S-SUBGRADIENT PROJECTION ALGORITHM WITH INERTIAL TECHNIQUE FOR NON-CONVEX SPLIT FEASIBILITY PROBLEMS

Jinzuo Chen[1], Kun Chen[2], Tzu-Chien Yin[3]

In this paper, an inertial relaxed S-subgradient projection algorithm is suggested to seek the solution of non-convex split feasibility problems in finite dimensional spaces. We obtain a convergence theorem for the sequence yielded by the proposed algorithm under implemented conditions on the step-size which does not rely on the spectral radius of the matrix.

Keywords: nonconvex split feasibility problems, inertial technique, S-subdifferentiable, S-subgradient projection.

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1. Introduction

In this research, we study the following split feasibility problem (abbr. SFP):

$$\text{find } x \in C \text{ such that } Ax \in Q, \tag{1}$$

where $C \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$ are non-empty closed convex subsets, $A : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix (a bounded and linear operator).

The SFP was first studied in Euclidean spaces by Censor [8] in 1994 for modeling inverse problems which arise from medical imageology and in modeling of Intensity-Modulated Radiation Therapy (IMRT) recently [3, 4, 5, 9], and extended to infinite dimensional spaces afterwards [25].

Many iterative algorithms closely related to the SFP, fixed point and optimization techniques have been investigated; please see, [4]-[44]. Byrne [4, 5], among them, employed the classical CQ algorithm:

$$x_{k+1} = P_C\left(x_k - \varsigma_k A^T (I - P_Q) A x_k\right), \quad k \geq 1, \tag{2}$$

where P_C and P_Q are the perpendicular projections onto C and Q, respectively, and the step-size $\varsigma_k \in (0, 2/\xi)$ with ξ (substitute ξ with $\|A\|^2$ equivalently) being the spectral radius of matrix $A^T A$. The subsets C and Q in the formula (2) can be discussed in another form, i.e., the level sets as follows:

$$C_0 = \{x \in \mathbb{R}^n : c(x) \leq 0\} \text{ and } Q_0 = \{y \in \mathbb{R}^m : q(y) \leq 0\}, \tag{3}$$

where c and q are convex functions from $\mathbb{R}^n$ and $\mathbb{R}^m$ to $\mathbb{R}$, respectively.

However, projections on non-empty closed convex sets and level sets have no closed form, which immensely affects the operation of algorithm (2). Regarding this question, Yang [26] presented relaxed CQ algorithm that the projections involved are on half-spaces

[1]School of Mathematics and Statistics, Lingnan Normal University, Zhanjiang, China, e-mail: chanjanegeoger@hotmail.com

[2]Computer Center, Taizhou Hospital of Zhejiang Province, Linhai, China, e-mail: chenkun0576@163.com

[3]Corresponding author. Research Center for Interneural Computing, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan, e-mail: yintzuchien@mail.cmuh.org.tw

containing the level sets instead of directly calculating the projections on the level sets. The relaxed CQ algorithm [26] is as follows:

$$x_{k+1} = P_{C_0^k}(x_k - \varsigma_k A^T(I - P_{Q_0^k})Ax_k), \quad k \geq 1, \tag{4}$$

with $\varsigma_k \in \big(0, 2/\|A\|^2\big)$,

$$C_0^k := \{x \in \mathbb{R}^n : \langle x - x_k, \phi_k \rangle + c(x_k) \leq 0\}$$

and

$$Q_0^k := \{y \in \mathbb{R}^m : \langle y - Ax_k, \varphi_k \rangle + q(Ax_k) \leq 0\},$$

here $\phi_k \in \partial c(x_k)$, $\varphi_k \in \partial q(Ax_k)$, respectively.

Projections on the above two half-spaces have closed form, however, the step-size ς_k in (4) is depends on matrix norm $\|A\|$, so, the algorithm suggested with such step-size is of little actual application and maneuverability, see [13]. To overcome this problem, López [15] rewrote algorithm (4) as

$$x_{k+1} = P_{C_0^k}\left(x_k - \varsigma_k \nabla f_k(x_k)\right), \quad k \geq 1, \tag{5}$$

with objection function

$$f_k(x) = \frac{1}{2}\|Ax - P_{Q_0^k}(Ax)\|^2 \tag{6}$$

and its gradient $\nabla f_k(x) = A^T\left(I - P_{Q_0^k}\right)Ax$.

