ON A CLASS OF MULTITIME VARIATIONAL PROBLEMS WITH ISOPERIMETRIC CONSTRAINTS

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În această lucrare, stabilim rezultate privind eficienţa şi dualitatea pentru o clasă de probleme de control multitemporal cu restricţii izoperimetrice, folosind tehnici de calcul variaţioanl. Mai întâi, introducem condiţii de optimalitate pentru o problemă scalară variaţională multitemporală (SCP). Apoi, studiem condiţii de eficienţă şi fundamentăm o teorie a programului dual pentru o problemă vectorială (VCP). Ambele probleme folosesc restricţii izoperimetrice, folosite frecvent când ne referim la resurse. În §1 şi §2, amintim unele rezultate şi precizăm punctul de plecare. În §3, studiem condiţii necesare de optim pentru problema (SCP). În §4, discutăm condiţii de eficienţă pentru problema (VCP) şi fundamentăm o teorie pentru programul dual. Această lucrare este o continuare a unor lucrări recente (a se vedea [11]*, Ştefan Mititelu şi* [22]*, Constantin Udrişte).*

This paper aims to establish some results of efficiency and duality for multitime control problems, thought as variational problems with isoperimetric constraints, mainly arising when we talk about resources. First, we introduce optimality conditions for a scalar multitime variational problem (SCP). Next, we study efficiency conditions and develop a duality theory for a vector multitime problem (VCP). In §1, we recall some notions, while in §2, we substantiate our starting point. In §3 we introduce our problem (SCP) and prove a result on necessary optimality conditions. In §4, we discuss efficiency conditions to our problem (VCP) and develop a dual program theory. Our work may be viewed as a natural continuation of some recent works (see [11]*, by Ştefan Mititelu;* [22]*, by Constantin Udrişte).*

Keywords: optimal variational problem, nonlinear programming, invex functional, duality.

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1. Introduction

Traditional control problems have found important applications in various areas of applied (experimental) sciences and technology ranging from Economics (processes control), Psychology (impulse control disorders), Medicine (bladder control) to Engineering (robotics and automation) and Biology (population ecosystems), see

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[3]. Such applications rely heavily on the temporal dependence of these problems. Taking into account both theoretical and applied viewpoints, multitime control problems have been intensively studied in the last few years [22]. Motivated by the work on this topic reported in [20], [22], [23], this paper aims to establish some results of efficiency and duality for multitime control problems thought as variational problems with isoperimetric constraints, mainly arising when we talk about resources. That is why, our current paper may be viewed as a natural continuation and extension of some recent works (see [11], by Ştefan Mititelu and [22], by Constantin Udrişte).

In the following, for two vectors $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)$, the relations $v = w$, $v < w$, $v \leqq w$ and $v \leq w$, are defined as:

$$v = w \Leftrightarrow v_i = w_i,\ i = \overline{1,n}; \quad v < w \Leftrightarrow v_i < w_i,\ i = \overline{1,n};$$
$$v \leqq w \Leftrightarrow v_i \leq w_i,\ i = \overline{1,n}; \quad v \leq w \Leftrightarrow v \leqq w \text{ and } v \neq w.$$

Throughout this paper, $t = (t^\alpha) \in \mathbb{R}^m_+$ is the multitime; $dv = dt^1 \cdots dt^m$ is the volume element in $\mathbb{R}^m_+$; Ω is the hyperparallelepiped in $\mathbb{R}^m_+$ defined by the closed interval $[0, t_0] = \{t \in \mathbb{R}^m_+ | 0 \leq t \leq t_0\}$, where $0 = (0, \dots, 0)$ and $t_0 = (t_0^1, \dots, t_0^m)$ are points in $\mathbb{R}^m_+$; $x(t) = (x^i(t))$, $i = \overline{1,n}$, is a C^2-class state vector; $u(t) = (u^a(t))$, $a = \overline{1,\ell}$, is a continuous control vector; the *running cost* $X(t, x(t), x_\gamma(t), u(t), u_\gamma(t))$ is a C^1-class function; $X^i_\alpha(t, x(t), x_\gamma(t), u(t), u_\gamma(t))$ are C^1-class functions satisfying the complete integrability conditions (m-flow type problem).

