THE CREDIBILITY MODEL FOR EXPONENTIAL PRINCIPLE WITH DEPENDENCE OVER RISKS AND ERRORS

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In this paper, we investigate the Bühlmann and Bühlmann-Straub credibility models with a certain dependence structure over risks and over errors under exponential principle. By means of orthogonal projection, the inhomogeneous Bühlmann credibility estimator of individual exponential premium is derived. The model is also extended to Bühlmann-Straub credibility. Our models generalize some well known existing results in credibility theory.

Keywords: Exponential principle, Credibility estimator, Dependence, Orthogonal projection.

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1. Introduction

In insurance practice, credibility theory is a set of quantitative methods which allows an insurer to adjust premium based on the policyholder's experience and the experience of the entire group of policyholders. It has been widely used in commercial property of liability insurance and group health or life insurance. The modern credibility theory is believed to be attributed to the remarkable contribution by Bühlmann [1], which is the first one that based the credibility theory on modern Bayes statistics. The credibility premium is represented by a credibility form

$$\text{credibilit premium} = Z \times \text{experience} + (1 - Z) \times \text{collective premium},$$

where $Z \in [0, 1]$ is called the credibility factor. It has been applied to broad insurance practices such as automobile insurance and workers compensation in loss reserving. For the recent detailed introduction, see [2, 3], which describes modern credibility theory comprehensively.

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Let $X_1, \cdots, X_K$ denote K risks under observation. The distribution of each X_i is characterized by its risk parameter Θ_i and contributes a sequence of claims experience $X_i = (X_{i1}, X_{i2}, \cdots, X_{iK})$ over n time period. The classical credibility theory introduced in [1, 3] and studied by many subsequent researchers, a common assumption is that the random vectors (X_i, Θ_i) (with $i = 1, \cdots, K$) are independent across individuals (independence over risks) and for each i, $X_{i1}, \cdots, X_{iK}$ are conditionally independent given Θ_i (conditional independence over time). While such independence assumptions may be at least approximately appropriate in some practical situations, it is far from a universal structure in this complex world. A rather special credibility model with claim dependence characterized by common effects was introduced in [4], the credibility premium under normally distributed claim amounts was derived. Wen [5] studied the credibility estimator under distribution-free cases with common effects, furthermore, the credibility models with dependence over risks were obtained, such as [6, 7, 8]. However, from the point of view of practical application, the credibility estimator can not be applied by insurance company. In general, in order to serve the purpose for insurance business, the insurance practice demands that the premium must be charged under some adaptable premium principle, see [9, 10, 11].

This present paper is an extension of the work of Wen [12], while they discussed the Bühlmann credibility model based on a certain assumption, i.e., random vectors (X_i, Θ_i) (with $i = 1, \cdots, K$) are independent and $X_{i1}, \cdots, X_{iK}$ are conditionally independent given Θ_i. This study consider Bühlmann and Bühlmann-Straub models with dependence structure over risks and errors. In this paper, we aim at studying the credibility model under exponential principle to account for not only a certain uniform conditional dependence for claim amounts each individual risk, but also a special dependence over risks. We introduce the individual premium under exponential principle, which is denoted by $H(X, \Theta_i) = \frac{1}{\alpha}\log[\mu(\Theta_i)]$, where $\mu(\Theta_i) = \mathrm{E}(\exp(\alpha X_{ij})|\Theta_i)$ and $\alpha > 0$. By means of orthogonal projection, we obtain the inhomogeneous Bühlmann credibility estimator with dependence structure over risks and errors, i.e., $\widehat{\mu(\Theta_i)}$. It can be written as a weighted sum of individual mean, total weighted mean and collective premium. Furthermore, the individual exponential premium is derived. Finally, the model is extended to Bühlmann-Straub credibility in which the natural weights among contracts are introduced. Though the inhomogeneous credibility estimator of $\mu(\Theta_i)$ under Bühlmann-Straub model is no longer universal form for the classical credibility premiums, it reveals an important fact that the weights in the formulae also can be added to 1.

The rest of the paper is arranged as follows. In Section 2, models and assumptions are introduced and some preliminaries are discussed. Section 3 derives the inhomogeneous Bühlmann credibility estimator with dependence over risks and errors under exponential principle. The result is extended to the

Bühlmann-Straub model in Section 4. Some conclusions are made in Section 5.

2. Model Assumptions and Preliminaries

Definition 2.1. *The individual premium under exponential principle is given by*

$$H(X,\Theta_i) = \frac{1}{\alpha}\log[\mu(\Theta_i)], \qquad \alpha > 0, \tag{1}$$

where $\mu(\Theta_i) = E(exp(\alpha X_{ij})|\Theta_i)$.

By analogy to the assumptions by Bühlmann [1], we consider the credibility models with dependence structure over risks and errors. The assumptions are stated as the following.

Assumption 2.1. For fixed contract i, given Θ_i, the X_{ij} follows the model: $\mathrm{e}^{\alpha X_{ij}} = \mu(\Theta_i) + \epsilon_{ij}$, and the errors are conditionally uniformly dependent, i.e, $\mathrm{Corr}(\epsilon_{il}, \epsilon_{it}) = \rho_i, l \neq t$ and $\rho_i < 1$. We also assume that $\mathrm{E}(\epsilon_{ij}|\Theta_i) = 0$ and $\mathrm{Var}(\epsilon_{ij}|\Theta_i) = \sigma^2(\Theta_i)$.

