TRANSMUTED UNIT RAYLEIGH QUANTILE REGRESSION MODEL: ALTERNATIVE TO BETA AND KUMARASWAMY QUANTILE REGRESSION MODELS

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In this paper, a new alternative unit distribution is presented. It consists of applying the quadratic transmutation scheme with the unit Rayleigh distribution. The quantile regression model of the proposed distribution is developed, as well as the maximum likelihood estimation of the unknown regression model parameters. We consider a real data application that links a measure of the educational attainment of OECD (Organization for Economic Co-operation and Development) countries with some of their Better Life Index such as life satisfaction, homicide rate, and voter turnout. It is shown that the proposed quantile regression model provides a better fit than well-known regression models in the literature when the unit response variable has skewed observations and outliers.

Keywords: unit Rayleigh distribution, quantile regression, residuals, transmuted unit distribution, educational attainment, OECD data sets.

MSC2010: AMS Classification codes: 60E05, 62E10, 62N05.

1. Introduction

The beta distribution is the first probability distribution that comes to mind for the modeling of the percentage and proportions. In certain situations, however, it is not precise enough to capture all of the information conveyed by the data. For this reason, using an appropriate transformation of the random variables (rvs), some alternative unit distributions have been proposed in the literature. The most popular are the Kumaraswamy (Kw) (see [15]), log-Lindley (see [10]), unit-Weibull (see [18]), unit generalized half normal (see [14]), log-weighted exponential (see [1]) and log-extended exponential geometric (LEEG) (see [11]) distributions.

On the other hand, when the response variable is observed on the unit interval, the beta regression model by [7] is generally applied to explain the

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mean response as a function of some values of the covariates. If the response variable has outliers, quantile regression modeling is preferable to classical regression modeling because the mean is sensitive to outliers and is delicately affected by biased data (see [13]). In these cases, the mean is not an appropriate choice for the measure of central tendency. Using a median-based method is more illustrative and robust than the mean-based method. The fundamentals of the unit quantile response regression modeling can be found in [4, 19, 11].

The aim of the paper is to provide a new alternative unit distribution with its quantile regression modeling for the percentages and proportions. The proposed distribution is based on the unit Rayleigh distribution by [3] combined with the quadratic transmutation scheme by [22]. Among its advantages, its cumulative distribution function (cdf) and probability density function (pdf) can be easily re-parameterized in terms of its quantile function (qf). It is also showing its modeling capacity with an application of interest, which relates educational attainment measurements of the OECD countries with some Better Life Index. On this topic, Reference [1] has related the educational attainment values of the OECD countries with Labor market insecurity and homicide rate variables via the mean response regression based on the log-weighted exponential distribution. It concluded that the Labor market insecurity and homicide rate affect educational attainment negatively. In light of the above, we will investigate whether the variables of life satisfaction, homicide rate, and voter turnout belonging to the OECD countries affect educational attainment via the median response regression based on the newly defined distribution.

The paper contains the following sections. Section 2 defines the new distribution. The related quantile regression model, its parameters estimation and its residual analysis are developed in Section 3. An actual data application is illustrated in Section 4. Finally, the paper ends with the conclusion in Section 5.

2. Transmuted unit Rayleigh distribution

Here, we present the transmuted unit Rayleigh distribution defined by the quadratic transmutation of the unit Rayleigh distribution by [3]. That is, its cdf and pdf are given as

$$F(x, \alpha, \beta) = e^{-\alpha(-\log x)^2}(1 + \beta - \beta e^{-\alpha(-\log x)^2})$$

and

$$f(x, \alpha, \beta) = \alpha \frac{(-\log x)}{x} e^{-\alpha(-\log x)^2}(1 + \beta - 2\beta e^{-\alpha(-\log x)^2}), \quad x \in (0, 1),$$

respectively, with the standard modifications when $x \notin (0, 1)$, where $\alpha > 0$ and $\beta \in [-1, 1]$. We denote this distribution as TUR or $TUR(\alpha, \beta)$. From the probabilistic point of view, the TUR distribution corresponds to the one of the rv X defined by $X = \inf(X_1, X_2)$ with probability $(1+\beta)/2$ and $X = \sup(X_1, X_2)$

with probability $(1-\beta)/2$, where X_1 and X_2 denote two independent rvs following the unit Rayleigh distribution. Thus, the parameter β operates as a flexible compromise between the two extreme order statistics of (X_1, X_2). In some senses, it amplifies the modeling capabilities of the unit Rayleigh distribution with moderate parametric complexity. This aspect will be illustrated later with a re-parameterized version of the TUR distribution (see Figure 1). In full generality, the benefits of using the quadratic transmutation scheme to extend existing distributions can be found in the survey of [21], and the references therein. Another interest of the TUR distribution is to have a manageable qf. After some algebraic operations, it is obtained as

$$Q(u,\alpha,\beta) = \exp\left[-\alpha^{-1/2}\sqrt{-\log\left(\frac{1+\beta-\sqrt{(1+\beta)^2-4\beta u}}{2\beta}\right)}\right], \quad u \in (0,1). \tag{1}$$

This expression will be at the center of our quantile regression analysis.

