HYPERSTABILITY OF MULTI-MIXED ADDITIVE-QUADRATIC JENSEN TYPE MAPPINGS

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In this article, we introduce the multi-mixed additive-quadratic Jensen type mappings and then unify the system of functional equations defining a multi-mixed additive-quadratic mapping to obtain a single equation. We show that under what conditions these mappings can be multi-additive, multi-quadratic and multi-additive-quadratic. Applying a fixed point theorem, we study the generalized Hyers-Ulam stability hyperstability of multi-mixed additive-quadratic Jensen type mappings.

Keywords: Hyers-Ulam stability; Multi-additive mapping; Multi-quadratic mapping; Multi-Mixed additive-quadratic mapping; Fixed point method.

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1. Introduction

In 1940, Ulam [35] raised the following question concerning stability of group homomorphisms: *under what condition does there exist an additive mapping near an approximately additive mapping*? Hyers [22] answered the problem of Ulam for Banach spaces. Later, the result of Hyers was significantly generalized by Aoki [1], Th. M. Rassias [33] (stability incorporated with sum of powers of norms), P. Găvruţa [21] (stability controlled by a general control function) and J. M. Rassias [32] (stability including mixed product-sum of powers of norms).

Throughout this paper, $\mathbb{N}$ and $\mathbb{Q}$ are the set of all positive integers and rationals, respectively, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}, \mathbb{R}_+ := [0, \infty)$. For any $l \in \mathbb{N}_0, n \in \mathbb{N}$, $t = (t_1, \ldots, t_n) \in \{-1, 1\}^n$ and $x = (x_1, \ldots, x_n) \in V^n$ we write $lx := (lx_1, \ldots, lx_n)$ and $tx := (t_1x_1, \ldots, t_nx_n)$, where lx stands, as usual, for the lth power of an element x of the commutative group V.

Let V be a commutative group, W be a linear space, and $n \geq 2$ be an integer. Recall from [19] that a mapping $f : V^n \longrightarrow W$ is called *multi-additive* if it is additive (satisfies Cauchy's functional equation $A(x+y) = A(x) + A(y)$) in each variable. Some facts on such mappings can be found in [23] and many other sources. In addition, f is said to be *multi-quadratic* if it is quadratic (satisfies quadratic functional equation $Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)$) in each variable [16]. Zhao et al. [39] showed that the mentioned mapping f is multi-quadratic if and only if the equation

$$\sum_{t\in\{-1,1\}^n} f(x_1 + tx_2) = 2^n \sum_{j_1,j_2,\ldots,j_n\in\{1,2\}} f(x_{1j_1}, x_{2j_2}, \ldots, x_{nj_n}) \tag{1}$$

holds, where $x_j = (x_{1j}, x_{2j}, \ldots, x_{nj}) \in V^n$ with $j \in \{1, 2\}$.

Prager and Schwaiger [30] introduced the notion of multi-Jensen mapping $f : V^n \longrightarrow W$ (satisfies Jensen's functional equation $J\left(\frac{x+y}{2}\right) = \frac{J(x)+J(y)}{2}$ in each variable) with the connection with generalized polynomials and obtained their general form. The aim of this

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note was to study the stability of the multi-Jensen equation. After that, the stability of multi-Jensen mappings in various normed spaces have been investigated by a number of mathematicians (see [17], [18], [31], [36] and [37]).

Let V and W be linear spaces, $n \in \mathbb{N}$ and $k \in \{0, \ldots, n\}$. A mapping $f : V^n \longrightarrow W$ is called k-Jensen and $n-k$-quadratic (briefly, multi-Jensen-quadratic) if f is Jensen in each of some k variables and is quadratic in each of the other variables (satisfies the Jensen type of quadratic equation $2\mathfrak{Q}\left(\frac{x+y}{2}\right) + 2\mathfrak{Q}\left(\frac{x-y}{2}\right) = \mathfrak{Q}(x) + \mathfrak{Q}(y)$). These mappings are introduced in [34]. Moreover, it is shown in [34] that the mapping f is multi-Jensen-quadratic mapping if and only if it satisfies the equation

$$2^n \sum_{q\in\{-1,1\}^{n-k}} f\left(\frac{x_1^k + x_2^k}{2}, \frac{x_1^{n-k} + qx_2^{n-k}}{2}\right) = \sum_{l_1,\ldots,l_n\in\{1,2\}} f(x_{l_1 1}, x_{l_2 2}, \ldots, x_{l_n n})$$

for all $x_i^k = (x_{i1}, \ldots, x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{ik+1} \ldots, x_{in}) \in V^{n-k}$ where $i \in \{1, 2\}$. For a different form of a multi-Jensen-quadratic mapping and its stability, we refer to [2].