Assumption 2.2. The risk parameters $\Theta_1, \Theta_2, \cdots, \Theta_K$ are dependent with the same structure distribution function $\pi(\theta)$. In addition, we assume $\mathrm{E}(\mu(\Theta_i))$ $= \mu,\ \mathrm{Var}(\mu(\Theta_i)) = \tau^2,\ \mathrm{E}(\sigma^2(\Theta_i)) = \sigma^2,\ \mathrm{Corr}(\mu(\Theta_i), \mu(\Theta_j)) = \eta_i\eta_j,\ i \neq j.$

Our goals are to estimate the individual exponential premium $H(X,\Theta_i) = \frac{1}{\alpha}\log[\mathrm{E}(\mu(\Theta_i)]$ of the i-th contracts for $i = 1, 2, \cdots, K$. Corresponding to inhomogeneous credibility estimators of $\mu(\Theta_i)$, we must solve the following optimal problem

$$\min_{c_0, c_{ij} \in R} \mathrm{E}\{\mu(\Theta_i) - c_0 - \sum_{i=1}^{K}\sum_{j=1}^{n} c_{ij}\exp(\alpha X_{ij})\}^2. \tag{2}$$

To simplify, we set $Y_{ij} = \exp(\alpha X_{ij}), i = 1, 2, \cdots, K,\ \ j = 1, 2, \cdots, n$ and $Y = (Y_1', Y_2', \cdots, Y_K')'$, here $Y_i = (Y_{i1}, Y_{i2}, \cdots, Y_{i,n})'$. Hence we can rewrite the optimal problem as

$$\min_{c_0,\, c_s} \mathrm{E}\{\mu(\Theta_i) - c_0 - \sum_{s=1}^{K} c_s' Y_s\}^2, \tag{3}$$

where $c_0 \in R,\ c_s \in R^{n_s}$.

Generally, the credibility estimator is defined as the best estimator which is a linear function of all observations in the portfolios. We denote the linear function as follows

$$L(Y,1) = \{c_0 + \sum_{s=1}^{K} c_s' Y_s, c_0 \in R, c_s \in R^{n_s}\}. \tag{4}$$

The inhomogeneous credibility premium of $\mu(\Theta_i)$ is defined, which is denoted by $\widehat{\mu(\Theta_i)}$.

The following lemma states some simple but fundamental features of the dependence structure just specified.

Lemma 2.1. *Under the assumption 2.1-2.2, we have*

(1) The means of Y_i are given by

$$E(Y_i) = \mu \boldsymbol{1}_n,\ i = 1, 2, \cdots, K, \tag{5}$$

where $\boldsymbol{1}_n$ is an n-vector with 1 in all of the n entries.

(2) The covariance of Y is given by

$$\sum\nolimits_{YY} = Cov(Y,Y) = diag(\Lambda_1, \cdots, \Lambda_K) + \tau^2(\eta_1 \boldsymbol{1}_n, \cdots, \eta_K \boldsymbol{1}_n)'(\eta_1 \boldsymbol{1}'_n, \cdots, \eta_K \boldsymbol{1}'_n), \tag{6}$$

where $\Lambda_i = ((1-\eta_i^2)\tau^2 + \rho_i\sigma^2)\boldsymbol{1}_n\boldsymbol{1}'_n + (1-\rho_i)\sigma^2 I_n$.

(3) The covariance between $\mu(\Theta_i)$ and the current claims is given by

$$\sum\nolimits_{\mu(\Theta_i)Y} = Cov(\mu(\Theta_i), Y) = \eta_i\tau^2(\eta_1, \cdots, \eta_K) \otimes \boldsymbol{1}'_n + (1-\eta_i^2)\tau^2 e_i \otimes \boldsymbol{1}'_n, \tag{7}$$

where e_i is a vector with 1 in the i-th entry and 0 in the other entries. Here, "$\otimes$" indicates the Kronecker product of matrices.

(4) The inverse of the variance matrix of Y is given by

$$\begin{aligned}\sum\nolimits_{YY}^{-1} =& diag(\Lambda_1^{-1}, \cdots, \Lambda_K^{-1}) - \frac{\tau^2}{1+na\tau^2}(\eta_1\lambda_i\boldsymbol{1}_n, \cdots, \eta_K\lambda_K\boldsymbol{1}_n)' \\ &(\eta_1\lambda_i\boldsymbol{1}'_n, \cdots, \eta_K\lambda_K\boldsymbol{1}'_n),\end{aligned} \tag{8}$$

where

$$\lambda_i = \frac{1}{(1-\rho_i)\sigma^2 + n((1-\eta_i^2)\tau^2 + \rho_i\sigma^2))}, \quad a = \sum_{i=1}^{K} \eta_i^2\lambda_i,$$

and

$$\Lambda_i^{-1} = \frac{1}{(1-\rho_i)\sigma^2}[I_n - \lambda_i((1-\eta_i^2)\tau^2 + \rho_i\sigma^2)]\boldsymbol{1}_n\boldsymbol{1}'_n.$$

Proof. (1) is straightforward.