3. An alternative quantile regression model

3.1. Motivation

It is well-known that the response variable is explained as the mean response modeling via certain values of the covariates according to classical regression modeling. The response variable can be defined in the unit interval. When this is valid, there is no doubt that the beta regression model of [7] can be used first for the modeling of the unit mean response. We note that some alternative unit regression models to beta regression modeling have been proposed by some researchers. See, for instance, [12, 5, 10, 16, 1, 2]. In contrast, the classical regression modeling approach is easily affected by outliers or response variables with skewed distribution. Appropriate inferences can not be obtained because the mean is disturbed by these situations precisely. Thus, robust methods are needed to relate the response variable to the covariates. Quantile regression modeling is the right alternative approach to classical regression modeling for the above issues (see [13]). In this way, the response variable is explained as the quantile response. In particular, median response modeling is preferred to classical regression modeling for obtaining robust estimations. One can refer to [20, 4, 19, 11, 8] concerning the unit median response regression modeling.

3.2. Proposed quantile regression model

This subsection aims to present an alternative quantile regression modeling based on the TUR distribution. We thus use the qf specified in Equation (1). In order to lighten the notations, let $\Upsilon(u,\beta) = [1+\beta-\sqrt{(1+\beta)^2-4\beta u}]/(2\beta)$.

Then, $\mu = Q(u, \alpha, \beta)$ entails that $\alpha = -\log[\Upsilon(u, \beta)]/(\log\mu)^2$. The re-parameterized cdf and pdf of the quantile TUR distribution are given by

$$G(y, \beta, \mu) = \Upsilon(u, \beta)^{\left(\frac{\log y}{\log \mu}\right)^2}\left(1 + \beta - \beta\Upsilon(u, \beta)^{\left(\frac{\log y}{\log \mu}\right)^2}\right) \tag{2}$$

and

$$g(y, \beta, \mu) = \frac{\log[\Upsilon(u, \beta)]}{(\log\mu)^2}\frac{\log y}{y}\Upsilon(u, \beta)^{\left(\frac{\log y}{\log \mu}\right)^2}\left(1 + \beta - 2\beta\Upsilon(u, \beta)^{\left(\frac{\log y}{\log \mu}\right)^2}\right), \tag{3}$$

respectively, where $y \in (0, 1)$, $\beta \in [-1, 1]$ is the shape parameter, $\mu \in (0, 1)$ is the quantile parameter, and $u \in (0, 1)$ is known. We denote this re-parameterized distribution as $TURQ$ or $TURQ(\beta, \mu, u)$. For instance, when $u = 0.5$ is taken, Equation (3) is reduced to the pdf of the median $TURQ$ distribution. Some possible shapes of the $TURQ$ distribution are shown in Figure 1. We see that these shapes can decrease and be unimodal with various skewed forms.

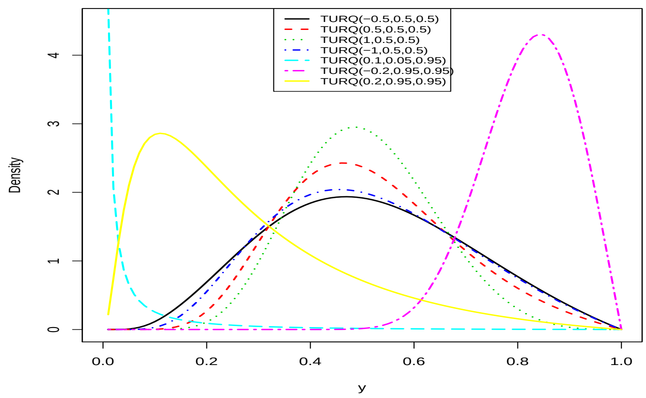

FIGURE 1. Some examples of the pdf shapes of the $TURQ$ distribution.

