ON THE NONLINEAR RESONANCE WAVE INTERACTION

Petre P.TEODORESCU[1], Veturia CHIROIU[2]

Această lucrare studiază interacţiunea dinamică a unei o bare lineare dispersive aşezată pe un mediu elastic continuu, cu un dispozitiv nelinear care este slab conectat la capătul din dreapta. Sunt studiate interacţiunile rezonante ale dispozitivului cu unda incidentă care se propagă prin bară folosind metoda cnoidală. Soluţiile sunt scrise ca o suma între o superpoziţie lineară şi una neliniară de vibraţii cnoidale.

This paper is studying the dynamic interaction of a dispersive linear rod resting on a continuous elastic foundation, with a nonlinear end attachment that is weakly connected to its right end. The resonant interactions of the attachment with incident traveling wave propagating in the rod are studied by using the cnoidal method. The solutions are written as a sum between a linear and a nonlinear superposition of cnoidal vibrations.

Keywords: resonant wave interaction, resonance capture, energy pumping.

1. Introduction

Resonant wave interaction is a nonlinear process in which energy is transferred between different natural modes of a system by resonance. For a nonlinear system, the motion is not simply a summation of the linear modes, but consists of the linear harmonics plus their nonlinear coupling [1], [2]. Under resonance conditions, the nonlinear coupling between different modes may lead to excitation of neutral modes. An interesting situation occurs in systems of coupled a main structure with a nonlinear attachment, where isolated resonance captures are resulting as a consequence of the energy pumping [3-5]. The energy pumping is an irreversible transfer of vibration energy from the main structure to its nonlinear attachment. It is interesting to note that this transient resonant interaction results in broadband passive absorption of energy by the attachment, in contrast to the linear vibration absorber whose effect is narrowband [6]. The interaction of incident traveling waves with a local defect can lead to phenomena, such as, speed up or slow down of the traveling wave, scattering of the wave to multiple independent wave packets, or trapping of the wave at the defect in the form of a localized wave [6], [7].

[1] Prof., University of Bucharest, Faculty of Mathematics and Informatics, Romania, e-mail: petre_teodorescu@yahoo.com
[2] Researcher, Institute of Solid Mechanics of Romanian Academy, Bucharest, Romania

In this paper, starting from the results obtained in [6], the energy exchange between a rod and a nonlinear end attachment is analyzed, for an external sine excitation applied on the assembly. The analytical solutions of the problem are obtained by using the cnoidal method [2].

2. The model

Consider an elastic rod of length L connected to a grounded local attachment of unit mass, viscous damping and stiffness nonlinearity. The connection between the rod and the nonlinear end attachment is made on the point $x = x_{\mathrm{A}}$ by means of a weak linear stiffness. Let us assume that $v(t)$ is the displacements of the attachment, the rod is initially at rest and that an external force $f(0,t) = A\sin\omega t$ is applied at the origin O of the coordinate system, at $t = 0$. The displacement $y(x_{\mathrm{A}}, t)$ of the rod at the point of attachment A, in the direction of $v(t)$, can be written as [6]

$$\begin{gathered} y(x_{\mathrm{A}},t) = \int_{-\infty}^{t} f(0,\tau) g_{\mathrm{AO}}(t-\tau)\mathrm{d}\tau - \\ -\int_{-\infty}^{t} \varepsilon\left(y(x_{\mathrm{A}},\tau) - v(\tau)\right) f(0,\tau) g_{\mathrm{AA}}(t-\tau)\mathrm{d}\tau, \end{gathered} \quad , \tag{2.1}$$

where the Green's function g_{AA} is the displacement at point A of the rod in the direction of $v(t)$, due to a unit impulse applied at the same point and the same direction, and the Green's function g_{AO} is the displacement at point A of the rod in the direction of $v(t)$, due to a unit impulse applied at origin O in the direction of the external force. Consequently, the motion equation of the attachment is given by

$$\ddot{v}(t) + \lambda\dot{v}(t) + \sum_{j=1}^{p} \alpha_j v^j(t) = \varepsilon[y(x_{\mathrm{A}},t) - v(t)], \quad v(0) = \dot{v}(0) = 0, \tag{2.2}$$

where $0 \le \varepsilon \ll 1$ scales the weak coupling, λ denotes the viscous damping coefficient, and α_j, $j = 1,\ldots,p$, the coefficients of the stiffness nonlinearity.