(2) Write $\Theta = (\Theta_1, \cdots, \Theta_K)'$. Then it follows from assumption 2.1, we have

$$\mathrm{E}(Y_i|\Theta) = \mu(\Theta_i)\mathbf{1}_n, \tag{9}$$

and

$$\mathrm{Cov}(Y_i, Y_i|\Theta) = (1-\rho_i)\sigma^2(\Theta_i)I_n + \rho_i\sigma^2(\Theta_i)\mathbf{1}_n\mathbf{1}'_n. \tag{10}$$

Using (10) implies

$$\mathrm{E}[\mathrm{Cov}(Y_i, Y_i|\Theta)] = (1-\rho_i)\sigma^2 I_n + \rho_i\sigma^2\mathbf{1}_n\mathbf{1}'_n. \tag{11}$$

From the dual expectation theorem of conditional covariance, we obtain

$$\begin{aligned}\mathrm{Cov}(Y_i, Y_j) &= \mathrm{E}[\mathrm{Cov}(Y_i, Y_j|\Theta)] + \mathrm{Cov}[\mathrm{E}(Y_i|\Theta), \mathrm{E}(Y_j|\Theta)] \\ &= \begin{cases} (1-\rho_i)\sigma^2 I_n + \rho_i\sigma^2\mathbf{1}_n\mathbf{1}'_n + \tau^2\mathbf{1}_n\mathbf{1}'_n, & i = j \\ \eta_i\eta_j\tau^2\mathbf{1}_n\mathbf{1}'_n, & i \neq j \end{cases}\end{aligned}$$

Consequently, we can get $\sum_{YY}$.

(3) Notice that $\mathrm{Cov}(\mu(\Theta_i), Y|\Theta) = 0$. Thus

$$\begin{aligned}\sum\nolimits_{\mu(\Theta_i)Y} &= \mathrm{E}[\mathrm{Cov}(\mu(\Theta_i), Y|\Theta)] + \mathrm{Cov}[\mathrm{E}(\mu(\Theta_i)|\Theta), \mathrm{E}(Y|\Theta)] \\ &= \mathrm{Cov}[\mathrm{E}(\mu(\Theta_i)|\Theta), \mathrm{E}(Y|\Theta)] \\ &= \mathrm{Cov}[\mu(\Theta_i), (\mu(\Theta_1)\mathbf{1}'_n, \cdots, \mu(\Theta_K)\mathbf{1}'_n)] \\ &= \eta_i\tau^2(\eta_1, \cdots, \eta_K) \otimes \mathbf{1}'_n + (1-\eta_i^2)\tau^2 e_i \otimes \mathbf{1}'_n\end{aligned}$$

(4) Using the well-known formula for matrix inverse (see [13])

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}. \tag{12}$$

we may derive

$$\Lambda_i^{-1} = \tfrac{1}{(1-\rho_i)\sigma^2}[I_n - \lambda_i((\mathbf{1} - \eta_i^2)\tau^2 + \rho_i\sigma^2)]\mathbf{1}_n\mathbf{1}'_n$$

Noting that

$$\mathbf{1}'_n\Lambda_i^{-1} = \frac{\mathbf{1}'_n}{(1-\rho_i)\sigma^2 + n((1-\eta_i^2)\tau^2 + \rho_i\sigma^2))},$$

$$\Lambda_i^{-1}\mathbf{1}_n = \frac{\mathbf{1}_n}{(1-\rho_i)\sigma^2 + n((1-\eta_i^2)\tau^2 + \rho_i\sigma^2))},$$

and

$$\mathbf{1}'_n\Lambda_i^{-1}\mathbf{1}_n = \frac{n}{(1-\rho_i)\sigma^2 + n((1-\eta_i^2)\tau^2 + \rho_i\sigma^2))}.$$

From (12), we obtain

$$\begin{aligned}\sum\nolimits_{YY}^{-1} = \; & \mathrm{diag}(\Lambda_1^{-1}, \cdots, \Lambda_K^{-1}) - \tfrac{\tau^2}{1+na\tau^2}(\eta_1\lambda_i\mathbf{1}_n, \cdots, \eta_K\lambda_K\mathbf{1}_n)' \\ & (\eta_1\lambda_i\mathbf{1}'_n, \cdots, \eta_K\lambda_K\mathbf{1}'_n).\end{aligned}$$

□

Now, we present the following lemmas on projections of random variables, which has been proven in [6].

Lemma 2.2. *The inhomogeneous credibility estimator is actually the orthogonal projection of a random variable Y on $L(X,1)$. The following formula hold*

$$proj(Y|L(X,1)) = E(Y) + \sum\nolimits_{YX}\sum\nolimits_{XX}^{-1}(X - E(X)), \tag{13}$$

where $proj(Y|X)$ represents the projection of Y on the linear space spanned by X.

3. **The Bühlmann's Credibility Estimator for Exponential Principle**

In this section, we proceed to drive the credibility estimator of $H(X, \Theta_i)$ under the model with dependence on risks and errors. We state the following theorem.

Theorem 3.1. *Under assumptions 2.1-2.2, the inhomogeneous credibility estimators of $\mu(\Theta_i)$ for $i = 1, \cdots, K$ are given by*

$$\widehat{\mu(\Theta_i)} = Z_{i1}\overline{Y_i} + Z_{i2}\overline{\overline{Y_\lambda}} + (1 - Z_{i1} - Z_{i2})\mu. \tag{14}$$

So an inhomogeneous credibility estimator of $H(X, \Theta_i)$ is

$$H\widehat{(X, \Theta}_i) = \frac{1}{\alpha}\log[Z_{i1}\overline{Y_i} + Z_{i2}\overline{\overline{Y_\lambda}} + (1 - Z_{i1} - Z_{i2})\mu], \tag{15}$$

where

$$Z_{i1} = (1-\eta_i^2)n\tau^2\lambda_i,\ Z_{i2} = \frac{n\tau^2\lambda\eta_i(1-(1-\eta_i^2)n\tau^2\lambda_i)}{1+na\tau^2}$$

are so-called the credibility factors. $\overline{Y_i} = \frac{\sum_{j=1}^n Y_{ij}}{n}$ *and* $\overline{\overline{Y_\lambda}} = \frac{\sum_{i=1}^K \eta_i\lambda_i\overline{Y_i}}{\lambda}$ *are individual mean and total weighted mean, respectively. In addition,* $\lambda = \sum_{i=1}^K \eta_i\lambda_i$.

