STOCHASTIC OPTIMIZATION AND RISK PROBLEMS

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The paper presents a procedure for solving stochastic optimization problems encountered in industrial control applications in which it is considered that random disturbances act directly on the parameters of the criterion function and/or on the restrictions that define the admissible domain. The objective is to construct this problem, and to reformulate it by an equivalent deterministic problem. This problem is introduced as a risk problem and is solved using appropriate non-linear mathematical programming techniques. In many cases the solution of the stochastic optimization problem represents the optimal decision for the control level in industrial applications.

Keywords: stochastic optimization, risk problems, non-linear problems, numerical example.

1. Introduction

The parametric optimization theory has been developed using the numerical means for solving linear or nonlinear mathematical programming problems.

Different numerical methods can be used for evaluating the solution of the standard linear or non-linear problem [1],[2],[3].

We consider the problem:

$$\left[\begin{array}{l}\max\left\{I(\mathbf{y})=\mathbf{c}^T \mathrm{f}(\mathbf{y})\right\} \\ \begin{cases}\mathbf{Ay}\le\mathbf{b} \\ \mathbf{y}\ge\mathbf{0}\end{cases}\end{array}\right. , \tag{1}$$

where I(y) is the criterion function defined of the admissible domain D_{adm}, limited by the inequalities' constraints in (1), $\mathbf{A}\in\mathbf{R}^{nr\times nc}$ $(nr\le nc)$ and $\mathbf{c},\mathbf{b},\mathbf{y}\in\mathbf{R}^{nc}$. We should highlight that in (1) vector f(**y**) is nonlinear with respect to the **y** components.

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In many practical applications, the optimization problems are expressed in a stochastic manner. In this case, the parameters of the optimization criterion and/or the associated constraints functions depend on a random process, ω. Thus, the problem described in (1) becomes a stochastic nonlinear programming problem:

$$\begin{bmatrix} \max\left\{ I(\mathbf{y}) = \mathbf{c}^T(\omega)\,\mathrm{f}(\mathbf{y}) \right\} \\ \begin{cases} \mathbf{A}(\omega)\mathbf{y} \le \mathbf{b}(\omega) \\ \mathbf{y} \ge \mathbf{0} \end{cases} \end{bmatrix} \tag{2}$$

As an example, we can consider a pyrolysis reactor, an important petrochemical installation, which is usually optimized by a criterion function which maximizes the concentration of the main product, ethylene. The quality of the used raw material, gasoline, depends on a stochastic variable, which can modify strongly the operating point of the plant [4].

In a thermo-energy installation, the efficient exploitation goal is to increase the energy transfer from the thermal agent to the product. The stochastic variable can be considered the quality of the thermal agent (which is expressed by two parameters: temperature and pressure), distributed by the provider which is, in this case, a thermal power plant. The thermal agent provides the heat exchanger with variable temperature and/or pressure, either due to the energy provider that does not deliver the agent to pre-specified parameters, or due to heat losses on the transport pipes. It is therefore necessary to maximize the heat transfer efficiency from the agent to the product in actual stochastic operating conditions.

It is known that a combustion process is described by an extremal characteristic that expresses the evolution of the temperature, depending on the air flow and the fuel flow. The extremal characteristics are parameterized by the used fuel flow values. A good control solution for the combustion process can ensure the correct fuel and air flow rate for optimal operation. At an industrial level, there is a random variable ω, that changes the position of the extremal point, namely the quality of the fuel, which is supplied at random calorific values. An efficient exploitation approach of the combustion process is thus obtained by solving a stochastic optimization problem [5].

A typical example can also be considered the process of preheating a blast furnace which ensures the air heating in a supplying facility. This installation uses a fuel that is a random mixture of stochastic components: coke oven gas, blast furnace gas and methane. By solving a stochastic optimization problem for the combustion regime, we can optimize and reduce the consumption of methane, which is the most expensive component, and therefore the price of the product [6].