Substituting (2.2) in (2.1), the following equation for the oscillation of the attachment is obtained

$$\ddot{v}(t) + \lambda\dot{v}(t) + \sum_{j=1}^{p} \alpha_j v^j(t) = \sum_{n=1}^{N} (-1)^{n-1}\varepsilon^n \Delta_n , \tag{2.3}$$

$$\Delta_n = f(0,t) * g_{\rm AO}(t) * g_{\rm AA}^{n-1}(t) + (-1)^n v(t) * g_{\rm AA}^{n-1}(t)\,.$$

The Green functions are expanded by a set of orthogonal polynomials $\varphi_n(t)$

$$g(t) = \sum_{n=0}^{\infty} \frac{1}{w_n} c_n \varphi_n(t)\,,\ c_n = \int_{-\infty}^{\infty} w(t)\varphi_n^*(t)\mathrm{d}t\,.$$

The polynomials $\varphi_n(t)$ satisfy the orthogonality and completeness conditions

$$\int_{-\infty}^{\infty} w(t)\varphi_n^*(t)\varphi_m(t)\mathrm{d}t = w_n\delta_{nm}\,,\ \delta(t-t_0) = \sum_{n=0}^{\infty} \frac{w(t)}{w_n}\varphi_n^*(t)\varphi_m(t_0)\,, \qquad (2.4)$$

and also a recurrence formula

$$.\,\varphi_{n+1}(t) = b_n(t-a_n)\varphi_n(t) - c_{n-1}\varphi_{n-1}(t)\,. \qquad (2.5)$$

The Hermite polynomials $H_n(t)$ are used in this paper for expanding the Green's functions. The motion equation of the rod is

$$\begin{gathered} y_{tt}(x,t) + \omega_0^2 y(x,t) - y_{xx}(x,t) = f(0,t)\,, \\ y_x(x_{\rm A},t) + \varepsilon[v(t) - y(x_{\rm A},t)] = 0\,,\ y(x_{\rm A} - L,t) = 0\,, \\ y(x,0) = y_t(x,0) = v(0) = \dot{v}(0) = 0\,, \end{gathered} \qquad (2.6)$$

where $x_A = L + e$ ($x = e$ and $x = L + e$ are the ends of the rod) and ω_0^2 is the normalized stiffness of the elastic foundation.

3. Solutions

The cnoidal method was proposed in [2] for solving the nonlinear equations, as a further extension of the Osborne method [8]. The set of equations (2.1)-(2.6) can be reduced to equations similar to Weierstrass equation with polynomials of higher order

$$\dot{\theta}^2 = \sum_{j=0}^{p} A_j(t)\theta^j\,, \qquad (3.1)$$

with θ a generic function of time, and $A_i > 0$ constants. The dot means differentiation with respect to the variable $t \to x - ct$, c a constant. We know that (3.1) admits a particular solution expressed as an elliptic Weierstrass function that is reduced, in this case, to the cnoidal function cn [9]

$$\theta \equiv f(t) = e_2 - (e_2 - e_3)\mathrm{cn}^2\left(\sqrt{(e_1 - e_3)}t; m\right), \tag{3.2}$$

where e_1, e_2, e_3 are the real roots of the equation $4y^3 - g_1 y - g_2 = 0$ with $e_1 > e_2 > e_3$, and $g_1, g_2 \in R$ expressed in terms of the constants A_i, $i = 1,2...5$, and satisfying the condition $g_1^3 - 27g_2^2 > 0$. The modulus m of the Jacobean elliptic function is $m = \dfrac{e_2 - e_3}{e_1 - e_3}$. For arbitrary initial conditions

$$\theta(0) = \theta_0, \quad \dot{\theta}(0) = \theta_{p0}, \tag{3.3}$$

the solution of (3.1) can be written as a sum between a linear and a nonlinear superposition of cnoidal vibrations

$$\theta = \theta_{\mathrm{lin}} + \theta_{\mathrm{nonlin}}, \tag{3.4}$$

where

$$\theta_{\mathrm{lin}} = 2\sum_{k=0}^{n} \alpha_k \mathrm{cn}^2\left(\omega_k t; m_k\right), \ \theta_{\mathrm{nonlin}} = \frac{\sum_{k=0}^{n} \beta_k \mathrm{cn}^2(\omega_k t; m_k)}{1 + \sum_{k=0}^{n} \gamma_k \mathrm{cn}^2(\omega_k t; m_k)}, \tag{3.5}$$

where the moduli $0 \le m_k \le 1$, the frequencies ω_k and amplitudes α_k depending on θ_0, θ_{p0} and A_k. Therefore, the solutions of (2.1)-(2.6) are written under the form (3.5)

$$v(t) = a_1 \sum_{k=1}^{n} \mathrm{cn}^2(t; m_k) + \frac{\sum_{k=0}^{n} \beta_{k1} \mathrm{cn}^2(t; m_k)}{1 + \sum_{k=0}^{n} \gamma_{k1} \mathrm{cn}^2(t; m_k)},$$

$$y(\xi) = a_2 \sum_{k=1}^{n} \mathrm{cn}^2(\xi; m_k) + \frac{\sum_{k=0}^{n} \beta_{k2} \mathrm{cn}^2(\xi; m_k)}{1 + \sum_{k=0}^{n} \gamma_{k2} \mathrm{cn}^2(\xi; m_k)}, \ \xi = kx - \omega t, \tag{3.6}$$

4. Results and conclusions

The calculations are carried out for $L = 200$, $\omega_0 = \sqrt{1.2}$, $f = 10$, $\lambda = 0.4$, $e = 1$. The response $v(t)$ of the attachment is displayed in fig.4.1. Instantaneous frequency of the nonlinear attachment is depicted in fig.4.2. These figures put into evidence the presence of four regimes of transient responses. The first regime (0-50-90s) describe the interaction of the nonlinear attachment with incoming traveling waves with frequency $\omega > \omega_0$. After a short transition, the attachment passes to periodic oscillation of the second regime (140-260s) with frequency nearly below ω_0, and after another short transition to a weakly oscillation of the third regime (340-500s) with frequency nearly above ω_0. The periodic motion of the second and third regimes are the consequence of energy pumping where the attachment engages in 1–1 resonance capture with a linear structural mode [6].

