ADAPTIVE GALERKIN FRAME METHODS FOR SOLVING OPERATOR EQUATIONS

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In this paper we use frames to construct corresponding trial spaces for an adaptive Galerkin scheme and design an algorithm in order to give an adaptive approximation solution to operator equations. We describe construction, prove error estimates for the resulting scheme and then investigate computational complexity. The adaptive Galerkin method that we analyze in this paper will iteratively produces finite sets $\Lambda_j \subset \Lambda,\ \ j \in \mathrm{N}$ *and the Galerkin approximations* u_{Λ_j} *to u of the Subspaces* $S_{\Lambda_j} = span(\{\varphi_\lambda\}_{\lambda\in\Lambda_j})$. *The error of this approximation is* $O(\#\Lambda_j^{-s})$, *(for some s>0) in the energy norm, where* $(\varphi_\lambda)_{\lambda\in\Lambda_j} \subset H$, *is a frame for a separable Hilbert space* H *and* $\#\Lambda_j$, *denotes the cardinality o f* Λ_j.

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1. Introduction

Assume that H is a separable Hilbert space with dual H^*, Λ is a countable set of indices and $\Psi = (\psi_\lambda)_{\lambda\in\Lambda} \subset H$ is a *frame* for H This means that there exist constants $0 < A \le B < \infty$ such that

$$A\|f\|_H^2 \le \sum_{\lambda\in\Lambda} |< f, \psi_\lambda >|^2 \le B\|f\|_H^2, \qquad \forall f \in H, \tag{1}$$

or equivalently (by the Riesz mapping),

$$A\|f\|_{H^*}^2 \le \|f(\Psi)\|_{\ell_2}^2 \le B\|f\|_{H^*}^2, \qquad \forall f \in H^*, \tag{2}$$

where $f(\Psi) = (f(\psi_\lambda))_{\lambda\in\Lambda} = (< f, \psi_\lambda >)_{\lambda\in\Lambda}$. Our problem is to find $u \in H$, such that

$$Lu = f, \tag{3}$$

where $L : H \to H^*$ is a symmetric, positive definite and bounded invertible linear operator. For example, we can consider a linear differential operator in variational form. In [2, 3], an iterative adaptive method for solving this system has been developed by wavelet bases.

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For an index set $\tilde{\Lambda} \subset \Lambda$, $(\psi_\lambda)_{\lambda\in\tilde{\Lambda}}$ is called a *frame sequence* if it is a frame for its closed span. We assume that $(\psi_\lambda)_{\lambda\in\tilde{\Lambda}}$ is a frame sequence with bounds $A, B,$ for all finite index sets $\tilde{\Lambda} \subset \Lambda$. The adaptive Galerkin method that we analyze in this paper will iteratively produce finite sets $\Lambda_j \subset \Lambda$, $j \in \mathrm{N}$, and the Galerkin approximation u_{Λ_j} to u of the subspaces $S_{\Lambda_j} = span(\{\psi_\lambda\}_{\lambda\in\Lambda_j})$, with error $O(\#\Lambda_j^{-s})$ (for some $s > 0$) in the energy norm , where $\#\Lambda_j$ denotes the cardinality of Λ_j. For the frame Ψ, let $T : \ell_2(\Lambda) \to H$ be the *synthesis operator*

$$T((c_\lambda)_\lambda) = \sum_{\lambda\in\Lambda} c_\lambda \psi_\lambda$$

and let $T^* : H \to \ell_2(\Lambda)$ (or $T^* : H^* \to \ell_2(\Lambda)$) be the *analysis operator*

$$T^*(f) = (< f, \psi_\lambda >)_\lambda.$$

Also let $S = TT^* : H \to H$ be the frame operator $S(f) = \sum_\lambda < f, \psi_\lambda > \psi_\lambda$.