2. **Starting point of our problem**

Consider the functional of multiple integral type

$$I(u) = \int_\Omega X(t, x(t), u(t))dv.$$

Recently, a multitime maximum principle for the following multitime optimal control problem, within the class of Dieudonné-Rashevsky type problems, has been stated by Professor Constantin Udrişte (see, [20], [22], [23])

$$(\text{MCP}) \begin{cases} \underset{u}{\text{Maximize}}\ I(u) \\ \text{subject to} \\ \dfrac{\partial x^i}{\partial t^\alpha} = X^i_\alpha(t, x(t), u(t)),\ i = \overline{1,n},\ \alpha = \overline{1,m}, \\ u(t) \in \mathcal{U},\ \forall t \in \Omega;\ x(0) = x_0,\ x(t_0) = x_1. \end{cases}$$

This kind of problems appears when we describe the torsion of prismatic bars in the elastic case [18], as well as in the elastic-plastic case [19]. Also, they arise when we think of optimization problems for convex bodies and the geometrical restictions, that is maximization of the surface for given width and diameter. These lead us again to Dieudonné-Rashevski type problems for support functions in spherical coordinates, [1], [2].

Our study follows the idea of Schrödinger to change a partial differential equation with an action using a multiple integral.

Starting from the study of the above mentioned problem and inspired by the ongoing research in optimal control, we introduce and study two multitime variational problems. The first problem is a scalar one and is thought as a necessary tool for pointing out our main results concerning a vectorial multitime multiobjective problem (Theorem 4.1, Theorem 4.2, Theorem 4.3 and Theorem 4.4 as well).

Our method of investigation is based on employing adequate variational calculus techniques in the study of the problems of optimal control. This fact is possible since the optimal control problems can be changed in variational problems. Moreover, the solutions of these problems belong to the interior of the problems domain.

3. **Scalar variational problem**

Let be given the functional of multiple integral type

$$I(x,u)=\int_{\Omega} X(t,x(t),x_{\gamma}(t),u(t),u_{\gamma}(t))dv.$$

Consider the functions $Y_{\beta}(t,x(t),x_{\gamma}(t),u(t),u_{\gamma}(t))$, $\beta=\overline{1,q}$, of C^1-class and connected to framework of problem (MCP), we introduce the following problem with mixed isoperimetric constraints

$$\text{(SCP)}\quad\begin{cases}\underset{x,u}{\text{Minimize}}\ I(x,u)\\ \text{subject to}\\ \displaystyle\int_{\Omega} X^i_{\alpha}(t,x(t),x_{\gamma}(t),u(t),u_{\gamma}(t))dv=0,\ i=\overline{1,n},\ \alpha=\overline{1,m},\\ \displaystyle\int_{\Omega} Y_{\beta}(t,x(t),x_{\gamma}(t),u(t),u_{\gamma}(t))\,dv\le 0,\ \beta=\overline{1,q},\\ x(0)=x_0,\ x(t_0)=x_1.\end{cases}$$

In the following, we introduce our necessary optimality conditions for the scalar problem (SCP). This result will be later used for finding and proving necessary optimality conditions for our multitime multiobjective vector problem. The proof of this theorem essentially uses Fritz-John conditions and the fundamental lemma of variational calculus.

Theorem 3.1 (Necessary conditions)**.** *Let (x,u) be an optimal solution of* (SCP)*. Then there are real scalars φ, λ^{α}_i, and $\mu\in\mathbb{R}^q$, which satisfy the conditions:*