In [19] and [16], Ciepliński studied the generalized Hyers-Ulam stability of multi-additive and multi-quadratic mappings in Banach spaces, respectively (see also [39]). Since then, the stability of multi-Cauchy-Jensen mappings in non-Archimedean spaces, Banach spaces, and multi-additive-quadratic mappings are studied in [3], [4], [5] and [6]. Recently, the stability of multi-cubic and multi-quartic mappings in Banach spaces via a fixed point theorem are investigated in [12] and [11], respectively. For other forms of multi-cubic mappings and functional equations which are recently studied, we refer to [29].

In [38], Zamani et al. introduced the mixed additive-quadratic functional equation

$$f(x + 2y) + f(x - 2y) + 8f(y) = 2f(x) + 4f(2y). \tag{2}$$

They determined the general solution of the equation (2) and studied its Hyers-Ulam stability in non-Archimedean Banach modules over a unital Banach algebra; for the general form of (2) see [10]. Some results on the stability of mixed additive-quadratic mappings can be found in [9], [25] and [27]. We also mention that some results on the stability of mixed type mappings can be found in [7], [8], [20], [24] and [26].

Here, we recall from [28] the following mixed additive-quadratic Jensen type functional equation

$$2\mathfrak{I}\left(\frac{x+y}{2}\right) + \mathfrak{I}\left(\frac{x-y}{2}\right) + \mathfrak{I}\left(\frac{y-x}{2}\right) = \mathfrak{I}(x) + \mathfrak{I}(y). \tag{3}$$

Motivated by the equation (3), in this paper we define multi-mixed additive-quadratic Jensen type mappings and include a characterization of such mappings. In other words, we prove that every multi-mixed additive-quadratic mapping can be shown a single functional equation and vice versa. Furthermore, we investigate the generalized Hyers-Ulam stability and hyperstability for such mappings by using a fixed point method which is taken from [14].

2. **Characterization of multi-mixed additive-quadratic Jensen type mappings**

Let V and W be vector spaces over $\mathbb{Q}$, $n \in \mathbb{N}$. Suppose that $n \geq 2$ and $x_i^n = (x_{i1}, x_{i2}, \ldots, x_{in}) \in V^n$, where $i \in \{1, 2\}$. We shall denote x_i^n by x_i if there is no risk of ambiguity.

We say the mapping $f : V^n \longrightarrow W$ is *n-multi-mixed additive-quadratic Jensen type* or briefly *multi-mixed additive-quadratic* if f is mixed additive-quadratic Jensen type in each variable (see the equation (3)).

Let $p, q \in \{-1, 1\}^n$ where $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$. In the following equation and the rest of the paper, as a convention, we put $p_k = q_l = 1$ whenever $p_k q_l = 1$ for $k, l \in \{1, \dots, n\}$.

In this section, we reduce the system of n equations defining the multi-mixed additive-quadratic mapping to obtain the single functional equation as follows:

$$\sum_{p\in\{-1,1\}^n} \sum_{q\in\{-1,1\}^n} f\left(\frac{px_1 + qx_2}{2}\right) = \sum_{l_1,\dots,l_n\in\{1,2\}} f(x_{l_1 1}, x_{l_2 2}, \dots, x_{l_n n}). \tag{4}$$

Put $\mathbf{m} := \{1, \dots, m\}$, $m \in \mathbb{N}$. For a subset $T = \{j_1, \dots, j_i\}$ of $\mathbf{m}$ with $1 \le j_1 < \ldots < j_i \le m$ and $x = (x_1, \dots, x_m) \in V^m$,

$${}_Tx := (0, \dots, 0, x_{j_1}, 0, \dots, 0, x_{j_i}, 0, \dots, 0) \in V^m$$

denotes the vector which coincides with x in exactly those components, which are indexed by the elements of T and whose other components are set equal zero. Note that ${}_\phi x = 0$, ${}_\mathbf{m}x = x$. We use these notations in the proof of upcoming lemma.

We shall to show that if a mapping $f : V^n \longrightarrow W$ satisfies the equation (4), then it is multi-mixed additive-quadratic mapping and vice versa. In order to do this, we need the next lemma.