Note that T is surjective, T^* is injective, T^* is the adjoint of T and because of (1) or (2) T is bounded, in fact we have

$$\|T\| = \|T^*\| \le \sqrt{B}. \tag{4}$$

It was shown in [1] that, for the frame $(\psi_\lambda)_{\lambda\in\Lambda}$, S is a positive invertible operator satisfying $AI_H \le S \le BI_H$ and $B^{-1}I_H \le S^{-1} \le A^{-1}I_H$. Also, the sequence

$$\tilde{\Psi} = (\tilde{\psi}_\lambda)_{\lambda\in\Lambda} = (S^{-1}\psi_\lambda)_{\lambda\in\Lambda}$$

is a frame (called the *canonical dual frame*) for H with bounds B^{-1}, A^{-1}. Every $f \in H$ has the expansion

$$f = \sum_\lambda < f, \tilde{\psi}_\lambda > \psi_\lambda = \sum_\lambda < f, \psi_\lambda > \tilde{\psi}_\lambda$$

Since $Ker(T) = Ran(T^*)^\perp$ we have $\ell_2(\Lambda) = Ran(T^*) \oplus Ker(T)$. Thus the orthogonal projection Q of a sequence $(c_\lambda)_{\lambda\in\Lambda} \in \ell_2(\Lambda)$ onto the $Ran(T^*)$ is given by

$$Q(c_\lambda)_{\lambda\in\Lambda} \in \ell_2(\Lambda) = (< \sum_\lambda c_\lambda S^{-1}\psi_\lambda, \psi_j >)_{j\in\Lambda},$$

that is, $Q = T^*S^{-1}T : \ell_2(\Lambda) \to \ell_2(\Lambda)$. For more details see [1].

Since L is bounded invertible, (3) has a unique solution u for any $f \in H$ and

$$\|Lu\|_{H^*} \cong \|u\|_H, \qquad u \in H,$$

where $a \cong b$ means that, there are constants d_1, d_2 such that $d_1 a \le b \le d_2 a$. Also the bilinear form a defined by

$$a(u,v) =< Lu, v >$$

is symmetric, positive definite and *elliptic* in the sense that

$$a(u,v) \cong \|v\|_H^2. \tag{5}$$

It follows that H is a Hilbert space with respect to the inner product a with an equivalent energy norm $\|.\|_a^2 = a(.\,,.)$.

2. The Equivalent ℓ_2-Problem

Let $\Psi = (\psi_\lambda)_{\lambda\in\Lambda}$ be a frame for H with bounds A and B, and $\tilde{\Psi} = (\tilde{\psi}_\lambda)_{\lambda\in\Lambda} = (S^{-1}\psi_\lambda)_{\lambda\in\Lambda}$ be its canonical dual frame. For the infinite matrix G defined by

$$G_{i,j} = < L\psi_j, \psi_i >,$$

where $<.,.>$ denotes the (H^*, H) duality product, we have the following theorem, in which T and T^* are as defined in section 1.

Theorem 2.1. The matrix G defines a bounded linear operator form $\ell_2(\Lambda)$ to $\ell_2(\Lambda)$ with $\|G\|_{\ell_2(\Lambda)\to\ell_2(\Lambda)} \le B\|L\|_{H\to H^*}$.

As an operator on $\ell_2(\Lambda)$ we have $G = T^*LT$. Also G as a map form $Ran(T^*)$ into itself is bounded and invertible.

Proof . By definition of G, for a sequence $C \in \ell_2(\Lambda)$ we have

$$=< \sum_k c_k L\psi_k, \psi_j >=< L\sum_k c_k\psi_k, \psi_j >=< LTC, \psi_j >$$

using (4) we have :

$$\|GC\|^2 = \sum_j \left|< LTC, \psi_j >\right|;$$

$$\le B\|LTC\|_{H^*}^2 \le B\|L\|^2\|TC\|_H^2$$

$$\le B^2\|L\|^2\|C\|_{\ell_2(\Lambda)}^2.$$

Therefore

$$\|G\| \le B\|L\|.$$

Let $C \in Ran(T^*)$, then $c_\lambda =< u, \psi_\lambda >$ for some $u \in H$. Also

$$(GC)_j = \sum_\lambda < L\psi_\lambda, \psi_j >< u, \psi_\lambda >$$

$$=< \sum_\lambda < u, \psi_\lambda > L\psi_\lambda, \psi_j >=< LSu, \psi_j >,$$

where S is the frame operator. This means that G maps $Ran(T^*)$ into $Ran(T^*)$ with $(< u, \psi_\lambda >)_\lambda \to (< LSu, \psi_\lambda >)_\lambda$.