Proof. Lemma 2.2 gives $\widehat{\mu(\Theta_i)} = \text{proj}(\mu(\Theta_i)|L(Y,1))$. We thus prove the theorem by computing $\text{proj}(\mu(\Theta_i)|L(Y,1))$.

Firstly, we need to get the following terms. Lemma 2.2 gives $\widehat{\mu(\Theta_i)} = \text{proj}(\mu(\Theta_i)|L(Y,1))$. We thus prove the theorem by computing $\text{proj}(\mu(\Theta_i)|L(Y, 1))$.

Firstly, we need to get the following terms.

$$\begin{aligned}
&\eta_i\tau^2(\eta_1,\cdots,\eta_K)\otimes \mathbf{1}_n' \textstyle\sum_{YY}^{-1}(Y-\mathrm{E}(Y)) \\
&= \eta_i\tau^2(\eta_1\mathbf{1}_n',\cdots,\eta_K\mathbf{1}_n')[\mathrm{diag}(\Lambda_1^{-1},\cdots,\Lambda_K^{-1}) - \frac{\tau^2}{1+na\tau^2}(\eta_1\lambda_i\mathbf{1}_n,\cdots,\eta_K \\
&\quad \lambda_K\mathbf{1}_n)'(\eta_1\lambda_i\mathbf{1}_n',\cdots,\eta_K\lambda_K\mathbf{1}_n')](Y-\mathrm{E}(Y)) \\
&= \eta_i\tau^2[\sum_{i=1}^K \eta_i\mathbf{1}_n'\Lambda_i^{-1}(Y_i-\mu\mathbf{1}_n) - \frac{a\tau^2}{1+na\tau^2}\sum_{i=1}^K \eta_i\lambda_i\mathbf{1}_n'(Y_i-\mu\mathbf{1}_n)] \\
&= \eta_i\tau^2[\sum_{i=1}^K \eta_i\mathbf{1}_n'\lambda_i n(\overline{Y_i}-\mu) - \frac{na\tau^2}{1+na\tau^2}\sum_{i=1}^K \eta_i\lambda_i n(\overline{Y_i}-\mu)] \\
&= \frac{n\eta_i\tau^2\lambda}{1+na\tau^2}(\overline{\overline{Y_\lambda}}-\mu)
\end{aligned}$$

and

$$\begin{aligned}
&(1-\eta_i^2)\tau^2\mathrm{e}_i\otimes\mathbf{1}_n'\sum\nolimits_{YY}^{-1}(Y-\mathrm{E}(Y)) \\
&= (1-\eta_i^2)\tau^2\mathrm{e}_i\otimes\mathbf{1}_n'[\mathrm{diag}(\Lambda_1^{-1},\cdots,\Lambda_K^{-1}) - \frac{\tau^2}{1+na\tau^2}(\eta_1\lambda_i\mathbf{1}_n,\cdots,\eta_K \\
&\quad \lambda_K\mathbf{1}_n)'(\eta_1\lambda_i\mathbf{1}_n',\cdots,\eta_K\lambda_K\mathbf{1}_n')](Y-\mathrm{E}(Y)) \\
&= (1-\eta_i^2)\tau^2\mathbf{1}_n'\Lambda_i^{-1}(Y_i-\mu\mathbf{1}_n) - \frac{(1-\eta_i^2)\tau^4 n\lambda_i\eta_i}{1+na\tau^2}\sum_{i=1}^K \eta_i\lambda_i n(\overline{Y_i}-\mu)] \\
&= (1-\eta_i^2)n\tau^2\lambda_i(\overline{Y_i}-\mu) - \frac{(1-\eta_i^2)\tau^4 n^2\lambda_i\lambda\eta_i}{1+na\tau^2}(\overline{\overline{Y_\lambda}}-\mu)
\end{aligned}$$

Then

$$\begin{aligned}
\widehat{\mu(\Theta_i)} &= \mathrm{proj}(\mu(\Theta_i)|L(Y,1)) \\
&= \mathrm{E}(\mu(\Theta_i)) + \sum\nolimits_{\mu(\Theta_i)Y} \sum\nolimits_{YY}^{-1} (Y - \mathrm{E}(Y)) \\
&= \mu + \frac{n\eta_i\tau^2\lambda}{1+na\tau^2}(\overline{\overline{Y_\lambda}} - \mu) + (1-\eta_i^2)n\tau^2\lambda_i(\overline{Y_i} - \mu) \\
&\quad - \tfrac{(1-\eta_i^2)\tau^4 n^2\lambda_i\lambda\eta_i}{1+na\tau^2}(\overline{\overline{Y_\lambda}} - \mu) \\
&= (1-\eta_i^2)n\tau^2\lambda_i\overline{Y_i} + \frac{n\tau^2\lambda\eta_i(1-(1-\eta_i^2)n\tau^2\lambda_i)}{1+na\tau^2}\overline{\overline{Y_\lambda}} \\
&\quad + (1-(1-\eta_i^2)n\tau^2\lambda_i - \frac{n\tau^2\lambda\eta_i(1-(1-\eta_i^2)n\tau^2\lambda_i)}{1+na\tau^2})\mu
\end{aligned}$$