Another significant example is associated with the combustion process of vehicles engine (Diesel engine), which can be controlled to reduce engine fuel consumption and pollution. An efficient solution for the combustion process is achieved by properly proportioning the fuel and the air flow which can significantly improve engine performances. However, the Diesel fuel (e.g. normal, or premium) quality is a random variable, ω, and an optimal operation of the engine is guaranteed by solving a stochastic problem [5].

The last example and perhaps most illustrative one is inspired by the exploitation of a wind turbine or photovoltaic panels. The objective is to maximize the quantity of electrical generated energy, from renewable sources, which depends on environmental parameters: speed and direction of the wind, solar radiation and temperature; the process of electrical generation is by its nature, a random process [7].

If we relate to photovoltaic generators, the functional characteristic that expresses the dependence of the generated power on the voltage at the generator terminals has an extreme representation with a maximum power point (MPP). This representation depends on a random variable, considered as the parameter ω, which represents the solar radiation. The problem can be solved with the help of stochastic programming [8].

In all these cases, the effect of the perturbation ω will lead to random variations of the criterion function's parameters and/or of the admissible domain D_{adm}, limited by constraints. The idea is to adapt the deterministic optimization algorithms to this stochastic case, or to find a deterministic equivalent representation for the stochastic problem.

In the first approach, the gradient algorithm was adapted to a stochastic gradient algorithm. The promoters of studies on the techniques of the stochastic gradient are mainly H. Robbins and D. Sigmund [9], J. Kieffer and J. Wolfowitz [10]. Based on this work, other authors have studied in some cases the performance of the stochastic gradient algorithm in terms of efficiency and asymptotic convergence.

From the more recent studies we can quote those of B.T. Polyak on the convergence and convergence speed for this type of algorithm [11]. In this case, B.T. Polyak has obtained an important result: the introduction of the stochastic gradient algorithm of the averaging technique which guaranteed a certain sense of optimality.

Taking into consideration that for the first approach there are known results in the specialized literature and that deterministic techniques for mathematical programming are available, the paper is focused on the second option consisting in the construction of a deterministic problem equivalent to the stochastic optimization problem.

After a short introduction, to illustrate that industrial applications demand to solve stochastic optimization problems, in the second section a non-linear stochastic optimization problem is detailed; depending on the requirements that need to be fulfilled by the solution we have either a totally or a partially admissible set of solutions. Section 3 formulates the risk problem and section 4 gives a numerical example to support the validity of the presented solution. In the last section, some concluding remarks are given.

2. Stochastic optimization problem

Let us consider the stochastic optimization problem given by (2), where the triplet $\{\mathbf{A}(\omega), \mathbf{b}(\omega), \mathbf{c}(\omega)\}$ is a set of stochastic variables defined for each elementary event $\omega \in \Omega$ (Ω being the events' space). Then (2) is a *non-linear stochastic program*.

For all the values $\mathbf{y} \in D_{adm}$, where D_{adm} is the admissible domain of the variables in (2), the expression:

$$\max_{\mathbf{y} \in D_{adm}} \left\{ I(\mathbf{y}) = \mathbf{c}^{\mathrm{T}}(\omega) \mathrm{f}(\mathbf{y}) \right\}, \tag{3}$$

has lost the optimization sense, since $I(\mathbf{y})$ is a random variable which is not submitted to the order relation. A similar consideration is valid for the constraints inequalities which define the domain D_{adm}.

Suppose that for each $\omega \in \Omega$ we can solve the problem given in (2), for a ω set, with inequalities constraints. Let $\mathbf{y}^*(\omega)$ be the optimal solution and $I^*(\mathbf{y}^*(\omega)) = \mathbf{c}^{\mathrm{T}} \mathrm{f}(\mathbf{y}^*(\omega))$ the optimal value of the criterion corresponding function $I(y)$. Since $\mathbf{y}^*(\omega)$ and $I^*((\mathbf{y}^*(\omega))$ are random variables, it results that $I^*(\mathbf{y}^*(\omega))$ cannot be calculated in an exact manner, but it can be expressed in statistical terms. In order to find the solution to the problem (3), we consider the case where this problem can be reformulated in a deterministic point of view.