Lemma 2.1. *If the mapping $f : V^n \longrightarrow W$ satisfies equation* (4)*, then $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero.*

Proof. Putting $x_1 = x_2 = (0, \dots, 0)$ in (4), we get

$$4^n f(0, \dots, 0) = 2^n f(0, \dots, 0). \tag{5}$$

Thus, $f(0, \dots, 0) = 0$. Letting $x_{1k} = x_{2k} = 0$ for all $k \in \{1, \dots, n\}\backslash\{j\}$ and $x_{1j} = x_{2j}$ in (4), we obtain

$$2 \times 4^{n-1} f(0, \dots, 0, x_{1j}, 0, \dots, 0) = 2^n f(0, \dots, 0, x_{1j}, 0, \dots, 0), \tag{6}$$

and so $f(0, \dots, 0, x_{1j}, 0, \dots, 0) = 0$. The above process can be repeated to obtain $2 \times 4^{n-1} f({}_kx_1) = 2^n f({}_kx_1)$, where $1 \le k \le n-1$. Hence, $f({}_kx_1) = 0$. This shows that $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero. □

We say the mapping $f : V^n \longrightarrow W$ is odd in the jth variable if

$$f(z_1, \dots, z_{j-1}, -z_j, z_{j+1}, \dots, z_n) = -f(z_1, \dots, z_{j-1}, z_j, z_{j+1}, \dots, z_n).$$

Moreover, f is even in the jth variable if

$$f(z_1, \dots, z_{j-1}, -z_j, z_{j+1}, \dots, z_n) = f(z_1, \dots, z_{j-1}, z_j, z_{j+1}, \dots, z_n).$$

We now prove the main result of this section.

Theorem 2.1. *The mapping $f : V^n \longrightarrow W$ is multi-mixed additive-quadratic mapping if and only if it satisfies equation* (4)*. Furthermore,*

(i) *if f is odd in a variable, then it is additive Jensen type in the same variable;*
(ii) *if f is even in a variable, then it is quadratic Jensen type in the same variable.*

Proof. (Necessity) Assume that f is a multi-mixed additive-quadratic mapping. We prove f satisfies the equation (4) by induction on n. For $n = 1$, it is trivial that f satisfies the equation (3). Assume that (4) is valid for some positive integer $n > 1$. Then,

$$\begin{aligned}
&\sum_{p\in\{-1,1\}^{n+1}}\sum_{q\in\{-1,1\}^{n+1}} f\left(\frac{px_1^{n+1}+qx_2^{n+1}}{2}\right)\\
&=\sum_{p\in\{-1,1\}^{n}}\sum_{q\in\{-1,1\}^{n}} f\left(\frac{px_1^{n}+qx_2^{n}}{2},x_{1n+1}\right)+\sum_{p\in\{-1,1\}^{n}}\sum_{q\in\{-1,1\}^{n}} f\left(\frac{px_1^{n}+qx_2^{n}}{2},x_{2n+1}\right)\\
&=\sum_{l_1,\dots,l_n\in\{1,2\}} f(x_{l_11},x_{l_22},\dots,x_{l_nn},x_{1n+1})+\sum_{l_1,\dots,l_n\in\{1,2\}} f(x_{l_11},x_{l_22},\dots,x_{l_nn},x_{2n+1})\\
&=\sum_{l_1,\dots,l_{n+1}\in\{1,2\}} f(x_{l_11},x_{l_22},\dots,x_{l_nn},x_{l_{n+1}n+1}).
\end{aligned}$$

This means that (4) holds for $n+1$.

(Sufficiency) Let $j\in\{1,\dots,n\}$ be arbitrary and fixed. Putting $x_{1k}=x_{2k}$ for all $k\in\{1,\dots,n\}\backslash\{j\}$ and using Lemma 2.1, we obtain

$$\begin{aligned}
&2^n f\left(x_{11},\dots,x_{1j-1},\frac{x_{1j}+x_{2j}}{2},x_{1j+1},\dots,x_{1n}\right)\\
&+2^{n-1} f\left(x_{11},\dots,x_{1j-1},\frac{x_{1j}-x_{2j}}{2},x_{1j+1},\dots,x_{1n}\right)\\
&+2^{n-1} f\left(x_{11},\dots,x_{1j-1},\frac{x_{2j}-x_{1j}}{2},x_{1j+1},\dots,x_{1n}\right)\\
&=2^{n-1}[f(x_{11},x_{12},\dots,x_{1n})+f(x_{11},\dots,x_{1j-1},x_{2j},x_{1j+1},\dots,x_{1n})].
\end{aligned}$$

Thus

$$\begin{aligned}
&2f\left(x_{11},\dots,x_{1j-1},\frac{x_{1j}+x_{2j}}{2},x_{1j+1},\dots,x_{1n}\right)\\
&+f\left(x_{11},\dots,x_{1j-1},\frac{x_{1j}-x_{2j}}{2},x_{1j+1},\dots,x_{1n}\right)\\
&+f\left(x_{11},\dots,x_{1j-1},\frac{x_{2j}-x_{1j}}{2},x_{1j+1},\dots,x_{1n}\right)\\
&=f(x_{11},x_{12},\dots,x_{1n})+f(x_{11},\dots,x_{1j-1},x_{2j},x_{1j+1},\dots,x_{1n}). \qquad (7)
\end{aligned}$$

Therefore, f is mixed additive-quadratic Jensen type in the jth variable.