$$\text{(SFJ)}\quad\begin{cases}\varphi\dfrac{\partial X}{\partial x^i}+\lambda^{\alpha}_j\dfrac{\partial X^j_{\alpha}}{\partial x^i}+\mu^{\beta}\dfrac{\partial Y_{\beta}}{\partial x^i}-D_{\gamma}\left(\varphi\dfrac{\partial X}{\partial x^i_{\gamma}}+\lambda^{\alpha}_j\dfrac{\partial X^j_{\alpha}}{\partial x^i_{\gamma}}+\mu^{\beta}\dfrac{\partial Y_{\beta}}{\partial x^i_{\gamma}}\right)=0\\ \varphi\dfrac{\partial X}{\partial u^a}+\lambda^{\alpha}_i\dfrac{\partial X^i_{\alpha}}{\partial u^a}+\mu^{\beta}\dfrac{\partial Y_{\beta}}{\partial u^a}-D_{\gamma}\left(\varphi\dfrac{\partial X}{\partial u^a_{\gamma}}+\lambda^{\alpha}_i\dfrac{\partial X^i_{\alpha}}{\partial u^a_{\gamma}}+\mu^{\beta}\dfrac{\partial Y_{\beta}}{\partial u^a_{\gamma}}\right)=0\\ \mu^{\beta}Y_{\beta}(t,x(t),x_{\gamma}(t),u(t),u_{\gamma}(t))=0\quad\text{(no summation)}\\ \varphi\ge 0,\ \mu^{\beta}\ge 0.\end{cases}$$

Proof. Let (x,u) be a minimal solution of problem (SCP) and the arbitrary vector functions $p(t)=(p^i(t))\in\mathbb{R}^n$ and $q(t)=(q^j(t))\in\mathbb{R}^q$, where $p,q\in C^1(\Omega)$, $p\big|_{\partial\Omega}=0$, $q\big|_{\partial\Omega}=0$. For $\varepsilon_1>0$ and $\varepsilon_2>0$, let

$$V=\{(\bar{x},\bar{u})\,|\,\bar{x}(t)=x(t)+\varepsilon_1 p(t),\ \bar{u}(t)=u(t)+\varepsilon_2 q(t)\}$$

be a neighborhood of (x,u). Consider the following functions defined as integrals:

$$f(\varepsilon_1,\varepsilon_2)=\int_\Omega X(t,x(t)+\varepsilon_1 p(t),x_\gamma(t)+\varepsilon_1 p_\gamma(t),u(t)+\varepsilon_2 q(t),u_\gamma(t)+\varepsilon_2 q_\gamma(t))\,dv;$$

$$g^i_\alpha(\varepsilon_1,\varepsilon_2)=\int_\Omega X^i_\alpha\,(t,x(t)+\varepsilon_1 p(t),x_\gamma(t)+\varepsilon_1 p_\gamma(t),u(t)+\varepsilon_2 q(t),u_\gamma(t)+\varepsilon_2 q_\gamma(t))\,dv;$$

$$h_\beta(\varepsilon_1,\varepsilon_2)=\int_\Omega Y_\beta(t,x(t)+\varepsilon_1 p(t),x_\gamma(t)+\varepsilon_1 p_\gamma(t),u(t)+\varepsilon_2 q(t),u_\gamma(t)+\varepsilon_2 q_\gamma(t))dv.$$

If (x,u) is a minimal solution of (SCP), then $(0,0)$ is a minimal solution of the following problem (PM1).

$$\text{(PM1)}\quad\begin{cases}\underset{\varepsilon_1,\,\varepsilon_2}{\text{Minimize}}\ f(\varepsilon_1,\varepsilon_2)\\ \text{subject to}\\ g^i_\alpha(\varepsilon_1,\varepsilon_2)=0,\ i=\overline{1,n},\ \alpha=\overline{1,m},\\ h_\beta(\varepsilon_1,\varepsilon_2)\le 0,\ \beta=\overline{1,q},\\ p\big|_{\partial\Omega}=0,\ q\big|_{\partial\Omega}=0.\end{cases}$$

Since $(0,0)$ is a minimum solution of (PM1), then there exist φ, λ^α_i and μ^β such that problem (PM1) satisfies the following Fritz-John conditions at $(0,0)$:

$$\text{(FJ)}\quad\begin{cases}\varphi\nabla f(0,0)+\lambda^\alpha_i\nabla g^i_\alpha(0,0)+\mu^\beta\nabla h_\beta(0,0)=0\\ \mu^\beta h_\beta(0,0)=0\\ \varphi\ge 0,\ \mu^\beta\ge 0.\end{cases}$$