If we have the repartition function F, the calculation of the probability P for which $I(y_j)$ belongs to the given interval $[a,b]$ is easily made. It is obvious that:

$$P(a \le I(\mathbf{y}) \le b) = \mathrm{F}(b) - \mathrm{F}(a), \tag{4}$$

In the case where we do not have the possibility to calculate F in an exact or even approximate manner, the probability P is expressed using the Laplace function:

$$L(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{\frac{-t^2}{2}} dt, \tag{5}$$

2.1. Stochastic optimization with totally admissible solutions

We specified that if vector $\mathbf{c}$ is a random vector, then the non-linear problem (3) has no sense in terms of optimization. This remark is also valid for representation (3) in the case of the triplet $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ being a random variable. In order to determine an optimal decision $\mathbf{y}^* \in D_{adm}$, it is possible to build an optimization problem in a deterministic sense, with totally admissible solutions, when $\mathbf{y}^*$ verifies all the realizations of the random variables $(\mathbf{A},\mathbf{b,c})$.

We consider the non-linear stochastic problem (3) with the associated triplet $(\mathbf{A},\mathbf{b,c})$ as random variables:

$$\left[\begin{array}{l}\max\left\{I(\mathbf{y})=\mathbf{c}^T(\omega)\,\mathrm{f}(\mathbf{y})\right\}\\ \begin{cases}\mathbf{A}(\omega)\mathbf{y}\le\mathbf{b}(\omega)\\ \mathbf{y}\ge\mathbf{0}\end{cases}\end{array}\right.,\tag{6}$$

and notice that, for each $\omega\in\Omega$, the inequalities:

$$\begin{cases}\mathbf{A}(\omega)\mathbf{y}\le\mathbf{b}(\omega)\\ \mathbf{y}\ge 0\end{cases},\tag{7}$$

give the following convex set:

$$D(\omega)=\left\{\mathbf{y}\;\middle|\;\mathbf{A}(\omega)\mathbf{y}\le\mathbf{b}(\omega),\ \mathbf{y}\ge 0\right\},\tag{8}$$

We define the global convex set through the following relation:

$$D=\bigcap_{\omega\in\Omega}D(\omega),\tag{9}$$

If ensemble D is non-empty, we name it the *set of totally admissible solutions* to problem (2). The domain D can be easily determined with $(\mathbf{A},\mathbf{b})$ as discrete random variables, only taking the values $\{\mathbf{A}_i,\mathbf{b}_i\}_{i\in\overline{1,l}}$ [12].

In this case:

$$D=\bigcap_{i=1}^{l}\{\mathbf{y}\mid\mathbf{A}_i\mathbf{y}\le\mathbf{b}_i,\ \mathbf{y}\ge 0\},\tag{10}$$

and the problem has totally admissible solutions corresponding to some types of applications [13].

2.2. Stochastic optimization with partially admissible solutions

Sometimes, the requirement the totally admissible solution $\mathbf{y}^*$ must fulfill is difficult to achieve. In this case, a richer set than the domain D must be employed.

Let us consider in this case the general problem below:

$$\left[\begin{array}{l}\max\left\{\left(\mathbf{c}^T\mathbf{f}\ (\mathbf{y})\right)\right\}\\ \begin{cases}\mathbf{Ay}=\mathbf{b}\\ \mathbf{Cy}=\mathbf{d}\\ \mathbf{y}\ge\mathbf{0}\end{cases}\end{array}\right.,\tag{11}$$

where (**A, b, c**) is a random triplet and (**C**, **d)** are constants.