The step-size in algorithm (5) is defined by

$$\varsigma_k := \lambda_k \frac{f_k(x_k)}{\|\nabla f_k(x_k)\|^2}, \qquad 0 < \lambda_k < 2. \tag{7}$$

The convergence of algorithm (5) with step-size (7) is guaranteed under the computation of metric projections onto half-spaces and not necessary to estimate the norm of matrix.

Recently, Dang [11] applied the inertial accelerated craftsmanship of Alvarez [1] to Yang's relaxed CQ algorithm (4) and suggested inertial accelerated relaxed CQ algorithm to solve the SFP as follows:

$$x_{k+1} = P_{C_0^k}\left(U_k(x_k + \theta_k(x_k - x_{k-1}))\right), \tag{8}$$

where x_1, x_2 be chosen arbitrary, $U_k = I - \gamma F_k$, $F_k = A^T(I - P_{Q_0^k})A$, $\gamma \in \big(0, 2/\|A\|^2\big)$.

On the other hand, let G_c and G_{f_k} be two subgradient projectors associated with $(c,\, 0)$ and $(f_k,\, 0)$, respectively, here the function c appears in the above formula (3) and the function f_k is mentioned in (6). Then Guo [13] proposed the following subgradient projection algorithm for solving the SFP,

$$x_{k+1} = G_c\left(R_{\lambda_k f_k}(x_k)\right) \tag{9}$$

where $R_{\lambda_k f_k} = I + \lambda_k\left(G_{f_k} - I\right)$, $\lambda_k \in (0, 2)$.

There is a natural question as follows:

Question: Can the algorithm (9) and its variants with inertial accelerated craftsmanship be combined with the step-size is used in (7)?

Motivated by the works of Dang [11] and Guo [13], we suggest in the paper a new form of subgradient projection algorithm to solve the SFP in which the step-size is used in (7) and combine with the inertial accelerated craftsmanship. Moreover, the functions c and q in (3) we consider are both continuous, S-subdifferential (see Definition 2.2), locally Lipschitzian, not necessarily convex instead of the original convex.

2. Preliminaries

Let $S \subseteq \mathbb{R}^n$ be nonempty closed subset, denote by P_S the orthogonal (metric) projection from $\mathbb{R}^n$ onto S; that is,

$$P_S(x) := \mathrm{argmin}_{y\in S}\|x-y\|, \quad x \in \mathbb{R}^n.$$

Definition 2.1 ([2]). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real function. Denote by $Lev f = \{x \in \mathbb{R}^n : f(x) \le \xi\}$ the level set of f.*

In order to define S-subgradient projector of a continuous function, we need to introduce the definition of S-subdifferential.

Definition 2.2 ([13]). *Given $S \subseteq \mathbb{R}^n$ and $r_f > 0$, a vector $u \in \mathbb{R}^n$ is said to be an S-subgradient of function $f : \mathbb{R}^n \to \mathbb{R}$ at x if*

$$\langle y-x, u\rangle + f(x) + \frac{r_f}{2}d_S^2(x) \le f(y) + \frac{r_f}{2}d_S^2(y), \quad y \in \mathbb{R}^n.$$

The set of all S-subgradients of function f at x is called S-subdifferential of f at x and is denoted by

$$\partial_{Sr_f} f(x) = \left\{u \in \mathbb{R}^n : \langle y-x, u\rangle + f(x) + \frac{r_f}{2}d_S^2(x) \le f(y) + \frac{r_f}{2}d_S^2(y), \quad y \in \mathbb{R}^n\right\} \tag{10}$$

where $d_S(x) = \inf_{y\in S}\|x-y\|$ is the usual distance from the point x to the set S.

If $r_f = 0$ in (10), the S-subdifferential is the following Fenchel subdifferential. So does $S = \mathbb{R}^n$.

Definition 2.3 ([2]). *Given a not necessarily convex function $f : \mathbb{R}^n \to \mathbb{R}$, define its Fenchel subdifferential at x,*

$$\partial f(x) := \{u \in \mathbb{R}^n : \langle y-x, u\rangle + f(x) \le f(y), \quad \forall y \in \mathbb{R}^n\}. \tag{11}$$

When f is convex, $\partial f(x)$ is the usual subdifferential.