Having in mind the forms of $\nabla f(0,0)$, $\nabla g^i_\alpha(0,0)$ and $\nabla h_\beta(0,0)$, the first condition (FJ) becomes

$$\int_\Omega\left[\left(\varphi\frac{\partial X}{\partial x^i}+\lambda^\alpha_j\frac{\partial X^j_\alpha}{\partial x^i}+\mu^\beta\frac{\partial Y_\beta}{\partial x^i}\right)p^i+\left(\varphi\frac{\partial X}{\partial x^i_\gamma}+\lambda^\alpha_j\frac{\partial X^j_\alpha}{\partial x^i_\gamma}+\mu^\beta\frac{\partial Y_\beta}{\partial x^i_\gamma}\right)p^i_\gamma\right]dv=0. \tag{1}$$

Integrating by parts in (1) and having in mind Theorem 8.2 in [25], we obtain

$$\int_\Omega\left[\left(\varphi\frac{\partial X}{\partial x^i}+\lambda^\alpha_j\frac{\partial X^j_\alpha}{\partial x^i}+\mu^\beta\frac{\partial Y_\beta}{\partial x^i}\right)-D_\gamma\left(\varphi\frac{\partial X}{\partial x^i_\gamma}+\lambda^\alpha_j\frac{\partial X^j_\alpha}{\partial x^i_\gamma}+\mu^\beta\frac{\partial Y_\beta}{\partial x^i_\gamma}\right)\right]p^i dv=0. \tag{2}$$

As p is arbitrary, according to a fundamental lemma of the variational calculus, from (2) it follows

$$\varphi\frac{\partial X}{\partial x^i}+\lambda^\alpha_j\frac{\partial X^j_\alpha}{\partial x^i}+\mu^\beta\frac{\partial Y_\beta}{\partial x^i}-D_\gamma\left(\varphi\frac{\partial X}{\partial x^i_\gamma}+\lambda^\alpha_j\frac{\partial X^j_\alpha}{\partial x^i_\gamma}+\mu^\beta\frac{\partial Y_\beta}{\partial x^i_\gamma}\right)=0,$$

that is the first condition of (SFJ).

In a similar manner, the second condition (FJ) implies the equality

$$\varphi\frac{\partial X}{\partial u^a}+\lambda_i^\alpha\frac{\partial X_\alpha^i}{\partial u^a}+\mu^\beta\frac{\partial Y_\beta}{\partial u^a}-D_\gamma\left(\varphi\frac{\partial X}{\partial u_\gamma^a}+\lambda_i^\alpha\frac{\partial X_\alpha^i}{\partial u_\gamma^a}+\mu^\beta\frac{\partial Y_\beta}{\partial u_\gamma^a}\right)=0,$$

that is the second condition of (SFJ).

From $\mu^\beta h_\beta(0,0)=0$ of (FJ), we get $\mu^\beta Y_\beta(t,x(t),x_\gamma(t),u(t),u_\gamma(t))dv=0$ ■

4. **Pareto variational problem**

In this section, we first introduce our vector problem. Based on the previous scalar problem and using Lemma 4.1, we establish necessary efficiency conditions for program (VCP). Next, using essentially the notion of invexity, we develop a dual program theory.

Let us consider now the vector function $(X_1,\dots,X_p)$, producing the running costs $X_1(t,x(t),x_\gamma(t),u(t),u_\gamma(t)),\dots,\ X_p(t,x(t),x_\gamma(t),u(t),u_\gamma(t))$. We denote

$$I_k(x,u)=\int_\Omega X_k(t,x(t),x_\gamma(t),u(t),u_\gamma(t))dv,\quad k=\overline{1,p}$$

and we consider the vector functional $I(x,u)=(I_1(x,u),\dots,I_p(x,u))$.

We introduce the following multitime control vector problem with isoperimetric constraints:

$$\text{(VCP)}\quad\begin{cases}\underset{x,u}{\text{Minimize}}\text{(Pareto) } I(x,u)\\ \text{subject to}\\ \displaystyle\int_\Omega X_\alpha^i(t,x(t),x_\gamma(t),u(t),u_\gamma(t))dv=0,\ i=\overline{1,n},\ \alpha=\overline{1,m},\\ \displaystyle\int_\Omega Y_\beta(t,x(t),x_\gamma(t),u(t),u_\gamma(t))\,dv\le 0,\ \beta=\overline{1,q},\\ x(0)=x_0,\ x(t_0)=x_1.\end{cases}$$

and denote Δ the domain of problem (VCP).