Let $v \in H^*$, since L and S are surjective, so there exists $u \in H$ such that $LSu = v$, thus G maps $(<u, \psi_\lambda>)_\lambda$ *to* $(<v, \psi_\lambda>)_\lambda$.
Also since L and S are injective, if $<LSu, \psi_\lambda> = <LSv, \psi_\lambda>$, for every $\lambda \in \Lambda$ then, because of completeness of the frame Ψ, $Lu = Lv$, therefore $u = v$, and $<u, \psi_\lambda> = <v, \psi_\lambda>$ for all $\lambda \in \Lambda$, i.e.; G is injective.

Consider $M = G|_{Ran(T^*)}$, then the equation (3) takes the following form:

There exists $U \in Ran(T^*)$ such that $u = TU$ (in terms of the canonical dual frame $\tilde{\Psi}$ we have $U = (<u, \tilde{\psi}_\lambda>)_{\lambda\in\Lambda}$), then $T^*LTU = T^*f$, that is

$$MU = F, \tag{6}$$

where $F = T^*f$. The solution U to (6) gives the solution u to (3).
The matrix M is symmetric and positive definite. We define its associated bilinear form a by

$$a(V, W) = <MV, W>_{\ell_2(\Lambda)}$$

with the associated norm

$$\|V\|^2 = a(V, V), \quad \forall V \in \ell_2(\Lambda).$$

From (5) and frame condition (1), we obtain $\|V\| \cong \|V\|_{\ell_2(\Lambda)}$. Thus there exist constants c_1, c_2 such that

$$c_1\|V\|^2_{\ell_2(\Lambda)} \le \|V\|^2 \le c_2\|V\|^2_{\ell_2(\Lambda)}, \tag{7}$$

therefore

$$c_1\|V\|_{\ell_2(\Lambda)} \le \|MV\|_{\ell_2(\Lambda)} \le c_2\|V\|_{\ell_2(\Lambda)}. \tag{8}$$

Now from (7) and (8) we have

$$\frac{c_1}{\sqrt{c_2}}\|V\| \le \|MV\|_{\ell_2(\Lambda)} \le \frac{c_2}{\sqrt{c_2}}\|V\|. \tag{9}$$

For a tolerance $\varepsilon > 0$ and a vector $V \in \ell_2(\Lambda)$ let V_N be the vector obtained by replacing all but the N largest coefficients in modulus of V by zeros, for the smallest $N \in \mathbb{N}$ such that

$$\|V - V_N\|_{\ell_2(\Lambda)} \le \varepsilon,$$

V_N is called the best N*-term approximation* for V. Now let $V = (v_\lambda)_{\lambda\in\Lambda} \in \ell_2(\Lambda)$, for each $n \ge 1$ let v_n^* be the n-th largest of the numbers

$|v_\lambda|$ and $V^* = (v_n^*)_{n=1}^{\infty}$. For each $0 < \tau < 2$ we let $\ell_\tau^\omega(\Lambda)$ denote the collection of all vectors $V \in \ell_2(\Lambda)$ for which

$$|V|_{\ell_\tau^\omega(\Lambda)} = \sup_{n \ge 1}\{n^{\frac{1}{\tau}} v_n^*\}$$

is finite. This expression defines a quasi norm for $\ell_\tau^\omega(\Lambda)$. Defining

$$\|V\|_{\ell_\tau^\omega(\Lambda)} = |V|_{\ell_\tau^\omega(\Lambda)} + \|V\|_{\ell_2(\Lambda)},$$

then there exists a constant c_τ such that

$$\|V + W\|_{\ell_\tau^\omega(\Lambda)} \le c_\tau(\|V\|_{\ell_\tau^\omega(\Lambda)} + \|W\|_{\ell_\tau^\omega(\Lambda)}). \tag{10}$$

Also

$$\sup_N N^s \|V - V_N\|_{\ell_2(\Lambda)} \cong \|V\|_{\ell_\tau^\omega(\Lambda)}$$

and there exists a constant C such that

$$\|V - V_N\|_{\ell_2(\Lambda)} \le C\|V\|_{\ell_\tau^\omega(\Lambda)} N^{-s}, \quad N \in \mathrm{N} \tag{11}$$

for each $s > 0$ such that $\tau = (\frac{1}{2} + s)^{-1}$. One can see [2,4,5] for further details on the quasi-Banach spaces $\ell_\tau^\omega(\Lambda)$.