(i) Let f be odd in the jth variable. It follows from relation (7) that

$$\begin{aligned}
&2f\left(x_{11},\dots,x_{1j-1},\frac{x_{1j}+x_{2j}}{2},x_{1j+1},\dots,x_{1n}\right)\\
&=f(x_{11},x_{12},\dots,x_{1n})+f(x_{11},\dots,x_{1j-1},x_{2j},x_{1j+1},\dots,x_{1n}).
\end{aligned}$$

(ii) Similar to the part (i), it follows from the assumption and (7) that

$$\begin{aligned}
&2f\left(x_{11},\dots,x_{1j-1},\frac{x_{1j}+x_{2j}}{2},x_{1j+1},\dots,x_{1n}\right)\\
&+2f\left(x_{11},\dots,x_{1j-1},\frac{x_{1j}-x_{2j}}{2},x_{1j+1},\dots,x_{1n}\right)\\
&=f(x_{11},x_{12},\dots,x_{1n})+f(x_{11},\dots,x_{1j-1},x_{2j},x_{1j+1},\dots,x_{1n}).
\end{aligned}$$

This means that f is quadratic in the jth variable. □

Recall that a mapping $f:V^n\longrightarrow W$ is called k-Jensen and $n-k$-quadratic (briefly, multi-Jensen-quadratic) if f is Jensen in each of some k variables and is quadratic in each of the other variables. In this note, we suppose for simplicity that f is Jensen in each of the first k variables, but one can obtain analogous results without this assumption. We also note that if f is odd in each of some k variables, then the concepts of multi-Jensen-quadratic and multi-additive-quadratic (which was introduced in [5]) for the mapping f coincide. This leads us to the next corollary.

Corollary 2.1. *Suppose that the mapping $f : V^n \longrightarrow W$ satisfies equation* (4)*.*

(i) *If f is odd in each variable, then it is multi-additive Jensen type;*

(ii) *If f is even in each variable, then it is multi-quadratic Jensen type;*

(iii) *If f is odd in each of some k variables and is even in each of the other variables, then it is multi-additive-quadratic.*

3. Stability Results for multi-mixed additive-quadratic mappings

In this section, we prove the generalized Hyers-Ulam stability of equation (4) by a fixed point result (Theorem 3.1) in Banach spaces. Throughout, for two sets A and B, the set of all mappings from A to B is denoted by B^A. Here, we introduce the upcoming three hypotheses:

(A1) Y is a Banach space, $\mathcal{S}$ is a nonempty set, $j \in \mathbb{N}$, $g_1, \ldots, g_j : \mathcal{S} \longrightarrow \mathcal{S}$ and $L_1, \ldots, L_j : \mathcal{S} \longrightarrow \mathbb{R}_+$,

(A2) $\mathcal{T} : Y^{\mathcal{S}} \longrightarrow Y^{\mathcal{S}}$ is an operator satisfying the inequality

$$\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| \leq \sum_{i=1}^{j} L_i(x) \|\lambda(g_i(x)) - \mu(g_i(x))\|, \quad \lambda, \mu \in Y^{\mathcal{S}}, x \in \mathcal{S},$$

(A3) $\Lambda : \mathbb{R}_+^{\mathcal{S}} \longrightarrow \mathbb{R}_+^{\mathcal{S}}$ is an operator defined through

$$\Lambda\delta(x) := \sum_{i=1}^{j} L_i(x)\delta(g_i(x)) \qquad \delta \in \mathbb{R}_+^{\mathcal{S}}, x \in \mathcal{S}.$$

Here, we highlight the following theorem which is a fundamental result in fixed point theory [14, Theorem 1]. This result plays a key tool to obtain our objecive in this section. We also remember that this fixed point method for the stability of functional equations was introduced and used for the first time by Brzdęk in [13].

Theorem 3.1. *Let hypotheses* (A1)-(A3) *hold and the function $\theta : \mathcal{S} \longrightarrow \mathbb{R}_+$ and the mapping $\phi : \mathcal{S} \longrightarrow Y$ fulfill the following two conditions:*

$$\|\mathcal{T}\phi(x) - \phi(x)\| \leq \theta(x), \quad \theta^*(x) := \sum_{l=0}^{\infty} \Lambda^l \theta(x) < \infty \qquad (x \in \mathcal{S}).$$

Then, there exists a unique fixed point ψ of $\mathcal{T}$ such that

$$\|\phi(x) - \psi(x)\| \leq \theta^*(x) \qquad (x \in \mathcal{S}).$$

Moreover, $\psi(x) = \lim_{l\to\infty} \mathcal{T}^l\phi(x)$ for all $x \in \mathcal{S}$.