Definition 4.1. Point $(x,u)\in\Delta$ is called *efficient solution* (Pareto minimum) for (VCP) if there is no $(\bar{x},\bar{u})\in\Delta$ such that $I(\bar{x},\bar{u})\le I(x,u)$.

The following Lemma shows the equivalence between the efficient solutions of (VCP) and the optimal solution of p scalar problems. This connection is needed in order to find necessary efficiency conditions. The proof of this result uses essentially the techniques of Jagannathan, [6].

Lemma 4.1. *The point* $(x^0,u^0)\in\Delta$ *is an efficient solution of problem* (VCP) *if and only if* (x^0,u^0) *is an optimal solution of each scalar problem* $(\text{SCP})_k$, $k=\overline{1,p}$, *where*

$$(\mathrm{SCP})_k \quad \begin{cases} \underset{x,u}{\text{Minimize}}\ I_k(x,u) \\ \text{subject to} \\ \displaystyle\int_\Omega X^i_\alpha(t,x(t),x_\gamma(t),u(t),u_\gamma(t))dv = 0,\ i=\overline{1,n},\ \alpha=\overline{1,m}, \\ \displaystyle\int_\Omega Y_\beta(t,x(t),x_\gamma(t),u(t),u_\gamma(t))\,dv \le 0,\ \beta=\overline{1,q}, \\ I_j(x,u) \le I_j(x^0,u^0), \quad j=\overline{1,p},\ j\neq k, \\ x(0)=x_0,\ x(t_0)=x_1. \end{cases}$$

This Lemma plays a role of paramount importance in suggesting the study of the efficient solutions of problem (VCP).

Applying Lemma 4.1 and Theorem 4.2 for each problem (SCP), we obtain

Theorem 4.1. *Let* $(x,u)\in\Delta$ *be an efficient solution of program* (VCP)*. Then there are* $\tau\in\mathbb{R}^p$*,* $\lambda^\alpha_i\in\mathbb{R}$ *and* $\mu\in\mathbb{R}^q$*, such that*

$$(\mathrm{VFJ}) \quad \begin{cases} \tau^k\dfrac{\partial X_k}{\partial x^i}+\lambda^\alpha_j\dfrac{\partial X^j_\alpha}{\partial x^i}+\mu^\beta\dfrac{\partial Y_\beta}{\partial x^i}-D_\gamma\left(\tau^k\dfrac{\partial X_k}{\partial x^i_\gamma}+\lambda^\alpha_j\dfrac{\partial X^j_\alpha}{\partial x^i_\gamma}+\mu^\beta\dfrac{\partial Y_\beta}{\partial x^i_\gamma}\right)=0 \\ \tau^k\dfrac{\partial X_k}{\partial u^j}+\lambda^\alpha_j\dfrac{\partial X^j_\alpha}{\partial u^j}+\mu^\beta\dfrac{\partial Y_\beta}{\partial u^j}-D_\gamma\left(\tau^k\dfrac{\partial X_k}{\partial u^a_\gamma}+\lambda^\alpha_i\dfrac{\partial X^i_\alpha}{\partial u^a_\gamma}+\mu^\beta\dfrac{\partial Y_\beta}{\partial u^a_\gamma}\right)=0 \\ \mu^\beta(t)Y_\beta(t,x(t),x_\gamma(t),u(t),u_\gamma(t))=0,\ \beta=\overline{1,q} \\ \tau=(\tau^k)\geqq 0,\ \mu=(\mu^\beta)\geqq 0. \end{cases}$$

A nontrivial situation arises when each component of vector τ is positive. In this case, we can consider $\tau^k=1$, for each $k=\overline{1,p}$, therefore we can introduce

Definition 4.2. The efficient solution (x^0,u^0) of (VCP) is called *normal* if $\tau^k=1$ for each $k=\overline{1,p}$.

Given programs (VCP) and (VCD), in the following we shall develop our dual program theory, stating weak, direct and converse duality theorems. The base of our research is the notion of ρ-invexity, [12], [17].