□

From the expression of the credibility factor, we can easily see that $0 \leq Z_{i1} \leq 1$, $0 \leq Z_{i2} \leq 1$ and

$$Z_{i1} \to 1,\ Z_{i2} \to 0 \text{ when } n \to \infty$$

In addition, from the central limit theorem, under Assumption 2.1 and 2.2 we have $\overline{Y_i} = \frac{\sum_{j=1}^n \exp(\alpha X_{ij})}{n} \to \mu(\Theta_i)$, a.s.. So $\widehat{\mu(\Theta_i)} = Z_{i1}\overline{Y_i} + Z_{i2}\overline{\overline{Y_\lambda}} + (1 - Z_{i1} - Z_{i2})\mu \to \mu(\Theta_i)$. Furthermore, the consistency of credibility estimator of (15) with individual premium $H(X,\Theta)$ follows from the continuity of the function $\frac{1}{\alpha}\log(x)$.

Remark 3.1. *Theorem 3.1 builds the inhomogeneous credibility estimator with dependence over risks and errors for Bühlmann model. In our model, if we assume that $\rho_1 = \cdots = \rho_K = 0$ and $Corr(\mu(\Theta_i), \mu(\Theta_j)) = 0$, then $\lambda_i = \frac{1}{\sigma^2+n\tau^2}$, and the credibility estimators of $\mu(\Theta_i)$ for $i = 1, \cdots, K$ are*

$$\widehat{\mu(\Theta_i)} = \frac{n\tau^2}{\sigma^2+n\tau^2}\overline{u_i} + \frac{\sigma^2}{\sigma^2+n\tau^2}\mu.$$

Furthermore, an inhomogeneous credibility estimators of $H(X,\Theta_i)$ is

$$\widehat{H(X,\Theta_i)} = \frac{1}{\alpha}\log[\frac{n\tau^2}{\sigma^2+n\tau^2}\overline{u_i} + \frac{\sigma^2}{\sigma^2+n\tau^2}\mu],$$

where $\overline{u_i} = \frac{1}{n}\sum\limits_{j=1}^{n} e^{\alpha X_{ij}}$, which are just the formulae in [12]. *Therefore, these credibility estimators are the special cases of our model.*

4. Credibility Estimator for Exponential Principle Under The Bühlmann-Straud Model

The credibility models with natural weights was developed by Bühlmann and Straud [3], and is known as the Bühlmann-Straud model. There have been broad applications of this model in insurance practice and thus it has been of the building blocks of credibility theory. In this section, we give the credibility estimators of Bühlmann-Straud with dependence structure over risks and error

under exponential principle. To be more specific, we are given a portfolio of K risks. The assumption is stated as follows.
Assumption 4.1 For fixed i, given Θ_i, the X_{ij} follows the model: $e^{\alpha X_{ij}} = \mu(\Theta_i)+\epsilon_{ij}$, and the errors are conditionally uniformly dependent, i.e. $\mathrm{Corr}(\epsilon_{il}, \epsilon_{it}|\Theta_i) = \rho_i$, $(l \neq t)$ and $\rho_i < 1$. We also assume that $\mathrm{E}(\epsilon_{ij}|\Theta_i) = 0$ and $\mathrm{Var}(\epsilon_{ij}|\Theta_i) = \frac{\sigma^2(\Theta_i)}{w_{ij}}$, where w_{ij} are known weights.

We will use the following notations regarding the weights

$$W_i = \sum_{j=1}^{n} w_{ij},\ W_a = \sum_{j=1}^{n} \sqrt{w_{ij}},\ \beta_i = \frac{1}{(1-\rho_i)\sigma^2}[W_i - \frac{\rho_i}{1+(n-1)\rho_i} W_a^2]$$

$$S_i = \frac{1}{1+\beta_i(1-\eta_i^2)\tau^2},\ \Lambda = \sum_{i=1}^{K} \eta_i^2 S_i \beta_i,\ \varphi_1 = \sum_{i=1}^{K} \frac{\eta_i S_i W_i}{(1-\rho_i)\sigma^2},$$

$$\varphi_2 = \sum_{i=1}^{K} \frac{\eta_i S_i \rho_i W_a^2}{(1+(n-1)\rho_i)(1-\rho_i)\sigma^2},\ \overline{Y_i^W} = \frac{\sum_{j=1}^{n} w_{ij} Y_{ij}}{W_i},\ \overline{Y_i^{W_a}} = \frac{\sum_{j=1}^{n} \sqrt{w_{ij}} Y_{ij}}{W_a},$$

$$\overline{\overline{Y_i^W}} = \frac{\sum_{i=1}^{K} \frac{\eta_i S_i W_i}{(1-\rho_i)\sigma^2} \overline{Y_i^W}}{\varphi_1},\ \overline{\overline{Y_i^{W_a}}} = \frac{\sum_{i=1}^{K} \frac{\eta_i S_i \rho_i W_a^2}{(1+(n-1)\rho_i)(1-\rho_i)\sigma^2} \overline{Y_i^{W_a}}}{\varphi_2}$$

Note that assumption 4.1 implies $\mathrm{Cov}(\epsilon_{il}, \epsilon_{it}|\Theta_i) = \frac{\sigma^2(\Theta_i)}{\sqrt{w_{il}}\sqrt{w_{it}}}$. Then we derive the following lemma.