From now on, for the mapping $f : V^n \longrightarrow W$, we consider the difference operator $\mathfrak{D}f : V^n \times V^n \longrightarrow W$ by

$$\mathfrak{D}f(x_1, x_2) = \sum_{p\in\{-1,1\}^n} \sum_{q\in\{-1,1\}^n} f\left(\frac{px_1 + qx_2}{2}\right) = \sum_{l_1,\ldots,l_n\in\{1,2\}} f(x_{l_1 1}, x_{l_2 2}, \ldots, x_{l_n n}).$$

We bring the oncoming lemma from [5] will be useful in the proof of our stability result. For simplicity, given an $m \in \mathbb{N}$, we write $S := \{0, 1\}^m$, and S_i stands for the set of all elements of S having exactly i zeros, i.e.,

$$S_i := \{(s_1, \ldots, s_m) \in S : \text{card}\{j : s_j = 0\} = i\}, \qquad i \in \{0, \ldots, m\}.$$

Lemma 3.1. *Let $m \in \mathbb{N}$, $l \in \mathbb{N}_0$ and $\psi : S \longrightarrow \mathbb{R}$. Then*

$$\sum_{v=0}^{m}\sum_{w=0}^{m}\sum_{s\in S_w}\sum_{t\in S_v}(2^l-1)^w\psi(st)=\sum_{i=0}^{m}\sum_{p\in S_i}(2^{l+1}-1)^i\psi(p)$$

In the sequel, S stands for $\{0,1\}^n$ and $S_i \subseteq S$ for $i \in \{0,\dots,n\}$. We have the following stability theorem for the functional equation (4) for the odd case.

Theorem 3.2. *Let V be a linear space and W be a Banach space. Suppose that $\phi : V^n \times V^n \longrightarrow \mathbb{R}_+$ is a mapping satisfying*

$$\lim_{l\to\infty}\left(\frac{1}{2^n}\right)^l\sum_{i=0}^{n}\sum_{p\in S_i}(2^l-1)^i\phi(2^lpx_1,2^lpx_2)=0 \tag{8}$$

for all $x_1, x_2 \in V^n$ and

$$\Phi(x)=:\sum_{l=0}^{\infty}\left(\frac{1}{2^n}\right)^{l+1}\sum_{i=0}^{n}\sum_{p\in S_i}(2^l-1)^i\phi(2^{l+1}px,0)<\infty \tag{9}$$

for all $x \in V^n$. Assume also $f : V^n \longrightarrow W$ is a mapping fulfilling the inequality

$$\|\mathfrak{D}f(x_1,x_2)\|\leqslant\phi(x_1,x_2) \tag{10}$$

for all $x_1, x_2 \in V^n$. If f is odd in each variable, then there exists a unique multi-additive Jensen type mapping $\mathfrak{J} : V^n \longrightarrow W$ such that

$$\|f(x)-\mathfrak{J}(x)\|\leq\Phi(x) \tag{11}$$

for all $x \in V^n$.

Proof. Replacing (x_1, x_2) by $(2x_1, 0)$ in (10) and using the oddness of f in each variable, we have

$$\left\|2^nf(x)-\sum_{s\in S}f(2sx)\right\|\leq\phi(2x,0)$$

for all $x = x_1 \in V^n$ and so

$$\left\|f(x)-\frac{1}{2^n}\sum_{s\in S}f(2sx)\right\|\leq\frac{1}{2^n}\phi(2x,0) \tag{12}$$

for all $x \in V^n$. Take $x \in V^n$ and let $\theta(x) := \frac{1}{2^n}\phi(2x,0)$, $\mathcal{T}\theta(x) := \frac{1}{2^n}\sum_{s\in S}\theta(2sx)$. Inequality (12) can be modified as

$$\|f(x)-\mathcal{T}f(x)\|\leq\theta(x) \tag{13}$$

for all $x \in V^n$. Define $\Lambda\eta(x) := \frac{1}{2^n}\sum_{s\in S}\eta(2sx)$ for all $\eta \in \mathbb{R}_+^{V^n}$, $x \in V^n$. We now see that Λ has the form described in (A3). Moreover, for each $\lambda, \mu \in W^{V^n}$ and $x \in V^n$, we get

$$\|\mathcal{T}\lambda(x)-\mathcal{T}\mu(x)\|=\left\|\frac{1}{2^n}\left[\sum_{s\in S}(\eta(2sx)-\mu(2sx)\right]\right\|\leq\frac{1}{2^n}\sum_{s\in S}\|\eta(2sx)-\mu(2sx)\|.$$