Let $f(t,x(t),x_\gamma(t),u(t),u_\gamma(t))$ be a scalar function of C^1-class. Consider the functional

$$F(x,u)=\int_\Omega f(t,x(t),x_\gamma(t),u(t),u_\gamma(t))\,dv.$$

Definition 4.3. The function $F(x,u)$ is called ρ-*invex* [*strictly* ρ-*invex*] at the point (x^*,u^*) if there exist the vector function $\eta(t)\in\mathbb{R}^n$ of C^1-class, with $\eta|_{\partial\Omega}=0$, $\xi(t)\in\mathbb{R}^k$ of C^0-class and the bounded vector function $\theta(x,u)\in\mathbb{R}^n$ such that $\forall(x,u)$

$[(x,u) \neq (x^*,u^*)]$,

$$F(x,u) - F(x^*,u^*) \;\geq\; [>] \int_\Omega \left(\eta_i \frac{\partial f}{\partial x^i}(t,x^*,u^*) + (D_\gamma \eta_i) \frac{\partial f}{\partial x^i_\gamma} + \xi_j \frac{\partial f}{\partial u^j}(t,x^*,u^*) + (D_\gamma \xi_a) \frac{\partial f}{\partial u^a_\gamma} \right) dv + \rho \|\theta(x,u)\|^2.$$

To develop our dual program theory, we consider the Lagrangian functions

$$L_k(t,x(t),x_\gamma(t),u(t),u_\gamma(t),\lambda,\mu) = X_k(t,x(t),x_\gamma(t),u(t),u_\gamma(t)) + \frac{1}{p}[\lambda^\alpha_i X^i_\alpha(t,x(t),x_\gamma(t),u(t),u_\gamma(t)) + \mu^\beta Y_\beta(t,x(t),x_\gamma(t),u(t),u_\gamma(t))]$$

where $k = \overline{1,p}$, which determine the vector function $L = (L_1,\ldots,L_p)$.

Let us introduce the following vector of multiple integrals

$$J(x,u,\lambda,\mu) = \int_\Omega L(t,x(t),x_\gamma(t),u(t),u_\gamma(t),\lambda,\mu)dv.$$

To problem (VCP), we associate the next dual vector multitime control problem:

$$(\text{VCD}) \begin{cases} \text{Maximize Pareto } J(x(t),u(t),\lambda,\mu) \\ \text{subject to} \\ \dfrac{\partial X_k}{\partial x^i} + \lambda^\alpha_j \dfrac{\partial X^j_\alpha}{\partial x^i} + \mu^\beta \dfrac{\partial Y_\beta}{\partial x^i} - D_\gamma \left(\dfrac{\partial X_k}{\partial x^i_\gamma} + \lambda^\alpha_j \dfrac{\partial X^j_\alpha}{\partial x^i_\gamma} + \mu^\beta \dfrac{\partial Y_\beta}{\partial x^i_\gamma} \right) = 0 \\ \dfrac{\partial X_k}{\partial u^j} + \lambda^\alpha_j \dfrac{\partial X^j_\alpha}{\partial u^j} + \mu^\beta \dfrac{\partial Y_\beta}{\partial u^j} - D_\gamma \left(\dfrac{\partial X_k}{\partial u^a_\gamma} + \lambda^\alpha_i \dfrac{\partial X^i_\alpha}{\partial u^a_\gamma} + \mu^\beta \dfrac{\partial Y_\beta}{\partial u^a_\gamma} \right) = 0 \\ \mu^\beta(t) Y_\beta(t,x(t),x_\gamma(t),u(t),u_\gamma(t)) = 0,\ \beta = \overline{1,q} \\ \mu = (\mu^\beta) \geqq 0,\ x(0) = x_0,\ x(t_0) = x_1. \end{cases}$$

Denote by $\mathcal{D}$ the domain of dual (VCD) and consider $(x,x_\gamma,u,u_\gamma,\lambda,\mu) = (x,x_\gamma,u,u_\gamma,\lambda^\alpha_i,\mu^\beta)$ the current point of $\mathcal{D}$.

Now we can introduce our duality theorems, as in the following.