Lemma 4.1. *Under the assumption 2.2 and 4.1 and with the notations above, we have the following results.*

(1) The means of Y_i are given by

$$E(Y_i) = \mu \boldsymbol{1}_n,\ i = 1, 2, \cdots, K. \tag{16}$$

(2)The covariance of Y is given by

$$\sum\nolimits_{YY} = diag(\Delta_1, \cdots, \Delta_K) + \tau^2(\eta_1 \boldsymbol{1}_n, \cdots, \eta_K \boldsymbol{1}_n)'(\eta_1 \boldsymbol{1}_n', \cdots, \eta_K \boldsymbol{1}_n'), \tag{17}$$

where

$$\begin{aligned}\Delta_i =&(1-\eta_i^2)\tau^2 \boldsymbol{1}_n \boldsymbol{1}_n' + (1-\rho_i)\sigma^2 diag(\frac{1}{w_{i1}}, \cdots, \frac{1}{w_{in}}) \\ &+ \rho_i \sigma^2 (\frac{1}{\sqrt{w_{i1}}}, \cdots, \frac{1}{\sqrt{w_{in}}})'(\frac{1}{\sqrt{w_{i1}}}, \cdots, \frac{1}{\sqrt{w_{in}}}).\end{aligned} \tag{18}$$

(3) The covariance between $\mu(\Theta_i)$ and Y is given by

$$\sum\nolimits_{\mu(\Theta_i)Y} = Cov(\mu(\Theta_i), Y) = \eta_i \tau^2 (\eta_1, \cdots, \eta_K) \otimes \boldsymbol{1}_n' + (1-\eta_i^2)\tau^2 e_i \otimes \boldsymbol{1}_n'. \tag{19}$$

(4) The inverse of the variance matrix of Y *is given by*

$$\sum\nolimits_{YY}^{-1} = diag(\Delta_1^{-1}, \cdots, \Delta_K^{-1}) - \frac{\tau^2}{1+\Lambda\tau^2} (\eta_1 S_1 E^{-1}\boldsymbol{1}_n, \cdots, \eta_K S_K E^{-1}\boldsymbol{1}_n)'(\eta_1 S_1 \boldsymbol{1}_n' E^{-1}, \cdots, \eta_K S_K \boldsymbol{1}_n' E^{-1}) \tag{20}$$

where

$$\Delta_i^{-1} = E^{-1} - E^{-1}\boldsymbol{1}_n \frac{1-\eta_i^2)\tau^2}{1+\beta_i 1-\eta_i^2)\tau^2}\boldsymbol{1}_n' E^{-1},$$

and

$$E^{-1} = \frac{1}{(1-\rho_i)\sigma^2}[diag(w_{i1}, \cdots, w_{in}) - \frac{\rho_i}{1+(n-1)\rho_i}(\sqrt{w_{i1}}, \cdots, \sqrt{w_{in}})'(\sqrt{w_{i1}}, \cdots, \sqrt{w_{in}})]. \tag{21}$$

Proof. (1) is straightforward.

(2) Write $\Theta = (\Theta_1, \cdots, \Theta_K)'$. Then from the dual expectation theorem of conditional covariance, we have

$$\mathrm{Cov}(Y_i, Y_i) = \tau^2 \mathbf{1}_n \mathbf{1}_n' + (1-\rho_i)\sigma^2 \mathrm{diag}(\frac{1}{w_{i1}}, \cdots, \frac{1}{w_{in}}) + \rho_i\sigma^2(\frac{1}{\sqrt{w_{i1}}}, \cdots, \frac{1}{\sqrt{w_{in}}})'(\frac{1}{\sqrt{w_{i1}}}, \cdots, \frac{1}{\sqrt{w_{in}}}) \tag{22}$$

and

$$\mathrm{Cov}(Y_i, Y_j) = \eta_i\eta_j\tau^2\mathbf{1}_n\mathbf{1}_n'. \tag{23}$$

Consequently, we can get $\sum_{YY}$.

(3) As part (3) of lemma 2.1.

(4) Write

$$\Delta_i = (1-\eta_i^2)\tau^2\mathbf{1}_n\mathbf{1}_n' + \mathrm{E}$$

where $\mathrm{E} = (1-\rho_i)\sigma^2\mathrm{diag}(\frac{1}{w_{i1}}, \cdots, \frac{1}{w_{in}}) + \rho_i\sigma^2(\frac{1}{\sqrt{w_{i1}}}, \cdots, \frac{1}{\sqrt{w_{in}}})'(\frac{1}{\sqrt{w_{i1}}}, \cdots, \frac{1}{\sqrt{w_{in}}})$.

Using (12), we can check that

$$\begin{aligned}\mathrm{E}^{-1} =& [(1-\rho_i)\sigma^2\mathrm{diag}(\frac{1}{w_{i1}}, \cdots, \frac{1}{w_{in}}) + \rho_i\sigma^2(\frac{1}{\sqrt{w_{i1}}}, \cdots, \frac{1}{\sqrt{w_{in}}})'(\frac{1}{\sqrt{w_{i1}}}, \cdots, \frac{1}{\sqrt{w_{in}}})]^{-1} \\ =& \frac{1}{(1-\rho_i)\sigma^2}[\mathrm{diag}(w_{i1}, \cdots, w_{in}) - \frac{\rho_i}{1+(n-1)\rho_i}(\sqrt{w_{i1}}, \cdots, \sqrt{w_{in}})'(\sqrt{w_{i1}}, \cdots, \sqrt{w_{in}})].\end{aligned} \tag{24}$$

