha_i : X \to [0, 1)$ such that the following conditions hold:
(i) $\alpha_i(fx) \leq \alpha_i(x)$ for all $x \in X$;
(ii) $(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4)(x) < 1$ for all $x \in X$;
(iii) $q(fx, fy) \preceq \alpha_1(x)q(x, y) + \alpha_2(x)q(x, fx) + \alpha_3(x)q(y, fy) + \alpha_4(x)q(x, fy)$ for all $x, y \in X$;
If f satisfies

$$\inf\{\|q(fx, y)\| + \|q(x, y)\| + \|q(x, fx)\| : x \in X\} > 0$$

for all $y \in X$ with $y \neq fy$, then f has a fixed point in X.

Proof. In Theorem 3.5, consider $\alpha_5(x) = 0$. Note that we only need to accuracy the relation (iii) in Corollary 3.7. □

Remark 3.8.
(i) Sometimes the constant numbers which satisfy Corollaries 3.2 and 3.6 are difficult to find. Thus, it is better to define such mappings $\alpha_i(x)$ as another auxiliary tool of the cone metric.
(ii) Theorems 2.1 and 2.3 of [2] cannot be applied to some examples, but by the following conditions

$$q(fx, fy) \preceq kq(gx, gy), \qquad k \in [0, 1),$$

$$q(fx, fy) \preceq k\Big(q(gx, fx) + q(gy, fy)\Big), \qquad k \in [0, \frac{1}{2})$$

for all $x, y \in X$ of Corollary 3.2, f and g have a common fixed point theorem. Thus, our Corollary 3.2 has generalized the main results of Abbas and Jungck's work [2] on c-distance in a cone metric space.

Example 3.9. Let $E = \mathbf{R}$ and $P = \{x \in E : x \geq 0\}$. Let $X = [0, 1)$ and define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define a function $q : X \times X \to E$ by $q(x, y) = d(x, y)$ for all $x, y \in X$. Then q is a c-distance. Also, let the mapping $f : X \to X$ defined by $f(x) = \frac{x^2}{4}$ for all $x \in X$. Define the mappings $\alpha_1(x) = \frac{x+1}{4}$, $\alpha_5(x) = \frac{x}{8}$ and $\alpha_2 = \alpha_3 = \alpha_4 = 0$ for all $x \in X$. Observe that:

(1) $\alpha_1(fx) = \frac{1}{4}\Big(\frac{x^2}{4} + 1\Big) \leq \frac{1}{4}\Big(x^2 + 1\Big) \leq \frac{x+1}{4} = \alpha(x)$ for all $x \in X$.

(2) $\alpha_5(fx) = \frac{x^2}{32} \leq \frac{x^2}{8} \leq \frac{x}{8} = \alpha_5(x)$ for all $x \in X$.

(3) $\alpha_i(fx) = 0 \leq 0 = \alpha_i(x)$ for all $x \in X$ and $i = 2, 3, 4$.

(4) $(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5)(x) = \frac{x+1}{4} + \frac{2x}{8} = \frac{2x+1}{4} < 1$ for all $x \in X$.

(5) For all comparable $x, y \in X$, we get

$$\begin{aligned} q(fx, fy) &= \Big|\frac{x^2}{4} - \frac{y^2}{4}\Big| \preceq \frac{|x + y||x - y|}{4} \\ &= \Big(\frac{x + y}{4}\Big)|x - y| \preceq \Big(\frac{x + 1}{4}\Big)|x - y| \\ &\preceq \alpha_1(x)q(x, y) + \alpha_2(x)q(x, fx) + \alpha_3(x)q(y, fy) \\ &\quad +\alpha_4(x)q(x, fy) + \alpha_5(x)q(y, fx). \end{aligned}$$

(6) Similarly, we have

$$\begin{aligned} q(fy, fx) &\preceq \alpha_1(x)q(y, x) + \alpha_2(x)q(fx, x) + \alpha_3(x)q(fy, y) \\ &\quad +\alpha_4(x)q(fy, x) + \alpha_5(x)q(fx, y) \end{aligned}$$

for all comparable $x, y \in X$.

(7) For any $x, y \in X$ with $y \neq Ty$, i.e., $y > 0$, we get

$$\inf\{\|q(fx, y)\| + \|q(x, y)\| + \|q(x, fx)\| : x \in X\} = 2|y - \frac{y^2}{4}| > 0.$$

Hence all the conditions of Theorem 3.5 are satisfied. Thus f has a fixed point $x = 0$.

Remark 3.10. For more details about fixed point results under c-distance in a cone metric space, see [6, 7, 10, 16, 29].

Remark 3.11. (i) Corollary 3.3 and Corollary 3.7 are same Theorem 3.1 and Corollary 3.1 of Kaewkhao et al. [16];
(ii) Corollary 3.4 is same Theorem 2.2 of Wang and Guo [31];
(iii) Corollary 3.6 is same Corollary 3.4 of Rahimi and Soleimani Rad [23].

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