Theorem 4.2 (weak duality). *Let $(x^*,u^*) \in \Delta$ and $(x,x_\gamma,u,u_\gamma,\lambda,\mu) \in \mathcal{D}$ be two feasible solutions of problems* (VCP) *and* (VCD). *Consider the functions λ^α_i and μ^β as in Theorem* 4.1 *and suppose that the following conditions are satisfied:*

a) *for each index $k \in \{1,\ldots,p\}$, the integral $\int_\Omega X_k\big(t,x(t),x_\gamma(t),u(t),u_\gamma(t)\big)dv$ is ρ_k-invex at (x,u);*

b) $\int_\Omega \lambda^\alpha_i X^i_\alpha\big(t,x(t),x_\gamma(t),u(t),u_\gamma(t)\big)dv$ *is ρ'-invex at (x,u);*

c) $\int_\Omega \mu^\beta Y_\beta\big(t,x(t),x_\gamma(t),u(t),u_\gamma(t)\big)dv$ *is ρ''-invex at (x,u);*

all with respect to η and ξ, as in Definition 4.3*;*

d) *at least one of the functionals from* a), b) *and* c) *is strictly ρ-invex;*

e) $\sum_{k=1}^{p} \rho_k + \rho' + \rho'' \geq 0$.

Then $I(x^*, u^*) \leq J(x, u, \lambda, \mu)$ *is false.*

Proof. Will be given in a forthcoming paper.

We would like to continue our study stating and proving a direct duality theorem. In this respect, let us consider (x^0, u^0) be a normal efficient solution of problem (VCP). According to Theorem 4.1, there are the real scalars $(\lambda_i^\alpha)^0$ and $(\mu^\beta)^0$ such that conditions (VFJ) are satisfied.

Theorem 4.3 (DIRECT DUALITY). *Suppose that the conditions of Theorem* 4.2 *are satisfied and* $(x^0, x_\gamma^0, u^0, u_\gamma^0, (\lambda_i^\alpha)^0, (\mu^\beta)^0)$ *is an efficient solution of dual control problem* (VCD).

Then $I(x^0, u^0) = J(x^0, u^0, (\lambda_i^\alpha)^0, (\mu^\beta)^0)$, *that is*

$$\min (\mathrm{VCP})(x^0, u^0) = \max (\mathrm{VCD})(x^0, x_\gamma^0, u^0, u_\gamma^0, (\lambda_i^\alpha)^0, (\mu^\beta)^0).$$

Proof. By the hypotheses of Theorem 4.2, it follows that inequality $I(x^0, u^0) \leq J(x^0, u^0, (\lambda_i^\alpha)^0, (\mu^\beta)^0)$ is not true. Therefore, $(x^0, x_\gamma^0, u^0, u_\gamma^0, (\lambda_i^\alpha)^0, (\mu^\beta)^0)$ is efficient for dual (VCD) and $\min (\mathrm{VCP})(x^0, u^0) = \max (\mathrm{VCD})(x^0, x_\gamma^0, u^0, u_\gamma^0, (\lambda_i^\alpha)^0, (\mu^\beta)^0)$ ∎

We finish this ongoing study with a converse duality theorem (the proof follows from Theorem 4.2).

Theorem 4.4 (CONVERSE DUALITY). *Let* $(x^0, x_\gamma^0, u^0, u_\gamma^0, (\lambda_i^\alpha)^0, (\mu^\beta)^0)$ *be an efficient solution of dual problem* (VCD). *Suppose that the following conditions hold:*

i) $(x^0, u^0) \in \Delta$;

ii) *conditions* a)÷e) *of Theorem* 4.2 *are satisfied at* (x^0, u^0).

Then (x^0, u^0) *is efficient solution of* (VCP) *and*

$$\min (\mathrm{VCP})(x^0, u^0) = \max (\mathrm{VCD})(x^0, x_\gamma^0, u^0, u_\gamma^0, (\lambda_i^\alpha)^0, (\mu^\beta)^0).$$

Conclusion

By introducing a new vector variational problem, employing isoperimetric constraints and a simplified scalar variational problem, we have derived necessary efficiency conditions. The notion of invexity allowed us to develop a dual program theory. The results of this paper are new and they complement previously known results. For other different viewpoints regarding the theory of efficiency and duality for optimum problems with constraints, we address the reader to the following research works: [7]÷[11], [14]÷[17], [20]÷[24].

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