We define:

$$D_1=\left\{\mathbf{y}\;\middle|\;\mathbf{Cy}=\mathbf{d},\ \mathbf{y}\ge 0\right\},\tag{12}$$

the set of all the solutions that verify the deterministic constraints. For each $\mathbf{y} \in D_1$, the constraints violation is tested through the expression:

$$\mathbf{Be} = \mathbf{b} - \mathbf{Ay}, \tag{13}$$

The elements of matrix $\mathbf{B}$ are set depending on the way one would like to have a correspondence in (13). For example, $\mathbf{B}$ can be expressed by the unit matrix $\mathbf{I}$, in order to have the natural correspondence.

The set:

$$\tilde{D} = \{ y \in D_1 \mid \forall \omega \in \Omega, \exists \mathbf{y}(\omega) \geq 0, \mathbf{Be}(\omega) = \mathbf{by}(\omega) - \mathbf{A}(\omega) \}, \tag{14}$$

gathers all the partially admissible solutions.

Let $M(\mathbf{Ay}, \mathbf{b}) = M(\mathbf{e})$ be the function expressing the losses associated to the original problem (as a result of constraints violation). This function will be defined and evaluated with the help of vector $\mathbf{e}$ from equation (13).

Problem (13) with losses, is formulated as follows:

$$\min_{\mathbf{y} \in D_1} \left\{ \left(\mathbf{c}^T f(\mathbf{y}) \right) + M(\mathbf{Ay}, \mathbf{b}) \right\}, \tag{15}$$

The most usual representations of $M(\mathbf{Ay}, \mathbf{b})$ are:

- in linear form,

$$M(\mathbf{Ay}, \mathbf{b}) = \mathbf{k}^T (\mathbf{b} - \mathbf{Ay}), \tag{16}$$

where $\mathbf{k}$ is a vector of constant weights; in this case, problem (15) is equivalent to:

$$\min_{\mathbf{y} \in D_1} \left\{ \mathbf{c}^T f(\mathbf{y}) + \mathbf{k}^T (\mathbf{b} - \mathbf{Ay}) \right\}, \tag{17}$$

- of first order,

$$M(\mathbf{Ay}, \mathbf{b}) = \mathbf{k}^T (\mathbf{b} - \mathbf{Ay}) + \left\| \mathbf{A}_1 (\mathbf{b} - \mathbf{Ay}) \right\|, \tag{18}$$

where $\mathbf{A}$ is a constant matrix a priori chosen; in this case, problem (15) is equivalent to:

$$\min_{\mathbf{y} \in D_1} \left\{ \mathbf{c}^T f(\mathbf{y}) + \mathbf{k}^T (\mathbf{b} - \mathbf{Ay}) + \left\| \mathbf{A}_1 (\mathbf{b} - \mathbf{Ay}) \right\| \right\}, \tag{19}$$

- of second order,

$$L(\mathbf{Ay}, \mathbf{b}) = \mathbf{k}^T (\mathbf{b} - \mathbf{Ay}) + \left\| \mathbf{A}_1 (\mathbf{b} - \mathbf{Ay}) \right\|^2, \tag{20}$$

and problem (15) is then equivalent to:

$$\min_{\mathbf{y} \in D_1} \left\{ \mathbf{c}^T f(\mathbf{y}) + \mathbf{k}^T (\mathbf{b} - \mathbf{Ay}) + \left\| \mathbf{A}_1 (\mathbf{b} - \mathbf{Ay}) \right\|^2 \right\}, \tag{21}$$

This type of problems generally leads to convex nonlinear programs, which are approached by the corresponding methods in order to provide the $\mathbf{y}^*$ solutions [14].

In some cases, the loss function is non-convex, and the optimization problem becomes difficult to solve. In this situation, solutions that have an imposed probability and that violate the constraints without losses are considered.

A minimal limit α_i of the probability is set, so that the constraint $i \in 1, nr$ can be verified for $y_j \in D_1$:

$$P\left(\sum_{j=1}^{nc} a_{ij} y_j \le b_i\right) > \alpha_i \,, \tag{22}$$

The set of the admissible solutions is defined below as:

$$\tilde{D}_a = \left\{ \mathbf{y} \in D_1 \middle| P\left(\sum_{j=1}^{nc} a_{ij} y_j \le b_i\right) \ge \alpha_i, \forall i \in \overline{1, nr} \right\}, \tag{23}$$

To remain in the lines of representation (11), this problem should be expressed like the one with totally admissible solutions (see section 2.1), but for D_a instead of D_{adm}. The domain D_a is easy to determine for a constant matrix $\mathbf{A}$ [13].