Note that $\mathbf{1}_n'\mathrm{E}^{-1}\mathbf{1}_n = \beta_i$, thus

$$\begin{aligned}\Delta_i^{-1} =& [\mathrm{E} + (1-\eta_i^2)\tau^2\mathbf{1}_n\mathbf{1}_n']^{-1} \\ =& \mathrm{E}^{-1} - \mathrm{E}^{-1}\mathbf{1}_n\frac{1-\eta_i^2)\tau^2}{1+\beta_i 1-\eta_i^2)\tau^2}\mathbf{1}_n'\mathrm{E}^{-1}.\end{aligned} \tag{25}$$

Furthermore $\mathbf{1}_n'\Delta_i^{-1} = \frac{\mathbf{1}_n'\mathrm{E}^{-1}}{1+\beta_i(1-\eta_i^2)\tau^2}$, $\Delta_i^{-1}\mathbf{1}_n = \frac{\mathrm{E}^{-1}\mathbf{1}_n}{1+\beta_i(1-\eta_i^2)\tau^2}$, and $\mathbf{1}_n'\Delta_i^{-1}\mathbf{1}_n = \frac{\beta_i}{1+\beta_i(1-\eta_i^2)\tau^2}$. Therefore, we can prove (4). □

Theorem 4.1. *Under the assumption 2.2 and 4.1, the inhomogeneous credibility estimators of $\mu(\Theta_i)$ for $i = 1, \cdots, K$, are given by*

$$\widehat{\mu(\Theta_i)} = Z_{i1}\overline{Y_i^W} - Zi2\overline{Y_i^{W_a}} - Z_{i3}\overline{\overline{Y_i^W}} + Z_{i4}\overline{\overline{Y_i^{W_a}}} + (1 - Z_{i1} + Zi2 + Z_{i3} - Z_{i4})\mu \quad (26)$$

where

$$Z_{i1} = \frac{(1-\eta_i^2)\tau^2 S_i W_i}{(1-\rho_i)\sigma^2},\ Z_{i2} = \frac{(1-\eta_i^2)\tau^2 S_i\rho_i W_a^2}{(1+(n-1)\rho_i)(1-\rho_i)\sigma^2}$$

$$Z_{i3} = \frac{\varphi_1\eta_i\tau^2((1-\eta_i^2)\beta_i S_i\tau^2 - 1)}{1+\Lambda\tau^2},\ Z_{i4} = \frac{\varphi_2\eta_i\tau^2((1-\eta_i^2)\beta_i S_i\tau^2 - 1)}{1+\Lambda\tau^2}$$

are called credibility factors.

So an inhomogeneous credibility estimator of $H(X, \Theta_i)$ is

$$\begin{aligned}H\widehat{(X,\Theta_i)} =& \frac{1}{\alpha}\log[Z_{i1}\overline{Y_i^W} - Zi2\overline{Y_i^{W_a}} - Z_{i3}\overline{\overline{Y_i^W}} + Z_{i4}\overline{\overline{Y_i^{W_a}}} \\ &+ (1 - Z_{i1} + Zi2 + Z_{i3} - Z_{i4})\mu].\end{aligned} \quad (27)$$

$$g(x^1, x^2, \ldots, x^n) = \mathrm{diag}\left(\frac{1}{g_1^2(x^1)}, \frac{1}{g_2^2(x^2)}, \cdots \frac{1}{g_n^2(x^n)}\right), \quad (28)$$

Proof. Here $\eta_i\tau^2(\eta_1\mathbf{1}_n', \cdots, \eta_K\mathbf{1}_n')\sum_{YY}^{-1}$ can be computed by

$$\begin{aligned}&\eta_i\tau^2(\eta_1\mathbf{1}_n', \cdots, \eta_K\mathbf{1}_n')\textstyle\sum_{YY}^{-1} \\ &= \eta_i\tau^2(\eta_1\mathbf{1}_n'\Delta_1^{-1}, \cdots, \eta_K\mathbf{1}_n'\Delta_K^{-1}) - \frac{\eta_i\tau^4}{1+\Lambda\tau^2}\sum_{i=1}^K \eta_i^2\mathbf{1}_n' S_i\mathrm{E}^{-1}\mathbf{1}_n \\ &(\eta_1 S_1\mathbf{1}_n'\mathrm{E}^{-1}, \cdots, \eta_K S_K\mathbf{1}_n'\mathrm{E}^{-1}) \\ &= \eta_i\tau^2(\eta_1 S_1\mathbf{1}_n'\mathrm{E}^{-1}, \cdots, \eta_K S_K\mathbf{1}_n'\mathrm{E}^{-1}) \\ &- \frac{\eta_i\tau^4}{1+\Lambda\tau^2}\sum_{i=1}^K \eta_i^2 S_i\beta_i(\eta_1 S_1\mathbf{1}_n'\mathrm{E}^{-1}, \cdots, \eta_K S_K\mathbf{1}_n'\mathrm{E}^{-1}) \\ &= \frac{\eta_i\tau^2}{1+\Lambda\tau^2}(\eta_1 S_1\mathbf{1}_n'\mathrm{E}^{-1}, \cdots, \eta_K S_K\mathbf{1}_n'\mathrm{E}^{-1}),\end{aligned}$$

and

$$\begin{aligned}&(1-\eta_i^2)\tau^2 e_i\otimes\mathbf{1}_n'\textstyle\sum_{YY}^{-1} \\ &= (1-\eta_i^2)\tau^2\mathbf{1}_n'\Delta_i^{-1} - \frac{\tau^4}{1+\Lambda\tau^2}(1-\eta_i^2)\beta_i\eta_i S_i(\eta_1 S_1\mathbf{1}_n'\mathrm{E}^{-1}, \cdots, \eta_K S_K\mathbf{1}_n'\mathrm{E}^{-1}) \\ &= (1-\eta_i^2)\tau^2\mathbf{1}_n' S_i\mathrm{E}^{-1} - \frac{\tau^4}{1+\Lambda\tau^2}(1-\eta_i^2)\beta_i\eta_i S_i(\eta_1 S_1\mathbf{1}_n'\mathrm{E}^{-1}, \cdots, \eta_K S_K\mathbf{1}_n'\mathrm{E}^{-1}).\end{aligned}$$