We are now able to describe our new quantile regression model. First, let $Y_1, Y_2, \ldots, Y_n$ be n rvs such that $Y_i \sim TURQ(\beta, \mu_i, u)$ for $i = 1, 2, \ldots, n$, where μ_i and β are unknown parameters and u is known. Consider observations of these rvs denoted by $y_1, y_2, \ldots, y_n$. Then, the $TURQ$ quantile regression model is defined as $g(\mu_i) = \mathbf{x}_i\boldsymbol{\theta}^T$, where $\boldsymbol{\theta} = (\theta_0, \theta_1, \ldots, \theta_p)^T$ and $\mathbf{x}_i = (1, x_{i1}, x_{i2}, \ldots, x_{ip})$ are the unknown regression parameter vector and known i^{th} vector of the covariates, respectively. By definition, the link function $g(x)$ aims to connect the covariates with the conditional quantile of the response variable. Its choice depends on the support of the distribution. We use the logit link function which is given by $g(x) = \text{logit}(x) = \log[x/(1-x)]$, for $i = 1, 2 \ldots n$, to connect covariates with the response variable, since the $TURQ$ distribution is defined on $(0, 1)$. If $u = 0.5$ is taken, the covariates are connected with the conditional median of the response variable.

3.3. Estimation of the model parameters

In this subsection, we estimate the unknown parameters via the maximum likelihood estimation (MLE) method for the $TURQ$ quantile regression modeling. In this regard, based on the link function $g(x)$, let $Y_1, Y_2, \ldots, Y_n$ be n independent rvs such that $Y_i \sim TURQ\,(\beta, \mu_i, u)$ for $i = 1, 2 \ldots, n$, where

$$\mu_i = \frac{e^{\mathbf{x}_i \boldsymbol{\theta}^T}}{1 + e^{\mathbf{x}_i \boldsymbol{\theta}^T}}. \tag{4}$$

For any known u, let $\boldsymbol{\Psi} = \left(\beta, \boldsymbol{\theta}^T\right)^T$ be the unknown parameter vector. Then, putting Equation (4) into Equation (3), the log-likelihood function of the $TURQ$ model is given by

$$\begin{aligned}\ell\left(\boldsymbol{\Psi}\right) = & n \log\left[-\log\left[\Upsilon(u,\beta)\right]\right] + \sum_{i=1}^{n} \log\left[\frac{-\log y_i}{y_i}\right] - 2\sum_{i=1}^{n} \log\left(\log \mu_i\right) \\ & + \sum_{i=1}^{n} \log\left[1 + \beta - 2\beta\Upsilon(u,\beta)^{\left(\frac{\log y_i}{\log \mu_i}\right)^2}\right] + \log\left(\Upsilon(u,\beta)\right)\sum_{i=1}^{n}\left(\frac{\log y_i}{\log \mu_i}\right)^2.\end{aligned} \tag{5}$$

The normal equations, which are requested by the author when they are needed, are obtained by routine procedure of the MLE method. Since they contain nonlinear functions according to model parameters, the log-likelihood function in Equation (5) can be maximized directly by software such as R, S-Plus and Matlab. Let us denote by $\hat{\boldsymbol{\Psi}} = \left(\hat{\beta}, \hat{\boldsymbol{\theta}}^T\right)^T$ the theoretical MLEs of $\boldsymbol{\Psi}$. Then, the asymptotic distribution behind $(\hat{\boldsymbol{\Psi}} - \boldsymbol{\Psi})$ is multivariate normal $\mathcal{N}_{p+1}\left(\mathbf{0}, I^{-1}\left(\boldsymbol{\Psi}\right)\right)$, where $I\left(\boldsymbol{\Psi}\right)$ is the expected information matrix. Practically, the observed information matrix is used for the estimation of $I\left(\boldsymbol{\Psi}\right)$. Its elements can be found numerically by the software. Here, we use the **maxLik** function of R software to maximize Equation (5). This function also gives asymptotic standard errors numerically, which are obtained by the observed information matrix.

3.4. Model validity

The residual analysis can be applied to check whether the regression model is valid. In order to understand this, the randomized quantile residuals can be discussed. The i^{th} randomized quantile residual is defined by $\hat{r}_i = \Phi^{-1}\left[G(y_i, \hat{\beta}, \hat{\mu}_i)\right]$, for $i = 1, \ldots, n$, where $G(y, \beta, \mu)$ is the cdf of the $TURQ$ distribution given by Equation (2), $\Phi^{-1}(x)$ is the qf of the standard normal distribution, and $\hat{\mu}_i$ is defined by Equation (4) with $\hat{\boldsymbol{\theta}}$ instead of $\boldsymbol{\theta}$. If the fitted model successfully processes the data set, the distribution of the random quantile residuals should be distributed as a normal distribution with a mean of 0 and a variance of 1. Further elements in this regard can be found in [6].