To define S-subgradient projector of a continuous function, we also need the following property.

Lemma 2.1 ([13]). *Let S be closed and convex and $C_\xi = Lev f$ be a non-empty set such that $C_\xi \subseteq S \subseteq \mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be S-subdifferential on $\mathbb{R}^n$ with respect to S. Then there exists a constant $r_f > 0$ and for any $x \notin C_\xi$ such that*

$$s_f(x) \in \partial_{Sr_f} f(x) \Rightarrow s_f(x) \ne 0.$$

Next, we can define the S-subgradient projector.

Definition 2.4 ([13]). *Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and S-subdifferential on $\mathbb{R}^n$ with respect to S. Let S be closed and convex and $C_\xi = Lev f$ be a non-empty set such that $C_\xi \subseteq S \subseteq \mathbb{R}^n$. Assume that $\partial_{Sr_f} f(x)$ is the S-subdifferential of f with respect to S and $s_f(x) \in \partial_{Sr_f} f(x)$. The S-subgradient projector onto C_ξ related to (f, ξ) is*

$$G_{S,f} : \mathbb{R}^n \to \mathbb{R}^n$$
$$x \mapsto \begin{cases} x, & x \in C_\xi, \\ x + \frac{\xi - f(x)}{\|s_f(x)\|^2} s_f(x), & x \notin C_\xi. \end{cases}$$

Lemma 2.2 ([13]). *Let $S \subseteq \mathbb{R}^n$ be closed and convex and $f : \mathbb{R}^n \to \mathbb{R}$ be S-subdifferential on $\mathbb{R}^n$ with respect to S. There exists a constant $r_f > 0$ such that*

$$u \in \partial_{Sr_f} f(x) \Leftrightarrow u \in \partial f(x) + r_f(I - P_S)(x).$$

Lemma 2.3 ([2]). *Let $a, b \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$. It then follows that*

$$\|ax + by\|^2 = a(a+b)\|x\|^2 + b(a+b)\|y\|^2 - ab\|x-y\|^2.$$

Lemma 2.4 ([1]). *Let $\{s_k\}$ and $\{t_k\}$ be two nonnegative real sequences such that*

$$s_{k+1} - s_k \le \sigma_k(s_k - s_{k-1}) + t_k,\ \Sigma_{k=1}^{\infty} t_k < \infty$$

where $\{\sigma_k\} \subset [0, \sigma]$ with $0 < \sigma < 1$. Then the sequence $\{s_k\}$ is convergent.

3. Split Feasibility Problem in Non-convex Case

3.1. Notions

We now consider the split feasibility problem in non-convex frames:

$$\text{find } x \in C_0 \text{ such that } Ax \in Q_0, \tag{12}$$

where $A : \mathbb{R}^n \to \mathbb{R}^m$, C_0 and Q_0 are mentioned in (3), however, the functions c and q are both assumed to be continuous, S-subdifferential, locally Lipschitzian, not necessarily convex instead of the original convex. Assume that the solution set $\Gamma := \{x \in C_0 : Ax \in Q_0\}$ of problem (12) is non-empty.

Some assumptions and conditions on problem (12) are listed as follows. Assume that

- $S_n \subseteq \mathbb{R}^n$ and $S_m \subseteq \mathbb{R}^m$ are two closed convex sets satisfying $C_0 \subseteq S_n$ and $Q_0 \subseteq S_m$, respectively.
- $\partial_{S_n r_c} c(x)$ and $\partial_{S_m r_q} q(y)$ are the S-subdifferential of c and q with respect to S_n and S_m, respectively, where $r_c > 0$ and $r_q > 0$ are two constants.
- $s_c(x) \in \partial_{S_n r_c} c(x)$ is the S-subgradient of c at $x \in \mathbb{R}^n$; $s_q(y) \in \partial_{S_m r_q} q(y)$ is the S-subgradient of q at $y \in \mathbb{R}^m$.

From the assumptions and conditions mentioned above, by Definition 2.2, Definition 2.4 and Lemma 2.1, we define the S-subgradient projector onto C_0 related to $(c, 0)$ as

$$G_{S_n,c} : \mathbb{R}^n \to \mathbb{R}^n$$
$$x \mapsto \begin{cases} x, & x \in C_0, \\ x + \frac{-c(x)}{\|s_c(x)\|^2} s_c(x), & x \notin C_0. \end{cases}$$

Another S-subgradient projector $G_{S_m,q} : \mathbb{R}^m \to \mathbb{R}^m$ is defined in the same way.