3. The risk problems

We can consider the following relation:

$$\max_{\mathbf{y} \in D_{adm}} \left\{ E\left(\mathbf{c}^T(\omega) \mathrm{f}(\mathbf{y})\right) \right\}, \tag{24}$$

where E is the mean (average) operator. For a random vector $\mathbf{c}(\omega)$ with a normal distribution, we have:

$$\begin{cases} m_i = E(c_i) \\ v_{ij} = E\left[(c_i - m_i)(c_j - m_j)\right] \end{cases}, \tag{25}$$

and problem (24) becomes:

$$\max_{\mathbf{y} \in D} \left\{ I_a(\mathbf{y}) = \mathbf{m}^T \mathbf{f}(\mathbf{y}) \right\}, \tag{26}$$

where **m** is a vector of m_i components.

The solution to (26), i.e. $\mathbf{y}_a^* \in D_{adm}$, is the solution of the average representation, respectively of problem (24) and it is calculated, as a deterministic problem, by means of an exact optimization method [1].

Based on the result from (26) associated with the problem given by (2), we propose the following problem:

$$\left[\begin{array}{l} \min\{\alpha\} \\ \begin{cases} P\left(\mathbf{c}^T \mathrm{f}(\mathbf{y}) < I_0\right) = \alpha \,, \\ \mathbf{y} \in D_{adm} \end{cases} \end{array}\right. \tag{27}$$

in order to minimize α, where α is the probability P that the criterion function $I(\mathbf{y}) = c^T f(\mathbf{y})$ takes a smaller value than an imposed value I_0 (I_0 is a value inferior to the value $I(\mathbf{y}_a^*)$). In this case we have:

$$P\left(I\left(\mathbf{y}\right) = \mathbf{c}(\omega)^T \mathbf{f(y)}\right) \leq I_0, with\, I_0 \leq I(y_a^*) \tag{28}$$

The α probability corresponds to the risk of not obtaining values of the function $I(\mathbf{y})$ smaller than I_0.

Problem (28) is usually considered as a problem of *minimal risk* and now, we build the minimal risk problem, in the following representation:

$$\begin{aligned}\min\{\alpha\} &= \min\left\{P\left(\mathbf{c}^T\mathbf{f(y)} < I_0\right)\right\} = \min\left\{P\left(\frac{\mathbf{c}^T\mathbf{f\ (y)\text{-}m}^T\mathbf{f(y)}}{\sqrt{\mathbf{y}^T\mathbf{Vy}}}) < \frac{I_0 - \mathbf{m}^T\mathbf{f(y)}}{\sqrt{\mathbf{y}^T\mathbf{Vy}}}\right)\right\} \\ &= \min\left\{L\left(\frac{I_0 - \mathbf{m}^T\mathbf{f(y)}}{\sqrt{\mathbf{y}^T\mathbf{Vy}}}\right)\right\}\end{aligned} \tag{29}$$

In equation (29), L is the Laplace function, $\mathbf{m}^T$ the vector of components $m_i = E\left(c_i\right)$ and $\mathbf{V}$ is the auto covariance matrix with elements $v_{ij} = E\left[\left(c_i - m_i\right)\left(c_j - m_j\right)\right]$.