The last regime (550-800s) consists in a weakly modulated periodic motions in the neighborhood of ω_0. The transition between the third and fourth regimes describes the case when the attachment can no longer sustain resonance capture, and escape from resonance capture occurs. The energy is radiated back to the rod and the instantaneous frequency decreases until it reaches a frequency ~ 0. By comparing our results with those obtained in [6] for impulse excitation and step initial displacement distribution, we observe that in the case of a sine external force, four regimes are depicted, and no three as in [6]. This can be explained by an oscillatory irreversible transfer of vibration energy from the rod to its nonlinear attachment. Two steps of energy pumping for 1-1 resonance capture with the linear structural mode are depicted, for two weakly modulated periodic motions with frequency nearly equal (below and above ω_0). We can term this phenomenon as an *oscillatory energy pumping*.

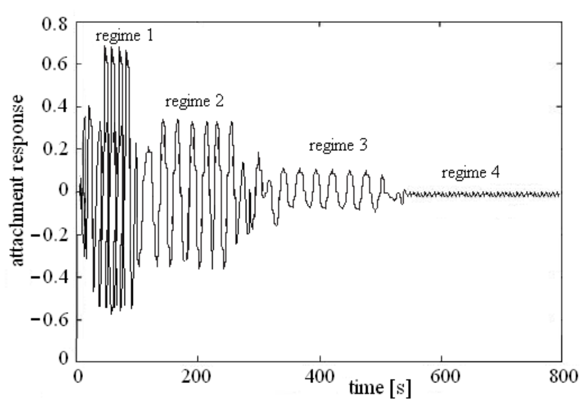

Fig. 4.1. The response of the attachment.

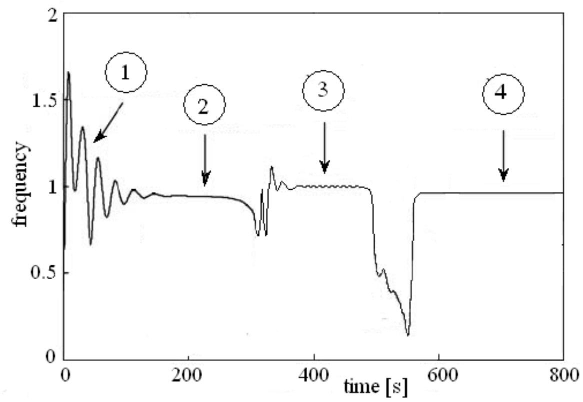

Fig. 4.2. Instantaneous frequency of the nonlinear attachment.

Acknowledgement

The authors gratefully acknowledge the financial support of the National Authority for Scientific Research (ANCS, UEFISCSU), through PN-II research project code ID_1391/2008.

R E F E R E N C E S

[1] *C.R.Gilson and M.C Ratter*, “Three-dimensional three-wave interactions: a bilinear approach”, in J. Phys. A: Math. Gen., **vol.31**, 1998, pp.349–367.

[2] *L.Munteanu and St.Donescu*, Introduction to Soliton Theory: Applications to Mechanics, Book Series “Fundamental Theories of Physics”, **vol.143**, Kluwer Academic Publishers, 2004.

[3] *L. I Manevitch, O.Gendelman, A.Musienko, A.F Vakakis and L.A. Bergman*, “Dynamic interaction of a semi-infinite linear chain of coupled oscillators with a strongly nonlinear end attachment”, Physica D., **vol.178**, no. 1-2, 2003, pp.1–18.

[4] *A. F. Vakakis, L. I. Manevitch, O. Gendelman, L.Bergman*, “Dynamics of linear discrete systems connected to local essentially nonlinear attachments”, J. Sound Vibration, **vol. 264**, 2003, pp.559–577.

[5] *A.F.Vakakis and O.Gendelman*, “Energy pumping in nonlinear mechanical oscillators ii: resonance capture”, J. Appl. Mech., **vol. 68**, no.1, 2001, pp.42–48.

[6] *A.F.Vakakis, L.I.Manevitch, A.I. Musienko, G.Kerschen, L.A.Bergman*, “Transient dynamics of a dispersive elastic wave guide weakly coupled to an essentially nonlinear end attachment, Wave Motion, **vol. 41**, 2005, pp.109–132.

[7] *R.H.Goodman, P.J.Holmes and M.I.Weinstein*, ”Interaction of sine-gordon kinks with defects: phase space transport in a two-mode model”, Physica D., **vol.161**, 2002, pp.21–44.

[8] *A.R.Osborne*, “Soliton physics and the periodic inverse scattering transform”, Physica D., **vol.86**, 1995, pp.81–89.

[9] *M.J.Ablowitz and P.A.Clarkson*, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge Univ. Press, Cambridge, 1991.