3. The Galerkin Algorithm

In this section we construct an adaptive Galerkin algorithm to give an adaptive approximation to solution U of (6), for $U \in \ell_\tau^\omega(\Lambda)$ (for some $s > 0$ and $\tau = (\frac{1}{2} + s)^{-1}$). We recall that $MU = F$ where $F = T^* f$ with this in mind we note that our algorithm generated nested finite sets Λ_j, $j \in \mathrm{N}$ and the approximate Galerkin solution U_{Λ_j} to U such that

$$\|U - U_{\Lambda_j}\| \prec (\#\Lambda_j)^{-s},$$

where $a \prec b$ means that there exists a constant d such that $a \le bd$. Let $S_\Gamma = span\{\psi_\gamma : \gamma \in \Gamma\}$ for $\Gamma \subset \Lambda$. The *approximate Galerkin solution* u_Γ from S_Γ is defined by conditions $a(u_\Gamma, v) = \langle f, v \rangle$, $v \in S_\Gamma$, that is equivalent to

$$M_\Gamma U_\Gamma = P_\Gamma F, \tag{12}$$

where $M=(m_{\gamma,\eta})_{\gamma,\eta\in\Gamma}$ and P_Γ is the orthogonal projector from $\ell_2(\Lambda)$ onto $\ell_2(\Gamma)$. Let $R_\Gamma = MU - MU_\Gamma = F - MU_\Gamma$ be the residual associated to Γ and $0<\alpha<1$ be the fixed real number. Following [2] we introduce the following routines at our disposal.

GALERKIN: $[\Gamma]\to U_\Gamma$

Solve the finite system of (12) for the finite set Γ and find the approximate Galerkin solution U_Γ to U.

GROW: $[\Gamma, U_\Gamma]\to \overline{\Gamma}$

by taking the N largest coefficients of R_Γ in absolute value, denote by Γ_N the set of these indices, find the smallest integer N such that

$$C\|R_\Gamma\|_{\ell_\tau^\omega(\Lambda)} N^{-s} \le (1-\alpha)\|R_\Gamma\|_{\ell_2(\Lambda)} \tag{13}$$

and define $\overline{\Gamma} = \Gamma \cup \Gamma_N$.

In order the algorithm **SOLVE** to meet our goal, we need some assumptions.

Assumption 1. Assume that the matrix M is s - *compressible* in the sense that for each $j\in\mathrm{N}$ there exist constants α_j and β_j and a matrix M_j having at most $\alpha_j 2^j$ nonzero entries per column, such that $\|M - M_j\| \le \beta_j$,

where $(\alpha_j)_{j\in\mathrm{N}}$ is summable and for any $s'<s$, $(\beta_j 2^{s'j})_{j\in\mathrm{N}}$ is summable.

Proposition 3.1. The s-compressible matrix M maps $\ell_\tau^\omega(\Lambda)$ bonded into itself for $\tau = (\frac{1}{2}+s)^{-1}$.

Proof. See[2].

Since $U, U_\Gamma \in \ell_\tau^\omega(\Lambda)$ so by Propsition.3.1. $R_\Gamma \in \ell_\tau^\omega(\Lambda)$. From (11) we have

$$\|R_\Gamma - R_{\Gamma_N} R_\Gamma\|_{\ell_2(\Lambda)} \le C\|R_\Gamma\|_{\ell_\tau^\omega(\Lambda)} N^{-s},$$

we may assume that $C\ge 1$. This inequality and (13) give

$$\|R_\Gamma - R_{\Gamma_N} R_\Gamma\|_{\ell_2(\Lambda)} \le (1-\alpha)\|R_\Gamma\|_{\ell_2(\Lambda)},$$

therefore

$$\|P_{\overline{\Gamma}} R_\Gamma\|_{\ell_2(\Lambda)} \ge \alpha \|R_\Gamma\|_{\ell_2(\Lambda)}. \tag{14}$$

Also because of (13)

$$N^{-s} \le \frac{(1-\alpha)\|R_\Gamma\|_{\ell_2(\Lambda)}}{C\|R_\Gamma\|_{\ell_\tau^\omega(\Lambda)}}.$$

