ON (FUZZY) ISOMORPHISM THEOREMS OF SOFT Γ-HYPERMODULES

Xueling Ma[1], Jianming Zhan[2], V. Leoreanu-Fotea[3]

Soft set theory, introduced by Molodtsov, has been considered as an effective mathematical tool for modeling uncertainties. In this paper, we apply soft sets to Γ*-hypermodules. The concept of soft* Γ*-hypermodules is first introduced. Then three isomorphism theorems of soft* Γ*-hypermodules are established. Finally, we derive three fuzzy isomorphism theorems of soft* Γ*-hypermodules.*

Keywords: Soft set; (fuzzy) isomorphism theorem; (normal) Γ-subhypermodule; soft Γ-hypermodule.
MSC2010: 20N20; 16Y60; 20N25.

1. Introduction

To solve complicated problems in economics, engineering, and environment, we cannot successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. Uncertainties cannot be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as the probability theory, the theory of (intuitionistic) fuzzy sets, the theory of vague sets, the theory of interval mathematics, and the theory of rough sets. One major problem shared by those theories is their incompatibility with the parameterization tools. To overcome these difficulties, Molodtsov [34] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Ali et al. [2] proposed some new operations on soft sets. Moreover, Cagman et al. [8] considered soft matrix theory and its decision making. Chen et al. [9] presented a new definition of soft set parametrization reduction, and compared this definition to the related concept of attribute reduction

[1]Department of Mathematics, Hubei University for Nationalities,Enshi, Hubei Province, 445000, P. R. China, E-mail: zjmhbmy@126.com

[2]Department of Mathematics, Hubei University for Nationalities,Enshi, Hubei Province, 445000, P. R. China, E-mail: zhanjianming@hotmail.com

[3]Faculty of Mathematics, Al.I Cuza University,Carol I Street, n.11, Iasi, Romania, E-mail: leoreanu2002@yahoo.com

in rough set theory. In particular, fuzzy soft set theory has been investigated by some researchers, for examples, see [28, 31]. Recently, the algebraic structures of soft sets have been studied increasingly, see [1].

On the other hand, the theory of algebraic hyperstructures (or hypersystems) is a well established branch of classical algebraic theory. In the literature, the theory of hyperstructure was first initiated by Marty in 1934 [32] when he defined the hypergroups and began to investigate their properties with applications to groups, rational fractions and algebraic functions. Later on, many people have observed that the theory of hyperstructures also have many applications in both pure and applied sciences, for example, semi-hypergroups are the simplest algebraic hyperstructures which possess the properties of closure and associativity. Some review of the theory of hyperstructures can be found in [10, 11, 14, 35], respectively. Krasner hyperrings [22], a well-known type of hyperrings, are essentially rings, with approximately modified axioms in which addition is a hyperoperation (i.e., $a + b$ is a set). Then this concept has been studied by a variety of authors, see [13, 14]. In particular, the relationships between the fuzzy sets and hyperrings have been considered by many researchers, for examples, see [23, 42, 43], while soft hypergroups were introduced by Leoreanu-Fotea and Corsini in [24].

The concept of Γ-rings was introduced by Barnes [6]. After that, this concept was discussed further by some researchers. The notion of fuzzy ideals in a Γ-ring was introduced by Jun and Lee in [21]. They studied some preliminary properties of fuzzy ideals of Γ-rings. Jun [19] defined fuzzy prime ideals of a Γ-ring and obtained a number of characterizations for a fuzzy ideal to be a fuzzy prime ideal. In particular, Dutta and Chanda [17] studied the structures of the set of fuzzy ideals of a Γ-ring. Ma et al. [25, 26] considered the characterizations of Γ-hemirings and Γ-rings, respectively. The notion of a Γ-module was introduced by Ameri et al. in [4]. They studied some preliminary properties of Γ-modules. Recently, some Γ-hyperstuctures have been studied by some researchers. Ameri et al. [3] considered the concept of fuzzy hyperideals of Γ-hyperrings. By a different way of [3], Yin et al. [37] investigated some new results on Γ-hyperrings. Ma et al. [27] considered the (fuzzy) isomorphism theorems of Γ-hyperrings. In the same time, Davvaz et al. [15, 16] considered the properties of Γ-hypernear-rings and Γ-H_v-rings, respectively. Recently, Wang et al. [36] investigated the characterizations of fuzzy isomorphism theorems of soft hypermodules.

In this paper, we discuss on soft Γ-hypermodules. In Section 2, we recall some basic concepts of Γ-hypermodules. In Section 3, we derive three isomorphism theorems of soft Γ-hypermodules. In particular, we establish three fuzzy isomorphism theorems of soft Γ-hypermodules in Section 4.