4. Data analysis

In this section, a real data application is performed in order to see the applicability and efficiency of the proposed quantile regression model. Three competitor regression models are considered for comparing the fitting results. They are the beta regression by [7] model as well as the Kw and LEEG quantile regression models, by [20] and [11], respectively. For the $TURQ$, Kw and LEEG regression models, the quantile level parameter u has been taken as 0.5. In such a way, they are unit median response models.

The considered data set is obtained from **OECD.Stat** with link

https://stats.oecd.org/. It includes data and metadata for the OECD countries and selected non-member economies. The **OECD.Stat** consists of themes such as Agriculture and Fisheries, Demography, Education and Training, Finance, Health, Labor, and Social Protection and Well-being etc. Each theme is divided into several topics. The used data set can be found in the indicator of the Better Life Index in the Social Protection and Well-being theme of the OECD.Stat. The reference year is 2017. It can be directly accessed via https://stats.oecd.org/index.aspx?DataSetCode=BLI link. The aim of the application is to relate the percentage of the educational attainment values of the OECD countries (variable y) with the percentage of the voter turnout (variable x_1), homicide rate (variable x_2), and life satisfaction (variable x_3). Educational attainment considers the number of adults aged 25 to 64 holding at least an upper secondary degree over the population of the same age.

The regression model based on μ_i is given by

$$\operatorname{logit}(\mu_i) = \theta_0 + \theta_1 x_{i1} + \theta_2 x_{i2}, + \theta_3 x_{i3} \quad i = 1, 2, \ldots, 38,$$

where μ_i is the mean for the beta model whereas it denotes the median for the LEEG, Kw and $TURQ$ quantile regression models. Some summary statistics of the response variable y and its box plot are given in Table 1 and Figure 2, respectively. From them, it is seen that the data are left skewed and five outliers are detected for the unit response variable. For these reasons, relating the unit response variable with covariates via the median quantile regression will be more useful for inferences, since the mean is affected by skewed data with the outliers precisely. Thus, it can be obtained as a more illustrative and more robust inference than the mean response regression.

Table 1. Some summary statistics of the response variable y

Minimum	Mean	Median	Maximum	Variance	Skewness	Kurtosis	n
0.3700	0.7724	0.8150	0.9500	0.0256	-1.2770	3.5361	38

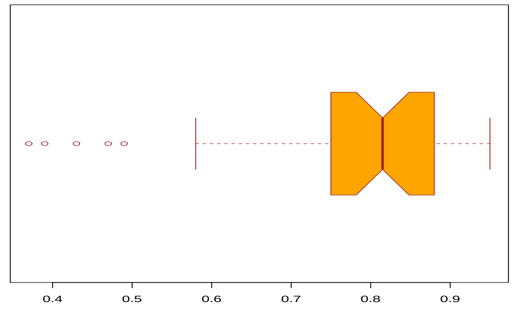

FIGURE 2. The box plot of the response variable y

We give the results of the regression analysis in Table 2. We see that the $TURQ$ regression model has lower values of the AIC (Akaike Information Criterion) and BIC (Bayesian Information Criterion) statistics with upper log-likelihood values ($\hat{\ell}$) than those of other regression models. Therefore, it can be decided that the proposed regression model is the best among the considered regression models in terms of the best modeling ability. Further, for the $TURQ$ regression model, all the parameters of the covariates have been seen as statistically significant at the level of 5%. The parameter θ_3 positively affected the median response while the parameters θ_1 and θ_2 negatively affected the median response. All covariates have marginal effects, negative for voter turnout and homicide rate, representing that an increase in voter turnout and homicide rate decreases the median of the response variable, and positive for life satisfaction, representing that an increase in life satisfaction increases the median of the response variable. It is concluded that when life satisfaction increases, the percentage of educational attainment increases, as expected. In addition, when the homicide rate and voter turnout decrease, the percentage of educational attainment also increases. The result of the voter turnout variable may be surprising.

TABLE 2. The results of fitted regression models with the considered model selection criteria.