Moreover, according to $s_c(x_k) \in \partial_{S_n r_c} c(x_k)$ and $s_q(Ax_k) \in \partial_{S_m r_q} q(Ax_k)$, we define the following half-spaces

$$C_0^k = \{u \in \mathbb{R}^n : \langle u - x_k, s_c(x_k)\rangle + c(x_k) \le 0\}, \quad k \ge 1,$$

and

$$Q_0^k = \{v \in \mathbb{R}^m : \langle v - Ax_k, s_q(Ax_k)\rangle + q(Ax_k) \le 0\}, \quad k \ge 1.$$

Set

$$f_k(x) = \frac{1}{2}\left\|x - P_{C_0^k}(x)\right\|^2 \text{ and } g_k(x) = \frac{1}{2}\left\|Ax - P_{Q_0^k}(Ax)\right\|^2$$

and can readily obtain that

$$\nabla f_k(x) = x - P_{C_0^k}(x) \text{ and } \nabla g_k(x) = A^T\left(Ax - P_{Q_0^k}(Ax)\right).$$

Denote the subgradient projector related to $(f_k, 0)$ by G_{f_k}, i.e.,

$$G_{f_k} : \mathbb{R}^n \to \mathbb{R}^n$$
$$x \mapsto \begin{cases} x, & x \in C_0^k, \\ x + \frac{-f_k(x)}{\|\nabla f_k(x)\|^2} \nabla f_k(x), & x \notin C_0^k. \end{cases} \tag{13}$$

Similarly, denote the subgradient projector associated with $(g_k,\, 0)$ by G_{g_k}, i.e.,

$$G_{g_k} : \mathbb{R}^n \to \mathbb{R}^n$$
$$x \mapsto \begin{cases} x, & Ax \in Q_0^k, \\ x + \frac{-g_k(x)}{\|\nabla g_k(x)\|^2} \nabla g_k(x), & Ax \notin Q_0^k. \end{cases} \tag{14}$$

Before constructing the iterative algorithm for solving the non-convex split feasibility problem (12), we mark

$$R_{\mu_k f_k} = I + \mu_k \left(G_{f_k} - I \right),$$

and

$$R_{\lambda_k g_k} = I + \lambda_k \left(G_{g_k} - I \right),$$

where $\lambda_k, \mu_k \in (0, 2)$ and $\{\sigma_k\} \subset [0, \sigma]$ with $0 < \sigma < 1$.

3.2. Convergence analysis

Next, we state an inertial relaxed S-subgradient projection algorithm below.

For given two initial points $x_0, x_{-1} \in \mathbb{R}^n$, compute the sequence $\{x_k\}$ by

$$\begin{cases} y_k = x_k + \sigma_k (x_k - x_{k-1}), \\ x_{k+1} = R_{\mu_k f_k} \left(R_{\lambda_k g_k}(y_k) \right), \quad k \geq 1. \end{cases} \tag{15}$$

We now give the convergence analysis of the algorithm (15) under the condition

$$\Sigma_{k=1}^{\infty} \sigma_k \|x_k - x_{k-1}\|^2 < \infty. \tag{16}$$

Theorem 3.1. *The sequence $\{x_n\}$ iteratively generated by algorithm* (15) *converges to $x^* \in \Gamma$ provided $\lambda_k, \mu_k \in (0, 2)$.*

Proof. Let $\tau \in \Gamma$ and select $s_q(Ax_k) \in \partial_{S_m r_q} q(Ax_k)$. From the assumption $Q_0 \subseteq S_m$, we obtain from (10) that

$$\langle A\tau - Ax_k, s_q(Ax_k) \rangle + q(Ax_k) \leq q(A\tau) + \frac{r_q}{2} d_{S_m}^2(A\tau) - \frac{r_q}{2} d_{S_m}^2(Ax_k) \leq 0$$

for any $A\tau \in Q_0$.

This shows that $A\tau \in Q_0^k$, i.e., $g_k(\tau) = 0$. In a similar vein, we have $f_k(\tau) = 0$.