2. **Preliminaries**

A *quasicanonical hypergroup* (not necessarily commutative) is an algebraic structure $(\mathcal{H}, +)$ satisfying the following conditions:

(i) for every $x, y, z \in \mathcal{H}, x + (y + z) = (x + y) + z$;

(ii) there exists a $0 \in \mathcal{H}$ such that $0 + x = x$, for all $x \in \mathcal{H}$;

(iii) for every $x \in \mathcal{H}$, there exists a unique element $x' \in \mathcal{H}$ such that $0 \in (x + x') \cap (x' + x)$ (we denote x' by $-x$ and we call it the opposite of x);

(iv) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.

Quasicanonical hypergroups are also called *polygroups.*

We note that if $x \in \mathcal{H}$ and A, B are non-empty subsets in $\mathcal{H}$, then by $A + B$, $A + x$ and $x + B$ we mean that $A + B = \bigcup_{a \in A, b \in B} a + b, A + x = A + \{x\}$ and $x + B = \{x\} + B$, respectively. Also, for all $x, y \in \mathcal{H}$, we have $-(-x) = x, -0 = 0$, where 0 is unique and $-(x + y) = -y - x$.

A sub-hypergroup $A \subset \mathcal{H}$ is said to be *normal* if $x + A - x \subseteq A$ for all $x \in \mathcal{H}$. A normal sub-hypergroup A of $\mathcal{H}$ is called *left (right) hyperideal* of $\mathcal{H}$ if $xA \subseteq A$ ($Ax \subseteq A$ respectively) for all $x \in \mathcal{H}$. Moreover A is said to be a *hyperideal* of $\mathcal{H}$ if it is both a left and a right hyperideal of $\mathcal{H}$. A *canonical hypergroup* is a commutative quasicanonical hypergroup.

Definition 2.1. [22]A *hyperring* is an algebraic structure $(R, +, \cdot)$, which satisfies the following axioms:

(1) $(R, +)$ is a canonical hypergroup;

(2) Related to the multiplication, $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, that is, $0 \cdot x = x \cdot 0 = 0$ for all $x \in R$;

(3) The multiplication is distributive with respect to the hyperoperation "+" that is, $z \cdot (x + y) = z \cdot x + z \cdot y$ and $(x + y) \cdot z = x \cdot z + y \cdot z$ for all $x, y, z \in R$.

Definition 2.2. [3]Let $(R, \oplus)$ and $(\Gamma, \oplus)$ be two canonical hypergroups. Then R is called a Γ-*hyperring*, if the following conditions are satisfied for all $x, y, z \in R$ and for all $\alpha, \beta \in \Gamma$,

(1) $x\alpha y \in R$;

(2) $(x \oplus y)\alpha z = x\alpha z \oplus y\alpha z, x\alpha(y \oplus z) = x\alpha y \oplus x\alpha z$;

(3) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Definition 2.3. [40] Let $(R, \oplus, \Gamma)$ be a Γ-hyperring and $(M, \oplus)$ be a canonical hypergroup. M is called a Γ-*hypermodule* over R if there exists $f : R \times \Gamma \times M \to M$ (the image of (r, α, m) being denoted by $r\alpha m$) such that for all $a, b \in R, m_1, m_2 \in M, \alpha, \beta \in \Gamma$, we have

(1) $a\alpha(m_1 \oplus m_2) = a\alpha m_1 \oplus a\alpha m_2$;

(2) $(a \oplus b)\alpha m_1 \subseteq a\alpha m_1 \oplus b\alpha m_1$;

(3) $a(\alpha \oplus \beta)m_1 \subseteq a\alpha m_1 \oplus a\beta m_1$;

(4) $(a\alpha b)\beta m_1 = a\alpha(b\beta m_1)$.

Throughout this paper, R and M are a Γ-hyperring and Γ-hypermodule, respectively, unless otherwise specified.

A subset A in M is said to be a Γ-*subhypermodule* of M if it satisfies the following conditions: (1) $(A, \oplus)$ is a subhypergroup of $(M, \oplus)$; (2) $r\alpha x \in A$, for all $r \in R$, $\alpha \in \Gamma$ and $x \in A$.

A Γ-subhypermodule A of M is called *normal* if $x \oplus A - x \subseteq A$, for all $x \in M$.

Definition 2.4. A fuzzy set μ of M is called a *fuzzy* Γ*-subhypermodule* of M if the following conditions hold:

(1) $\min\{\mu(x), \mu(y)\} \leq \inf_{z \in x \oplus y} \mu(z)$, for all $x, y \in M$;

(2) $\mu(x) \leq \mu(-x)$, for all $x \in M$;

(3) $\mu(x) \leq \mu(r\alpha x)$, for all $r \in R$, $x \in M$ and $\alpha \in \Gamma$.

A fuzzy Γ-subhypermodule μ of M is called *normal* if $\mu(y) \leq \inf_{z \in x \oplus y - x} \mu(z)$, for all $x, y \in M$.