Since N is the smallest integer that satisfies (13) we have

$$N \geq \left(\frac{C\|R_\Gamma\|_{\ell_\tau^\omega(\Lambda)}}{(1-\alpha)\|R_\Gamma\|_{\ell_2(\Lambda)}}\right)^{\frac{1}{s}} \geq (N-1).$$

So for $N \geq 2$,

$$N \leq \left(\frac{C\|R_\Gamma\|_{\ell_\tau^\omega(\Lambda)}}{(1-\alpha)\|R_\Gamma\|_{\ell_2(\Lambda)}}\right)^{\frac{1}{s}} + 1 \leq 2\left(\frac{C\|R_\Gamma\|_{\ell_\tau^\omega(\Lambda)}}{(1-\alpha)\|R_\Gamma\|_{\ell_2(\Lambda)}}\right)^{\frac{1}{s}}.$$

Hence for a constant q_1

$$\#(\overline{\Gamma}-\Gamma) = N \leq q_1\left(\frac{\|R_\Gamma\|_{\ell_\tau^\omega(\Lambda)}}{\|R_\Gamma\|_{\ell_2(\Lambda)}}\right)^{\frac{1}{s}}. \qquad (15)$$

Lemma 3.2. The output $\overline{\Gamma}$ of ***GROW*** satisfies

$$\|U-U_{\overline{\Gamma}}\| \leq \theta\|U-U_\Gamma\| \qquad \text{where } \theta = \left(1-\alpha^2\left(\frac{c_1}{c_2}\right)^3\right)^{\frac{1}{2}}.$$

Proof. From (9) we have

$$\begin{aligned}\|U_{\overline{\Gamma}}-U_\Gamma\| &\geq \frac{c_1^{\frac{1}{2}}}{c_2}\|M(U_{\overline{\Gamma}}-U_\Gamma)\|_{\ell_2(\Lambda)} \geq \frac{c_1^{\frac{1}{2}}}{c_2}\|M(U_{\overline{\Gamma}}-U_\Gamma)\|_{\ell_2(\overline{\Gamma})}\\ &= \frac{c_1^{\frac{1}{2}}}{c_2}\|M(U_{\overline{\Gamma}}-U_\Gamma)\|_{\ell_2(\overline{\Gamma})} = \frac{c_1^{\frac{1}{2}}}{c_2}\|P_{\overline{\Gamma}}R_\Gamma\|_{\ell_2(\Lambda)}\\ &\geq \alpha\frac{c_1^{\frac{1}{2}}}{c_2}\|R_\Gamma\|_{\ell_2(\Lambda)} = \alpha\frac{c_1^{\frac{1}{2}}}{c_2}\|M(U-U_\Gamma)\|_{\ell_2(\Lambda)},\end{aligned}$$

where the last inequality obtains from (14), therefore by (9)

$$\|U_{\overline{\Gamma}}-U_\Gamma\| \geq \alpha\left(\frac{c_1}{c_2}\right)^{\frac{3}{2}}\|U-U_\Gamma\|. \qquad (16)$$

Now because of orthogonality of the Galerkin solutions with respect to the energy inner product,

$$\|U-U_\Gamma\|^2 = \|U-U_{\overline{\Gamma}}\|^2 + \|U_\Gamma-U_{\overline{\Gamma}}\|^2,$$

then by (16)

$$\|U-U_{\overline{\Gamma}}\|^2 = \|U-U_\Gamma\|^2 - \|U_\Gamma-U_{\overline{\Gamma}}\|^2$$

$$\leq \|U-U_\Gamma\|^2-\alpha^2(\frac{c_1}{c_2})^3\|U-U_\Gamma\|^2=\|U-U_\Gamma\|^2(1-\alpha^2(\frac{c_1}{c_2})^3),$$

hence

$$\|U-U_{\bar{\Gamma}}\|\leq\|U-U_\Gamma\|(1-\alpha^2(\frac{c_1}{c_2})^3)^{\frac{1}{2}}.$$

Now we are ready to present our algorithm.