Using Lemma 2.2 and Lemma 4.1, we obtain

$$\begin{aligned}\widehat{\mu(\Theta_i)} &= \mathrm{E}(\mu(\Theta_i)) + \sum_{\mu(\Theta_i)Y}\sum_{YY}^{-1}(Y - \mathrm{E}(Y)) \\ &= \mu + (1-\eta_i^2)\tau^2\mathbf{1}_n' S_i \mathrm{E}^{-1}(Y_i - \mu\mathbf{1}_n) \\ &\quad -\tfrac{\eta_i\tau^2}{1+\Lambda\tau^2}((1-\eta_i^2)\beta_i S_i\tau^2 - 1)\sum_{i=1}^{K}\eta_i S_i \mathbf{1}_n' \mathrm{E}^{-1}(Y_i - \mu\mathbf{1}_n) \\ &= Z_{i1}\overline{Y_i^W} - Z_{i2}\overline{Y_i^{W_a}} - Z_{i3}\overline{\overline{Y_i^W}} + Z_{i4}\overline{\overline{Y_i^{W_a}}} \\ &\quad +(1 - Z_{i1} + Zi2 + Z_{i3} - Z_{i4})\mu.\end{aligned}$$

Then

$$\begin{aligned}\widehat{H(X,\Theta_i)} = \; & \tfrac{1}{\alpha}\log[Z_{i1}\overline{Y_i^W} - Z_{i2}\overline{Y_i^{W_a}} - Z_{i3}\overline{\overline{Y_i^W}} + Z_{i4}\overline{\overline{Y_i^{W_a}}} \\ & +(1 - Z_{i1} + Zi2 + Z_{i3} - Z_{i4})\mu].\end{aligned}$$

□

From (26), the inhomogeneous credibility estimators of $\mu(\Theta_i)$ are not the strict weight from any longer. However, since the credibility factors still satisfy $Z_{i1} - Z_{i2} - Z_{i3} + Z_{i4} + (1 - Z_{i1} + Z_{i2} + Z_{i3} - Z_{i4}) = 1$, we can think these from the general credibility estimators. If we take all $w_{ij} = 1$, then (27) are degenerated to (15).

5. Conclusions

In this paper, we have studied the Bühlmann and Bühlmann-Straub credibility models under exponential principle with uniform dependence on risks and errors. The inhomogeneous Bühlmann credibility estimator of $\widehat{\mu(\Theta_i)}$ is derived. The result is also extended to Bühlmann-Straub model. However, as is shown in section 4, the Bühlmann-Straub credibility estimator has only the generalized form of credibility. In addition, the Bühlmann and Bühlmann-Straub credibility estimators of individual exponential premium are obtained. Our model extends the results of [5, 12]. For the models established in this paper to be taken into practical use, such parameters ρ_i and η_i still need to be further estimated, which will be the investigation goal of future works.

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REFERENCES

[1] *H. Bühlmann*, Experience Rating and Credibility, Astin Bulletin, **4** (1967), No. 3, 199-207.

[2] *H. Bühlmann, and A. Gisler*, A Course in Credibility Theory and its Application, Netherlands: Springer, 2005.

[3] *H. Bühlmann and E. Straub*, Glaubwürdigkeit für Schadensätze (credibility for loss ratios), Bulletin of the Swiss Association of Actuaries, **70** (1970), No. 1, 111-133.

[4] *K. L. Yeo and E. A. Valdez*, Claim dependence with common effects in credibility models, Insur. Math. Econ, **38** (2006), No. 3, 609-629.

[5] *L. M. Wen, X. Y. Wu and X. Zhou*, The credibility premiums for models with dependence induced by common effects, Insur. Math. Econ , **44** (2009), No. 1, 19-25.

[6] *L. M. Wen, and W. L. Deng*, The credibility models with equal correlation risks, J. Syst Sci Complex, **24** (2011), No. 3, 532-539.

[7] *Q. Zhang,, L. J. Wu and Q. Q. Cui*, The balanced credibility estimators with correlation risk and inflation factor, Stat Papers, (2015), (DOI 10.1007/s00362-015-0719-6).

[8] *N. Kolev and D. Paiva*, Random sums of exchangeable variables and actuarial applications, Insur. Math. Econ, **42** (2008), No. 1, 147-153.

[9] *H. U. Gerber*, 1979. An introduction to mathematical risk theory (Vol. 8), Philadelphia: S.S. Heubner Foundation Monograph Series 8, 1979.

[10] *V. R. Young*, Premium principles, Encyclopedia of Actuarial Science, Wiley, New York, (2004), No. 3, 1322-1331.

[11] *M. Denuit*, 1999. The exponential premium calculation principle revisited, Astin Bulletin, **29** (1999), No. 2, 215-226.

[12] *L. M. Wen, W. Wang and J. L. Wang*, The credibility premiums for exponential principle, J. Acta Mathematical Sinica, English Series, **27** (2011), No. 11, 2217-2228.

[13] *R. Rao and H. Toutenburg*, Linear Models, Springer, New York, 1995.