Definition 2.5. [40]If M and M' are Γ-hypermodules, then a mapping $f : R \longrightarrow R'$ such that

$$f(x \oplus y) = f(x) \oplus f(y) \quad \text{and} \quad f(r\alpha x) = r\alpha f(x),$$

for all $r \in R$, $x, y \in M$ and $\alpha \in \Gamma$, is called a Γ*-hypermodule homomorphism.*

Clearly, a Γ-hypermodule homomorphism f is an isomorphism if f is injective and surjective. We write $M \cong M'$ if M is isomorphic to M'.

3. Isomorphism Theorems

In what follows, let M be a Γ-hypermodule and A be a non-empty set. A set-valued function $F : A \to \mathcal{P}(M)$ can be defined as $F(x) = \{y \mid (x, y) \in \rho, y \in M\}$ for all $x \in A$, where ρ is a subset of $A \times M$. Then the pair (F, A) is called a soft set over M.

For a soft set (F, A) over M, the set $\mathrm{Supp}(F, A) = \{x \mid F(x) \neq \varnothing, x \in A\}$ is called the support of the soft set (F, A). A soft set (F, A) is non-null if $\mathrm{Supp}(F, A) \neq \varnothing$.

Definition 3.1. Let (F, A) be a non-null soft set over M. Then (F, A) is called a *soft* Γ*-hypermodule over* M if $F(x)$ is a Γ-subhypermodule of M for all $x \in \mathrm{Supp}(F, A)$.

First of all, since any fuzzy structure (H, μ) can be seen as a soft structure $(F, [0, 1])$, by considering $F(\alpha)$ equal to the α-level set for μ, it follows that any fuzzy Γ-hypermodule is a soft Γ-hypermodule.

Example 3.1. Let us consider the ring $R = Z_2$, $\Gamma = \{\cdot\}$ and let L be a modular lattice with 0. If we define the next hyperoperation on L :

$$x \oplus y = \{z \mid z \vee y = z \vee x = x \vee y\},$$

then $(L, \oplus)$ is a canonical hypergroup, in which the scalar identity is 0 and the opposite of any element x of L is just x. (see [11], page 129).

Define $\hat{1} \cdot x = x$ and $\hat{0} \cdot x = 0 \in L$. Then L is a Γ-hypermodule.

Consider now $A = L$ and define $F(x)$ as the ideal $I(x)$ generated by x, that means $I(x) = \{u \in L \mid u \leq x\}$.

Now, notice that any ideal I of the lattice L is a Γ-subhypermodule. Indeed, if $x, y \in I$, then for all $z \in x \oplus y$ we have $z \leq x \vee y$ and since $x \vee y \in I$, it follows that $z \in I$. Hence $x \oplus y \subseteq I$. On the other hand, if $r \in Z_2$ and $x \in I$, then $r \cdot x \in \{x, 0\} \subseteq I$. Then (F, L) is a soft Γ-hypermodule.

Moreover, in the above example, notice that if I is a Γ-subhypermodule of L, then it is an ideal of the lattice L. Indeed, if $x, y \in I$, then $x \vee y \in x \oplus y \subseteq I$ and if $x \in I, u \in L$, such that $u \leq x$, then $u \in x \oplus x \subseteq I$, whence I is an ideal of the lattice L.

Definition 3.2. Let M_1 and M_2 be two Γ-hypermodules, (F, A) and (G, B) be soft Γ-hypermodules over M_1 and M_2, respectively, and $f : M_1 \to M_2$ and $g : A \to B$ be two functions. Then (f, g) is called a *soft* Γ*-hypermodule homomorphism* if the following conditions hold:

(1) f is a Γ-hypermodule homomorphism;
(2) g is a mapping;
(3) for all $x \in A$, $f(F(x)) = G(g(x))$.

If there is a soft Γ-hypermodule homomorphism (f, g) between (F, A) and (G, B), we say that (F, A) is soft Γ-hypermodule homomorphic to (G, B), denoted by $(F, A) \sim (G, B)$. Furthermore, if f is a monomorphism (epimorphism, isomorphism) and g is a injective (surjective, bijective) mapping, then (f, g) is called a soft monomorphism (epimorphism, isomorphism respectively), and (F, A) is soft monomorphic (epimorphic, isomorphic respectively) to (G, B). We use $(F, A) \cong (G, B)$ to denote that (F, A) is soft Γ-hypermodule isomorphic to (G, B).

If N is a normal Γ-subhypermodule of M, then we define the relation N^* by

$$xN^*y \Longleftrightarrow (x - y) \cap N \neq \emptyset.$$

This is a congruence relation on M.