The above relation shows that the hypothesis (A2) holds. By induction on l, one can check for any $l \in \mathbb{N}_0$ and $x \in V^n$ that

$$\Lambda^l\theta(x):=\left(\frac{1}{2^n}\right)^l\sum_{i=0}^{n}(2^l-1)^i\sum_{p\in S_i}\theta(2^lpx). \tag{14}$$

Fix an $x \in V^n$. Here, we adopt the convention that $0^0 = 1$. Hence, the relation (14) is trivially true for $l = 0$. Next, assume that (14) holds for a $l \in \mathbb{N}_0$. Using Lemma 3.1 for $m = n$ and $\psi(s) := \theta(2^{l+1}sx)$ $(s \in S)$, we get

$$\begin{aligned}
\Lambda^{l+1}\theta(x) = \Lambda(\Lambda^l\theta)(x) &= \frac{1}{2^n}\sum_{v=0}^{n}\sum_{t\in S_v}(\Lambda^l\theta)(2tx)\\
&= \left(\frac{1}{2^n}\right)^{l+1}\sum_{v=0}^{n}\sum_{t\in S_v}\sum_{w=0}^{n}(2^l-1)^w\sum_{s\in S_w}\theta(2^{l+1}stx)\\
&= \left(\frac{1}{2^n}\right)^{l+1}\sum_{v=0}^{n}\sum_{w=0}^{n}\sum_{s\in S_w}\sum_{t\in S_v}(2^l-1)^w\theta(2^{l+1}stx)\\
&= \left(\frac{1}{2^n}\right)^{l+1}\sum_{i=0}^{n}\sum_{p\in S_i}(2^{l+1}-1)^i\theta(2^{l+1}px).
\end{aligned}$$

Therefore, (14) holds for any $l \in \mathbb{N}_0$ and $x \in V^n$. Now, relations (9) and (14) necessitate that all assumptions of Theorem 3.1 are satisfied. Hence, there exists a mapping $\mathfrak{J} : V^n \longrightarrow W$ such that

$$\mathfrak{J}(x) = \lim_{l\to\infty}(\mathcal{T}^l f)(x) = \frac{1}{2^n}\sum_{s\in S}\mathfrak{J}(2sx) \qquad (x \in V^n),$$

and also (11) holds. We shall to show that

$$\|\mathfrak{D}(\mathcal{T}^l f)(x_1, x_2)\| \le \left(\frac{1}{2^n}\right)^l \sum_{i=0}^{n}\sum_{p\in S_i}(2^l-1)^i\phi(2^l px_1, 2^l px_2) \tag{15}$$

for all $x_1, x_2 \in V^n$ and $l \in \mathbb{N}_0$. We argue by induction on l. The inequality (15) is valid for $l = 0$ by (10). Assume that (15) is true for an $l \in \mathbb{N}_0$. For each $x_1, x_2 \in V^n$, we have

$$\begin{aligned}
&\|\mathfrak{D}(\mathcal{T}^{l+1} f)(x_1, x_2)\|\\
&= \frac{1}{2^n}\left\|\sum_{s\in S}\mathfrak{D}(\mathcal{T}^l f)(2sx_1, 2sx_2)\right\|\\
&\le \left(\frac{1}{2^n}\right)^{l+1}\sum_{s\in S}\sum_{i=0}^{n}\sum_{t\in S_i}(2^l-1)^i\phi(2^{l+1}stx_1, 2^{l+1}stx_2)\\
&= \left(\frac{1}{2^n}\right)^{l+1}\sum_{i=0}^{n}\sum_{p\in S_i}(2^{l+1}-1)^i\phi(2^{l+1}px_1, 2^{l+1}px_2)
\end{aligned}$$

for all $x_1, x_2 \in V^n$. We note that the last equality follows from Lemma 3.1 with $m := n$ and $\psi(s) := \phi(2^{l+1}sx_1, 2^{l+1}sx_2)$ $(s \in S)$. Letting $l \to \infty$ in (15) and applying (8), we arrive at $\mathfrak{D}\mathfrak{J}(x_1, x_2) = 0$ for all $x_1, x_2 \in V^n$. This means that the mapping $\mathfrak{J}$ satisfies (4). Now, the part (i) of Corollary 2.1 implies that $\mathfrak{J}$ is a multi-additive Jensen type mapping. Lastly, assume that $\mathfrak{I} : V^n \longrightarrow W$ is another multi-Jensen mapping satisfying the equation (4) and inequality (11), and fix $x \in V^n$, $j \in \mathbb{N}$. By Lemma 2.1 and (9), we have

$$\begin{aligned}
&\|\mathfrak{J}(x) - \mathfrak{I}(x)\| \\
&= \left\| \left(\frac{1}{2^n}\right)^j \mathfrak{J}(2^j x) - \left(\frac{1}{2^n}\right)^j \mathfrak{I}(2^j x) \right\| \\
&\leq \left(\frac{1}{2^n}\right)^j (\|\mathfrak{J}(2^j x) - f(2^j x)\| + \|\mathfrak{I}(2^j x) - f(2^j x)\|) \\
&\leq 2 \left(\frac{1}{2^n}\right)^j \Phi(2^j x) \\
&\leq 2 \left(\frac{1}{2^n}\right)^j \sum_{l=j}^{n} \left(\frac{1}{2^n}\right)^{l+1} \sum_{i=0}^{n} \sum_{p \in S_i} (2^l - 1)^i \phi(2^{l+1} px, 0).
\end{aligned}$$