Parameters	**Beta**			**Kw**			**LEEG**			**TURQ**		
	Estimate	SE	p-value	Estimate	SE	p-value	Estimate	SE	p-value	Estimate	SE	p-value
θ_0	0.9615	0.9685	0.3208	1.6247	1.1740	0.1664	0.3275	1.0754	0.7607	-0.3469	1.0407	0.7389
θ_1	-2.9211	1.0176	0.0041	-4.1197	1.3892	0.0030	-4.0917	1.4520	0.0048	-1.7892	0.8534	0.0360
θ_2	-0.0470	0.0178	0.0084	-0.0404	0.0168	0.0159	-0.0477	0.0145	0.0010	-0.0673	0.0237	0.0046
θ_3	0.3794	0.1492	0.0110	0.4237	0.2546	0.0960	0.6214	0.1745	< 0.001	0.4754	0.1356	< 0.001
β	11.5900	2.6100	< 0.0001	6.2167	1.0787	< 0.0001	7.8378	1.7365	< 0.0001	-0.4272	0.5474	0.4352
$\hat{\ell}$		30.9024			29.4339			28.6480			32.9941	
AIC		-51.8048			-48.8677			-47.2961			-55.9881	
BIC		-43.6169			-40.6798			-39.1082			-47.8002	

Moreover, the Vuong likelihood ratio test (see [23]) can be used for comparison of the non-nested two regression models to whether there is any significant difference in the fit models. The Vuong statistic is defined as $T = [1/(\omega\sqrt{n})] \sum_{i=1}^{n} \log\left[g(y_i, \theta)/f(y_i, \lambda)\right]$, where

$$\omega^2 = (1/n)\sum_{i=1}^{n} \left\{\log\left(g(y_i,\theta)/f(y_i,\lambda)\right)\right\}^2 - \left\{(1/n)\sum_{i=1}^{n} \log\left(g(y_i,\theta)/f(y_i,\lambda)\right)\right\}^2,$$

$g(y, \theta)$ and $f(y, \lambda)$ are the corresponding competitor pdfs calculated at the MLEs. It is noticed that, while $n \to \infty$, the asymptotic distribution behind T is the standard normal distribution. When this test was applied to the rival $TURQ$ quantile regression model with the Kw quantile regression model, the value of the T statistic was found as 1.1964 (with p-value 0.1158). Hence, the test rejects the null hypothesis at 12% level of significance, in favor of the hypothesis that the $TURQ$ quantile regression model is not equivalent to the Kw quantile regression model. In addition, if the $LEEG$ quantile regression model is taken as the rival, the test rejects the null hypothesis at 10% level of significance, in favor of the $TURQ$ quantile regression model.

Since the randomized quantile residuals have a theoretical standard normal distribution, one may see whether they fit this corresponding distribution. The results of the Kolmogorov-Smirnov (KS), Anderson-Darling (A^*) and Cramer-von Mises (W^*) goodness-of-fits statistics are given in Table 3. From this table, it is clear that the results based on the $TURQ$ quantile regression model of the randomized quantile residuals are more suitable than those of other regression models as well as its model fitting is acceptable.

Table 3. The goodness-of-fit results of the randomized quantile residuals for the regression models.

Models	KS	p-value	A^*	p-value	W^*	p-value
TURQ	0.0891	0.8971	0.3134	0.9273	0.0367	0.9512
Beta	0.1240	0.5613	0.6425	0.6077	0.1008	0.5833
LEEG	0.1198	0.6033	0.8716	0.4313	0.1377	0.4308
Kw	0.1291	0.5095	0.7516	0.5163	0.1302	0.4578

5. Conclusions and future work

The following findings have been obtained by this paper.

(1) A new alternative unit distribution model and its quantile regression model for the analysis of measures of proportions and percentages have been proposed.
(2) Educational attainment measurement of the OECD countries has been related with covariates, which are some Better Life Index such as life

satisfaction, homicide rate, and voter turnout. All covariates have been seen statistically significant at the level of 5% for the median response.
(3) For describing the median of the data, the quantile regression analysis application has indicated that the proposed model has provided better fits than the famous beta and Kumaraswamy regression models based on a skewed unit response variable with outlier observations.
(4) It has been seen that the proposed modeling strategy is suitable for illustrating its potential usages.

This is how we define a new alternative unit distribution model as a quantile model for a unit response variable that distributes the TUR distribution. Hence, it can be a remarkable addition to applications of applied sciences and data scientists.

Future research would be another work about the combination of the quadratic transmutation scheme and the unit Weibull distribution [18]. This distribution and its regression modeling will propose strong competitors of the unit distributions by generalizing the TUR and $TURQ$ models.

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