Taking into account (14), we consider two cases: $Ay_k \in Q_0^k$ and $Ay_k \notin Q_0^k$.

If $Ay_k \in Q_0^k$, we have

$$\langle G_{g_k}(y_k) - \tau, G_{g_k}(y_k) - y_k \rangle = \langle G_{g_k}(y_k) - \tau, y_k - y_k \rangle = 0.$$

If $Ay_k \notin Q_0^k$, it follows from (10), (11), (14) and $g_k(\tau) = 0$ that

$$\begin{aligned} \langle G_{g_k}(y_k) - \tau, G_{g_k}(y_k) - y_k \rangle =& \frac{g_k(y_k)}{\|\nabla g_k(y_k)\|^2} \langle \tau - y_k, \nabla g_k(y_k) \rangle + \frac{g_{q,k}^2(y_k)}{\|\nabla g_k(y_k)\|^2} \\ \leq& \frac{g_k(y_k)}{\|\nabla g_k(y_k)\|^2} \left(g_k(\tau) - g_k(y_k) \right) + \frac{g_{q,k}^2(y_k)}{\|\nabla g_k(y_k)\|^2} \\ =& 0. \end{aligned}$$

Summarily,

$$\langle G_{g_k}(y_k) - \tau, G_{g_k}(y_k) - y_k \rangle \leq 0. \tag{17}$$

Set $w_k = R_{\lambda_k g_k}(y_k)$. Following the similar arguments in (17), we conclude

$$\langle G_{f_k}(w_k) - \tau, G_{f_k}(w_k) - w_k \rangle \leq 0, \tag{18}$$

By (17), we achieve

$$\begin{aligned}\|w_k-\tau\|^2 =&\|y_k-\tau\|^2+2\lambda_k\left\langle y_k-G_{g_k}(y_k),G_{g_k}(y_k)-y_k\right\rangle\\&+2\lambda_k\left\langle G_{g_k}(y_k)-\tau,G_{g_k}(y_k)-y_k\right\rangle\\&+\lambda_k^2\left\|G_{g_k}(y_k)-y_k\right\|^2\\\leq&\|y_k-\tau\|^2-\lambda_k(2-\lambda_k)\left\|G_{g_k}(y_k)-y_k\right\|^2.\end{aligned}$$

This together with (15) and (18), we get

$$\begin{aligned}\|x_{k+1}-\tau\|^2 =&\|w_k-\tau\|^2+2\mu_k\left\langle w_k-G_{f_k}(w_k),G_{f_k}(w_k)-w_k\right\rangle\\&+2\mu_k\left\langle G_{f_k}(w_k)-\tau,G_{f_k}(w_k)-w_k\right\rangle\\&+\mu_k^2\left\|G_{f_k}(w_k)-w_k\right\|^2\\\leq&\|y_k-\tau\|^2-\lambda_k(2-\lambda_k)\left\|G_{g_k}(y_k)-y_k\right\|^2\\&-\mu_k(2-\mu_k)\left\|G_{f_k}(w_k)-w_k\right\|^2.\end{aligned}\tag{19}$$

Now, by Lemma 2.3 that

$$\begin{aligned}\|y_k-\tau\|^2 =&\|(1+\sigma_k)(x_k-\tau)-\sigma_k(x_{k-1}-\tau)\|^2\\=&(1+\sigma_k)\|x_k-\tau\|^2-\sigma_k\|x_{k-1}-\tau\|^2\\&+\sigma_k(1+\sigma_k)\|x_k-x_{k-1}\|^2\\\leq&(1+\sigma_k)\|x_k-\tau\|^2-\sigma_k\|x_{k-1}-\tau\|^2\\&+2\sigma_k\|x_k-x_{k-1}\|^2.\end{aligned}$$

This together with (19) shows that

$$\|x_{k+1}-\tau\|^2-\|x_k-\tau\|^2\leq\sigma_k(\|x_k-\tau\|^2-\|x_{k-1}-\tau\|^2)+2\sigma_k\|x_k-x_{k-1}\|^2.$$