SOLVE $[\varepsilon, M, F]\to[\Lambda_\varepsilon, U_{\Lambda_\varepsilon}]$

(i) $\Lambda_0=\phi,\ \ R_{\Lambda_0}=F$; **(ii)** while $\theta^j\|F\|_{\ell_2(\Lambda)}\dfrac{c_2^{\frac{1}{2}}}{c_1}>\varepsilon.$

(ii,1) GALERKIN: $[\Lambda_j]\to U_{\Lambda_j}$. **(ii,2) GROW:** $[\Lambda_j, U_{\Lambda_j}]\to\Lambda_{j+1}$

(ii,3) $j\to j+1.$ **(iii)** $\Lambda_\varepsilon=\Lambda_{j+1},\ \ U_{\Lambda_\varepsilon}=U_{\Lambda_{j+1}}$.

Theorem 3.3. The output U_{Λ_ε} of the algorithm satisfies

$$\|U-U_{\Lambda_\varepsilon}\|\leq\varepsilon.$$

Proof **.** Using Lemma.3.2 for Λ_j and Λ_{j+1}

$$\|U-U_{\Lambda_{j+1}}\|\leq\theta\|U-U_{\Lambda_j}\|, \tag{17}$$

repeartedly

$$\|U-U_{\Lambda_\varepsilon}\|\leq\theta^k\|U\|, \tag{18}$$

for a constant $k\in\mathrm{N}$. Combining this with (9) we have

$$\|U-U_{\Lambda_\varepsilon}\|\leq\theta^k\|U\|\leq\theta^k\frac{c_2^{\frac{1}{2}}}{c_1}\|F\|_{\ell_2(\Lambda)}\leq\varepsilon.$$

Assumption 2. Assume that $U\in\ell_\tau^\omega(\Lambda)$, $\tau=(\dfrac{1}{2}+s)^{-1}$ and the inverse matrices $M_{\Lambda_j}^{-1}$ are uniformly bounded on $\ell_\tau^\omega(\Lambda)$, i.e.; there exists a constant C_0 such that

$$\|M_{\Lambda_j}^{-1}\|_{\ell_\tau^\omega(\Lambda)}\leq C_0,\quad j\in\mathrm{N}.$$

Theorem 3.4. The outputs U_{Λ_j} and Λ_j in step ***(ii)*** in the algorithm ***SOLVE*** satisfy

$$\left\|U-U_{\Lambda_j}\right\| \prec (\#\Lambda_j)^{-s}.$$

Proof. Since $U\in\ell_\tau^\omega(\Lambda)$ then by Proposition.3.1 $F\in\ell_\tau^\omega(\Lambda)$. Therefore

$$\left\|P_{\Lambda_j}F\right\|_{\ell_\tau^\omega(\Lambda)} \le \|F\|_{\ell_\tau^\omega(\Lambda)} \le \|M\|_{\ell_\tau^\omega(\Lambda)}\|U\|_{\ell_\tau^\omega(\Lambda)},$$

combining this inequality with **Assumption 2** and (12) we obtain

$$\left\|U_{\Lambda_j}\right\|_{\ell_\tau^\omega(\Lambda)} = \left\|U_{\Lambda_j}\right\|_{\ell_\tau^\omega(\Lambda_j)} \le C_0\left\|P_{\Lambda_j}F\right\|_{\ell_\tau^\omega(\Lambda)} \le \|M\|_{\ell_\tau^\omega(\Lambda)} C_0\|U\|_{\ell_\tau^\omega(\Lambda)},$$

hence

$$\left\|R_{\Lambda_j}\right\|_{\ell_\tau^\omega(\Lambda)} \le \|M\|_{\ell_\tau^\omega(\Lambda)}\left\|U-U_{\Lambda_j}\right\|_{\ell_\tau^\omega(\Lambda)},$$

$$\le \|M\|_{\ell_\tau^\omega(\Lambda)} c_\tau(\|U\|_{\ell_\tau^\omega(\Lambda)} + \left\|U_{\Lambda_j}\right\|_{\ell_\tau^\omega(\Lambda)}) \le \|M\|_{\ell_\tau^\omega(\Lambda)} c_\tau\|U\|_{\ell_\tau^\omega(\Lambda)}(1+C_0\|M\|_{\ell_\tau^\omega(\Lambda)}),$$

where the second inequality takes from (10). Thus the residuals R_{Λ_j} are uniformly bounded, that means there exists a constant q_2 such that $\left\|R_{\Lambda_j}\right\|_{\ell_\tau^\omega(\Lambda)} \le q_2$.