Consequently, letting $j \to \infty$ and using the fact that series (9) is convergent for all $x \in V^n$, we obtain $\mathfrak{J}(x) = \mathfrak{I}(x)$ for all $x \in V^n$. This completes the proof. □

We have the next result which is analogous to Theorem 3.2 for the functional equation (4) in the even case.

Theorem 3.3. *Let V be a linear space and W be a Banach space. Suppose that $\phi : V^n \times V^n \longrightarrow \mathbb{R}_+$ is a mapping satisfying*

$$\lim_{l \to \infty} \left(\frac{1}{4^n}\right)^l \sum_{i=0}^{n} \sum_{p \in S_i} (2^l - 1)^i \phi(2^l px_1, 2^l px_2) = 0$$

for all $x_1, x_2 \in V^n$ and

$$\Psi(x) =: \sum_{l=0}^{\infty} \left(\frac{1}{4^n}\right)^l \sum_{i=0}^{n} \sum_{p \in S_i} (2^l - 1)^i \phi(2^{l+1}(px, 0)) < \infty$$

for all $x \in V^n$. Assume also $f : V^n \longrightarrow W$ is a mapping fulfilling the inequality

$$\|\mathfrak{D} f(x_1, x_2)\| \leqslant \phi(x_1, x_2) \tag{16}$$

for all $x_1, x_2 \in V^n$. If f is even in each variable, then there exists a unique multi-quadratic mapping $\mathcal{Q} : V^n \longrightarrow W$ such that

$$\|f(x) - \mathcal{Q}(x)\| \leq \Psi(x)$$

for all $x \in V^n$.

Proof. Replacing (x_1, x_2) by $(2x_1, 0)$ in (16) and using the evenness of f in each variable, we get

$$\left\| f(x) - \frac{1}{4^n} \sum_{s \in S} f(2sx) \right\| \leq \frac{1}{4^n} \phi(2x, 0) \tag{17}$$

for all $x \in V^n$. Take $x \in V^n$ and let $\theta(x) := \frac{1}{4^n}\phi(2x, 0)$, $\mathcal{T}\theta(x) := \frac{1}{4^n} \sum_{s \in S} \theta(2sx)$. The relation (17) can be rewritten as

$$\|f(x) - \mathcal{T} f(x)\| \leq \theta(x)$$

for all $x \in V^n$. The rest of the proof is similar to the proof of Theorem 3.2 and so we omit it. □

The following corollary is a direct consequence of Theorem 3.2 concerning the stability of (4) for the even case.

Corollary 3.1. *Let $\delta > 0$. Let also V be a normed space and W be a Banach space. Suppose that $f : V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\mathfrak{D}f(x_1, x_2)\| \leq \delta$$

for all $x_1, x_2 \in V^n$. If f is even in each variable, then there exists a unique multi-quadratic mapping $\mathcal{Q} : V^n \longrightarrow W$ such that

$$\|f(x) - \mathcal{Q}(x)\| \leq \frac{\delta}{2^n(2^n - 1)}$$

for all $x \in V^n$.

Proof. Setting the constant function $\phi(x_1, x_2) = \delta$ for all $x_1, x_2 \in V^n$, and applying Theorem 3.3, we have

$$\begin{aligned}
\Phi(x) &= \sum_{l=0}^{n} \left(\frac{1}{4^n}\right)^{l+1} \sum_{i=0}^{n} \sum_{p \in S_i} (2^l - 1)^i \phi(2^{l+1}px, 0) \\
&= \delta \sum_{l=0}^{\infty} \left(\frac{1}{4^n}\right)^{l+1} \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} (2^l - 1)^i \times 1^{n-i} \\
&= \frac{\delta}{4^n} \sum_{l=0}^{\infty} \left(\frac{1}{4^n}\right)^{l} 2^{nl} = \frac{\delta}{4^n} \sum_{l=0}^{\infty} \left(\frac{1}{2^n}\right)^{l} = \frac{\delta}{2^n(2^n - 1)}.
\end{aligned}$$