Since the Laplace function L is increasing and $\mathbf{V}$ is a positive definite matrix, the problem (29) is reduced to the new optimization problem for the argument of L:

$$\min_{\mathbf{y}\in D}\left\{\frac{I_0\,\text{-}\mathbf{m}^T\mathbf{f(y)}}{\sqrt{\mathbf{y}^T\mathbf{Vy}}}\right\}, \tag{30}$$

In any case, our intent is to solve the stochastic optimization problem with an imposed risk α_0, and let us consider the problem:

$$\max_{y\in D_{adm}} I(\mathbf{y}) = \mathbf{c}^T(\omega)\mathrm{f}(\mathbf{y}), \tag{31}$$

and $P\{I(\mathbf{y}) = \mathbf{c}^T(\omega)\mathrm{f}(\mathbf{y}) < I_0\} = \alpha_0$, is now the imposed probability P that the criterion function $I(\mathbf{y}) = c^T\,\mathrm{f}(\mathbf{y})$ takes a smaller value than a value I_0, inferior to the value $I(\mathbf{y}_a^*)$.

Using the results from (29) and (30), we have:

$$\alpha_0 = P\left(\mathbf{c}^T\mathbf{y} < I_0\right) = P\left(\frac{\mathbf{c}^T\mathbf{f(y)\text{-}m}^T\mathbf{f(y)}}{\sqrt{\mathbf{y}^T\mathbf{Vy}}}) < \frac{I_0 - \mathbf{m}^T\mathbf{f(y)}}{\sqrt{\mathbf{y}^T\mathbf{Vy}}}\right), \tag{32}$$

For α_0 imposed risk and L a bijective function, it results:

$$\frac{I(\mathbf{y})\text{-}I_a(\mathbf{y}_a^*)}{\sqrt{\mathrm{f}^T(\mathbf{y})\mathbf{V}\,\mathrm{f}(\mathbf{y})}} = \beta_0 = \mathrm{L}^{-1}(\alpha_0), \tag{33}$$

The equivalent deterministic problem for (33), will be:

$$\max\{I(\mathbf{y}) = I_a(\mathbf{y}_a^*) + \beta_0\sqrt{\mathrm{f}^T(\mathbf{y})\mathbf{V}\,\mathrm{f}(\mathbf{y})}\}, \tag{34}$$

The α_0 probability corresponds to the imposed risk of not obtaining the values of the function $I(\mathbf{y})$ smaller than the I_0 value. The problem (34) is usually considered as a problem of *imposed risk*.

4. Numerical example

Over the last decade, there have been major concerns in reorienting policies to produce renewable energy, from solar and wind power sources. The results are quite promising, but the production from renewable sources is still costly and strongly dependent on environmental parameters. In this context, automated control tools can provide effective solutions for the exploitation of renewable energy sources.

Let us consider the characteristic of a photovoltaic generator, that expresses the dependence of the electrical power by the generated current and voltage, $P = f(U, I)$, illustrated in Figure 1. In this case the random variable ω is the solar radiation. Based on the instances of ω, a set of corresponding functional characteristics is generated.

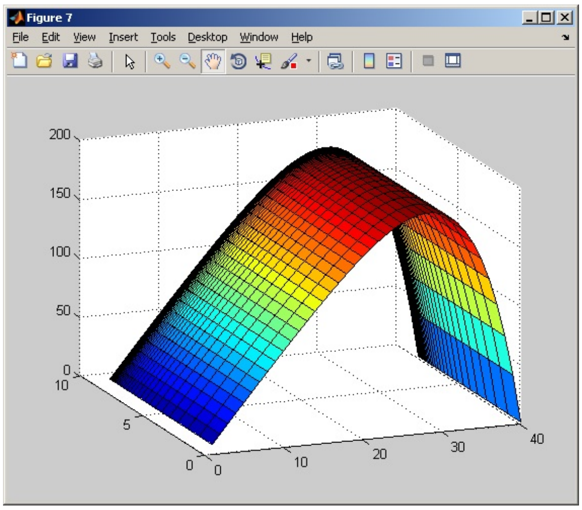

Fig. 1. 3D characteristic, power as a function of voltage and current

We consider a region on the side of the previous characteristic, in which the dependence $P = f(U, I)$ can be approximated by a linearized criterion function and the optimization problem becomes linear stochastic.