This inequality with (15) and (9) give

$$\#\Lambda_j \le \#\Lambda_{j-1} + q_1 q_2^{\frac{1}{s}}\left\|M(U-U_{\Lambda_{j-1}})\right\|_{\ell_2(\Lambda)}^{\frac{-1}{s}}$$

$$\le \#\Lambda_{j-1} + q_1 q_2^{\frac{1}{s}}\left(\frac{c_1}{\sqrt{c_2}}\right)^{\frac{-1}{s}}\left\|U-U_{\Lambda_{j-1}}\right\|^{\frac{-1}{s}}.$$

Then by (17)

$$\#\Lambda_j\left\|U-U_{\Lambda_j}\right\|^{\frac{1}{s}} \le (\#\Lambda_{j-1} + q_1 q_2^{\frac{1}{s}}\left(\frac{c_1}{\sqrt{c_2}}\right)^{\frac{-1}{s}}\left\|U-U_{\Lambda_{j-1}}\right\|^{\frac{-1}{s}})(\theta^{\frac{1}{s}}\left\|U-U_{\Lambda_{j-1}}\right\|^{\frac{1}{s}})$$

$$=\theta^{\frac{1}{s}}(\#\Lambda_{j-1}\left\|U-U_{\Lambda_{j-1}}\right\|^{\frac{1}{s}} + q_1 q_2^{\frac{1}{s}}\left(\frac{c_1}{\sqrt{c_2}}\right)^{\frac{-1}{s}}),$$

iterating the procedure, we have

$$\#\Lambda_j\left\|U-U_{\Lambda_{j-1}}\right\|^{\frac{1}{s}} \le \#\Lambda_1\left\|U-U_{\Lambda_1}\right\|^{\frac{1}{s}}\theta^{\frac{j-1}{s}} + q_3\sum_{i=1}^{j-1}\theta^{\frac{i}{s}}, \tag{19}$$

where $q_3 = q_1 q_2^{\frac{1}{s}}\left(\frac{c_1}{\sqrt{c_2}}\right)^{\frac{-1}{s}}$. Also using (15) and (9) for Λ_0 and Λ_1 give

$$\#\Lambda_1\left\|U-U_{\Lambda_1}\right\|^{\frac{1}{s}} \le \#\Lambda_0 + q_1\left\|R_{\Lambda_0}\right\|_{\ell_\tau^\omega(\Lambda)}^{\frac{1}{s}}\left\|M(U-U_{\Lambda_0})\right\|_{\ell_\tau(\Lambda)}^{\frac{-1}{s}}$$

$$\leq q_1 \|MU\|_{\ell_\tau(\Lambda)}^{\frac{-1}{s}} \|F\|_{\ell_\tau^\omega(\Lambda)}^{\frac{1}{s}},$$

this inequality with Proposition 3.1 induce

$$\#\Lambda_1 \|U - U_{\Lambda_1}\|^{\frac{1}{s}} \prec \|F\|_{\ell_\tau^\omega(\Lambda)}^{\frac{1}{s}},$$

hence by (19) we obtain

$$\#\Lambda_j \|U - U_{\Lambda_j}\|^{\frac{1}{s}} \leq \#\Lambda_1 \|U - U_{\Lambda_1}\|^{\frac{1}{s}} + kq_3$$

$$\prec \|U\|_{\ell_\tau^\omega(\Lambda)}^{\frac{1}{s}} + q_3,$$

where k is the constant in (18). Thus

$$\|U - U_{\Lambda_j}\| \prec (\#\Lambda_j)^{-s} (\|U\|_{\ell_\tau^\omega(\Lambda)}^{\frac{1}{s}} + q_3)^s.$$

That means

$$\|U - U_{\Lambda_j}\| \prec (\#\Lambda_j)^{-s}.$$

6. Conclusions

In this paper we have used frames instead of Riesz bases, in order to give an adaptive algorithm for the numerical solution of an operator equation. The scheme is based on approximated iterations of the Galerkin method. Using frames instead of Riesz bases does not spoil the optimal convergence of the Galerkin method. Convergence and computational complexity can be theoretically proved and numerically verified.

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