□

Theorem 3.4. *Let V be a linear space and W be a Banach space. Suppose that $\phi : V^n \times V^n \longrightarrow \mathbb{R}_+$ is a mapping satisfying*

$$\lim_{l \to \infty} \left(\frac{1}{2^n}\right)^{l} \sum_{i=0}^{n} \sum_{p \in S_i} (2^l - 1)^i \phi(2^l px_1, 2^l px_2) = 0$$

for all $x_1, x_2 \in V^n$ and

$$\Gamma(x) =: \sum_{l=0}^{\infty} \left(\frac{1}{2^{2n-k}}\right)^{l} \sum_{i=0}^{n} \sum_{p \in S_i} (2^l - 1)^i \phi(2^{l+1}px, 0) < \infty$$

for all $x \in V^n$. Assume also $f : V^n \longrightarrow W$ is a mapping fulfilling the inequality

$$\|\mathfrak{D}f(x_1, x_2)\| \leqslant \phi(x_1, x_2) \tag{18}$$

for all $x_1, x_2 \in V^n$. If f is odd in each of some k variables and is even in each of the other variables, then there exists a unique multi-additive-quadratic mapping $\mathcal{F} : V^n \longrightarrow W$ such that

$$\|f(x) - \mathcal{F}(x)\| \leq \Gamma(x)$$

for all $x \in V^n$.

Proof. Similar to the proof of preceding theorems, interchanging (x_1, x_2) by $(2x_1, 0)$ in (18) and using the assumptions, we find

$$\left\| 2^{2n-k} f(x) - \sum_{s \in S} f(2sx) \right\| \leq \phi(2x, 0)$$

for all $x \in V^n$ and so

$$\left\| f(x) - \frac{1}{2^{2n-k}} \sum_{s \in S} f(2sx) \right\| \leq \frac{1}{2^{2n-k}} \phi(2x, 0) \tag{19}$$

for all $x \in V^n$. Choose $x \in V^n$ and let $\theta(x) := \frac{1}{2^{2n-k}}\phi(2x,0)$, $\mathcal{T}\theta(x) := \frac{1}{2^{2n-k}}\sum_{s\in S}\theta(2sx)$. The relation (19) converts to

$$\|f(x) - \mathcal{T}f(x)\| \leq \theta(x)$$

for all $x \in V^n$. The rest of the proof is similar to the proof of Theorem 3.2. □

Let A be a nonempty set, (X, d) a metric space, $\psi \in \mathbb{R}_+^{A^n}$, and $\mathcal{F}_1, \mathcal{F}_2$ operators mapping a nonempty set $D \subset X^A$ into X^{A^n}. We say that operator equation

$$\mathcal{F}_1\varphi(a_1, \ldots, a_n) = \mathcal{F}_2\varphi(a_1, \ldots, a_n) \tag{20}$$

is ψ-hyperstable provided every $\varphi_0 \in D$ satisfying inequality

$$d(\mathcal{F}_1\varphi_0(a_1, \ldots, a_n), \mathcal{F}_2\varphi_0(a_1, \ldots, a_n)) \leq \psi(a_1, \ldots, a_n), \qquad a_1, \ldots, a_n \in A,$$

fulfils (20); this definition is introduced in [15]. In other words, a functional equation $\mathcal{F}$ is *hyperstable* if any mapping f satisfying the equation $\mathcal{F}$ approximately is a true solution of $\mathcal{F}$.

Corollary 3.2. *Suppose that $\alpha_{ij} > 0$ for $i \in \{1, 2\}$ and $j \in \{1, \ldots, n\}$ fulfill $\sum_{i=1}^{2}\sum_{j=1}^{n}\alpha_{ij} < n$. Let $f : V^n \longrightarrow W$ be a mapping fulfilling the inequality*

$$\|\mathfrak{D}f(x_1, x_2)\| \leq \prod_{i=1}^{2}\prod_{j=1}^{n}\|x_{ij}\|^{\alpha_{ij}}$$

for all $x_1, x_2 \in V^n$, where V is a normed space and W is a Banach space.

(i) *If f is odd in each variable, then it is multi-additive Jensen type;*
(ii) *If f is even in each variable, then it is multi-quadratic Jensen type;*
(iii) *If f is odd in each of some k variables and is even in each of the other variables, then it is multi-additive-quadratic.*

Proof. The results follow from Theorems 3.2, 3.3 and 3.4 by taking

$$\phi(x_1, x_2) = \prod_{i=1}^{2}\prod_{j=1}^{n}\|x_{ij}\|^{\alpha_{ij}}.$$

□

4. Conclusions

In this paper, the authors introduced the multi-mixed additive-quadratic Jensen type mappings. It is shown that such mappings can be described by an equation. Using a fixed point theorem, it is proved that the multi-mixed additive-quadratic mappings can be stable and hyperstable.

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