Applying Lemma 2.4 in above inequality, we have the existence of $\lim_{k\to\infty}\|x_k-\tau\|$. This leads to the boundedness of $\{x_k\}$ and therefore $\{y_k\}$ is bounded. Furthermore,

$$\begin{aligned}&\lambda_k(2-\lambda_k)\left\|G_{g_k}(y_k)-y_k\right\|^2+\mu_k(2-\mu_k)\left\|G_{f_k}(w_k)-w_k\right\|^2\\&\leq\|x_k-\tau\|^2-\|x_{k+1}-\tau\|^2+\sigma_k(\|x_k-\tau\|^2-\|x_{k-1}-\tau\|^2)\\&\quad+2\sigma_k\|x_k-x_{k-1}\|^2,\end{aligned}$$

which implies that

$$\lim_{k\to\infty}\|G_{g_k}(y_k)-y_k\|=\lim_{k\to\infty}\|G_{f_k}(w_k)-w_k\|=0.\tag{20}$$

Note that

$$\|G_{g_k}(y_k)-y_k\|=\left\|y_k+\frac{-g_k(y_k)}{\|\nabla g_k(y_k)\|^2}\nabla g_k(y_k)-y_k\right\|=\frac{g_k(y_k)}{\|\nabla g_k(y_k)\|},\tag{21}$$

where

$$\|\nabla g_k(y_k)\|=\|\nabla g_k(y_k)-\nabla g_k(\tau)\|\leq\|A\|^2\|y_k-\tau\|.$$

We have that $\{\nabla g_k(y_k)\}$ is bounded. Combining (20) and (21), we get

$$\lim_{k\to\infty}\left\|Ay_k-P_{Q_0^k}(Ay_k)\right\|=0.\tag{22}$$

On the other side, the locally boundedness of ∂q holds for the assumption of q, which is locally Lipschitzian. Clearly, ∂q is bounded on bounded sets and so is $I-P_{S_m}$.

Using Lemma 2.2, we conclude that $\partial_{S_m r_q} q$ is bounded on bounded sets. Therefore,

$$q(Ay_k) \leq \left\langle Ay_k - P_{Q_0^k}(Ay_k), s_q(Ay_k) \right\rangle \leq \eta \left\| Ay_k - P_{Q_0^k}(Ay_k) \right\| \tag{23}$$

with $\eta > 0$ satisfying $\|s_q(Ay_k)\| \leq \eta$.

Since $\{x_k\}$ is bounded, there exists a subsequence $\{x_{k_i}\} \subset \{x_k\}$ such that $x_{k_i} \to x^*$. By (15) and (16), we have

$$\lim_{k\to\infty} \|y_k - x_k\| = \lim_{k\to\infty} \sigma_k \|x_k - x_{k-1}\| = 0,$$

which implies that $y_{k_i} \to x^*$. The continuity assumption of function q yields

$$q(Ax^*) = \lim_{i\to\infty} q(Ay_{k_i}) \leq 0, \tag{24}$$

which means that $Ax^* \in Q_0$.

Next, we show that $x^* \in C_0$. By $w_k = R_{\lambda_k g_k}(y_k)$ and (20), we have

$$\lim_{i\to\infty} \|w_{k_i} - y_{k_i}\| = 0, \tag{25}$$

which implies that $w_{k_i} \to x^*$.

According to (13), we need to consider two cases: $w_{k_i} \in C_0^{k_i}$ and $w_{k_i} \notin C_0^{k_i}$.

If $w_{k_i} \in C_0^{k_i}$. $c(x^*) \leq 0$, i.e., $x^* \in C_0$ is obtained from the similar arguments of (23), (24) and (25).

If $w_{k_i} \notin C_0^{k_i}$. Using the parallel discussions of (21), we have

$$\lim_{i\to\infty} \left\| w_{k_i} - P_{C_0^{k_i}}(w_{k_i}) \right\| = 0.$$

The analogous analyses of (23), (24) and (25) yield $x^* \in C_0$.

Consequently, we find an element x^* satisfying $x^* \in C_0$ and $Ax^* \in Q_0$. The proof is done. □

4. Conclusion

In this paper, we investigate the nonconvex SFP in finite dimensional spaces. We suggest an inertial relaxed S-subgradient projection algorithm to seek the solution of nonconvex split feasibility problems in finite dimensional spaces. We obtain a convergence theorem for the sequences yielded by the proposed algorithm under implemented conditions on the step-size which does not rely on the spectral radius of the matrix.

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