By using the previous considerations, the stochastic optimization problem can be constructed:

$$\begin{bmatrix} \max\left\{ I(\mathbf{y}) = \mathbf{c}(\omega)^T \mathbf{y} \right\} \\ \begin{cases} \mathbf{Ay} \le \mathbf{b} \\ \mathbf{y} \ge \mathbf{0} \end{cases} \end{bmatrix} \tag{35}$$

In relation (35), the criterion function I(y) represents the electrical generated power; y is the vector of the generated electric variables: voltage(y1) and current(y2), and the solar radiation is the stochastic variable ω. For this problem, we can find a deterministic equivalent, if we accept the hypotheses:

- the random vector $\mathbf{c}(\omega)$ has a normal non-degenerated distribution with,

$$\begin{cases} m_i = E(c_i) \\ v_{ij} = E\left[(c_i - m_i)(c_j - m_j)\right] \end{cases}, \tag{36}$$

- the optimization problem,

$$\max_{\mathbf{y}\in D} I_a(\mathbf{y}) = \left\{ \mathbf{m}^T \mathbf{y} \right\}, \tag{37}$$

with $\mathbf{m}^T$ of components $m_i = E(c_i)$ is a deterministic vector.

The deterministic problem (37) admits the solution $\mathbf{y}_a^*$ and the maximum value of the criterion function will be $I_a(\mathbf{y}_a^*) = \mathbf{m}^T \mathbf{y}_a^*$, for which we choose $I_0 < I_a(\mathbf{y}_a^*)$.

For the initial problem (35), we computed, using ω instantiations, the set (A, b):

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \tag{38}$$

The parameters vector $\mathbf{c}$ is a random vector with a normal repartition of an average value $\mathbf{m^T} = [2 \quad 3]$, and the auto covariance matrix $\mathbf{V}$, based on relations (36), it results:

$$\mathbf{V} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \tag{39}$$

By using an adequate method, [14],[15], the initial problem (37) admits the following optimal solution:

$$\begin{cases} y_{1a}^* = 1.5 \\ y_{2a}^* = 1 \end{cases}, \ I_a(\mathbf{y}_a^*) = \mathbf{m^T} \mathbf{y}_a^* = 6.1 \tag{40}$$

We propose $I_0 = 1.25$, a value which respects the condition $I_a(\mathrm{y}_a^*) \ge I_0 = 1.25$, and consider the equivalent problem with (34) according to (29):

$$\left[\begin{array}{l} \min\left\{\dfrac{I_0 - \mathbf{m}^T\mathbf{y}}{\sqrt{\mathbf{y}^T\mathbf{V}\mathbf{y}}}\right\} \\ \begin{cases} \mathbf{A}\mathbf{y} \le \mathbf{b} \\ \mathbf{y} \ge \mathbf{0} \end{cases} \end{array}\right. \tag{41}$$

Relation (43), becomes:

$$\left[\begin{array}{l} \max\left\{\dfrac{2y_1 + 3y_2 - 1,25}{\begin{bmatrix} y_1 & y_2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}\right\} \\ \begin{cases} y_1 + y_2 \le 2 \\ -3y_1 - 2y_2 \le 3 \\ y_1, y_2 \ge 0 \end{cases} \end{array}\right. \tag{42}$$

The solution of this problem, evaluated by means of the corresponding gradient method [14],[15], is:

$$\begin{cases} y_1^* = 0.556 \\ y_2^* = 1.455 \end{cases}, \tag{43}$$

where y_1^* and y_2^* are the voltage and the current values in the optimal exploitation point. The minimal risk $\hat{\alpha}$ was calculated from relation (29):

$$\hat{\alpha} = L\left(\frac{1.25 - 2\cdot 0.556 - 3\cdot 1.455}{\sqrt{\begin{bmatrix} 0.556 & 1.455 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0.556 \\ 1.455 \end{bmatrix}}}\right) = 0.045 \tag{44}$$

The value $\hat{\alpha} = 0.045$ represents the minimal risk and $1 - \hat{\alpha} = 0.955$, represents respectively, the level of confidence for the criterion function value $I(\mathbf{y}^*) = 5.477$, which is superior to the imposed value, $I_0 = 1.25$. For computational ease, in this numerical example, normalized values are used.

5. Conclusions

The paper presents a methodology for the mathematical solution of stochastic optimization problems encountered in industrial applications, where random disturbances act directly on the parameters of the criterion function and/or on the constrains.

We proposed to solve a stochastic optimization problem by reformulating it as an equivalent nonlinear optimization deterministic problem. This problem is considered as a minimal risk problem or as an imposed risk problem, and it is solved by appropriate non-linear mathematical programming techniques.

In order to validate the proposed mathematical solution, a calculus on an illustrative numerical example, i.e. solar generated renewable energy, for a linear stochastic optimization problem is presented. This example can be adapted for linear/nonlinear stochastic problems in order to efficiently manage the optimal decisions in modern control of industrial applications.

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R E F E R E N C E S

[1]. *P. Borne, D. Popescu D., F.G. Filip, D. Stefanoiu*, Optimization for Engineering Techniques, Exact Methods, John Wiley Press, London, 2012.
[2]. *D. Popescu, A. Gharbi, D. Stefanoiu, P. Borne*, Process Control Design for Industrial Applications, John Wiley Press, London, 2017.
[3]. *C. Dimon, G. Dauphin-Tanguy, D. Popescu, I.A. Tache*, Numerical control for hydrodynamic traffic flow models, *1st International Conference on Systems and Computer Science - ICSCS2012*, Villeneuve d'Ascq, France, 2012.
[4]. *X. Zeng, M. Anitescu, C. Pereira, M. Regalbuto*, A framework for chemical plat safety assessment under uncertainty, Studies in Informatics and Control, vol 18(1), pp. 7-20, 2009.
[5]. *D. Popescu, F. Ionescu, R. Dobrescu, D. Stefanoiu*, Modeling in Engineering Industrial Processes, Ed. AGIR, Bucharest, 2011.
[6]. *D. Popescu, D. Stefanoiu, C. Lupu, C. Petrescu, B. Ciubotaru, C. Dimon*, Automatica industriala (Industrial Automation), Editura Agir, Bucuresti, 2006.
[7]. *A. Pintea, H. Wang, N. Christov, P. Borne, D. Popescu*, Modelling and power regulation of horizontal variable speed wind turbines, Studies in Informatics and Control, vol 20(3), pp. 305-312, 2011.
[8]. *D. Stefanoiu, P. Borne, D. Popescu, F.G. Filip, A. El Kamel*, Optimization in Engineering Sciences, Metaheuristics, Stochastic Methods and Decision Support, John Wiley Press, London, 2014.
[9]. *H. Robbins, D. Siegmund*, A Convergence Theorem for Nonnegative Almost Supermartingale and Some Applications, Optimizing Methods in Statistics, Academic Press, New York, 1971.
[10]. *K. Kieffer, W. Wolfovitz*, Stochastic Estimation of the Maximum of a Regression Function, Annals of Mathematical Satistics, 1992.
[11]. *B.T. Polyak*, Acceleration of Stochastic Approximation by Averaging, SIAM Review, vol. 30, 1992.
[12]. *J.M. Dion, D. Popescu*, Optimisation des Systèmes-Commande Optimale, Diderot, Paris, 1997.
[13]. *B. Garner, E. Welzl*, Linear Programming-Randomization and Abstract Framework, Springer Verlag, Berlin, Germany, 1996.
[14]. *F.G. Filip, D. Popescu, M. Mateescu*, Optimal Decisions for Complex Systems-software Packages, Mathematics and Computers in Simulation, 76, pp. 422-429, 2009.
[15]. *D. Popescu, M. Serbanescu*, Software Package for Optimal Decisions Large Scale Systems, IFAC Symposium on Large Scale Systems, LSS Bucharest, (pp. 225 – 230), 2001.