Let N be a normal Γ-subhypermodule of M. Then, for $x, y \in N$, the following are equivalent:

(1) $(x - y) \cap N \neq \emptyset$,
(2) $x - y \subseteq N$,
(3) $y \in x + N$.

The class $x + N$ is represented by x and we denote it with $N^*(x)$. Moreover, $N^*(x) = N^*(y)$ if and only if $x \equiv y(modN)$. We can define M/N as follows:

$$M/N = \{N^*(x) | x \in M\}.$$

Define a hyperoperation $\boxplus$ and an operation $\odot_\alpha$ on M/N by
$N^*(x) \boxplus N^*(y) = \{N^*(z) | z \in N^*(x) \oplus N^*(y)\}$;
$r \odot_\alpha N^*(x) = N^*(r\alpha x)$, for all $r \in R$, $N^*(x) \in M/N$.
Then, $(M/N, \boxplus, \odot_\alpha)$ is a Γ-hypermodule, see [40].
The following proposition is straightforward.

Proposition 3.1. *Let N be a normal Γ-subhypermodule of M, and (F, A) be a soft Γ-hypermodule over M, then (F, A) is soft Γ-hypermodule epimorphic to $(F/N, A)$, where $(F/N)(x) = F(x)/N$ for all $x \in A$, and $N \subseteq F(x)$ for all $x \in Supp(F, A)$ (if $x \in A-$ $Supp(F, A)$, we mean that $(F/N)(x) = \varnothing$).*

Next, we establish three isomorphism theorems of soft Γ-hypermodules.

Theorem 3.1. *(First Isomorphism Theorem).Let M_1 and M_2 be two Γ-hypermodules, (F, A) and (G, B) be soft Γ-hypermodules over M_1 and M_2, respectively. If (f, g) is a soft Γ-hypermodule epimorphism from (F, A) to (G, B) with kernel N such that N is a normal Γ-subhypermodule of M_1 and $N \subseteq F(x)$ for all $x \in supp(F, A)$, then*

(1) $(F/N, A) \simeq (f(F), A)$;

(2) if g is bijective, then $(F/N, A) \simeq (G, B)$.

Proof. (1) It is clear that $(F/N, A)$ and $(f(F), A)$ are soft Γ-hypermodules over M_1/N and M_2, respectively.

Define $\overline{f} : M_1/N \to M_2$ by $\overline{f}(N^*[x]) = f(x)$, for all $x \in M_1$. If xN^*y, we have $(x - y) \cap N \neq \emptyset$, that is, there exists $z \in (x - y) \cap N$. Hence $f(z) = 0$ and $f(z) \in f(x) - f(y)$. It follows that $f(x) = f(y)$. So $\overline{f}$ is well-defined.

Since f is surjective, it is clear that $\overline{f}$ is surjective. To show that $\overline{f}$ is injective, assume that $f(x) = f(y)$, then we have $0 \in f(x - y)$. Thus, there exists $z \in x - y$ such that $z \in ker f$. It follows that $(x - y) \cap N \neq \emptyset$, which implies $N^*[x] = N^*[y]$. Therefore $\overline{f}$ is injective. Furthermore, we have

(1) $\overline{f}(N^*[x] \boxplus N^*[y]) = \overline{f}(\{N^*[z] \mid z \in N^*[x] \oplus N^*[y]\}) = \{f(z) \mid z \in N^*[x] \oplus N^*[y]\} = f(N^*[x]) \oplus f(N^*[y]) = f(x) \oplus f(y) = \overline{f}([N^*[x]) \oplus \overline{f}(N^*[y])$,

(2) $\overline{f}(r \odot_\alpha N^*[x]) = \overline{f}(N^*[r\alpha x]) = f(r\alpha x) = r\alpha f(x) = r\alpha\overline{f}(N^*[x])$.

Thus, $\overline{f}$ is a Γ-hypermodule isomorphism.

Define $\overline{g} : A \to A$ by $\overline{g}(x) = x$ for all $x \in A$, then $\overline{g}$ is a bijective mapping. Furthermore, $\overline{f}(F(x)/N) = f(F(x)) = f(F(\overline{g}(x)))$ for all $x \in A$.

Therefore, $(\overline{f}, \overline{g})$ is a soft Γ-hypermodule isomorphism, and $(F/N, A) \cong (f(F), A)$.

(2) Since $\overline{f}$ is an isomorphism, g is bijective and for all $x \in A$, $\overline{f}(F(x)/N) = f(F(x)) = G(g(x))$. Hence, $(\overline{f}, g)$ is a soft Γ-hypermodule isomorphism. So we have $(F/N, A) \cong (G, B)$. □

Theorem 3.2. *(Second Isomorphism Theorem). Let N and K be two Γ-subhypermodules of M, with N normal in M. If (F, A) is a soft Γ-hypermodule of K, then we have*

$$(F/(N \cap K), A) \simeq ((N + F)/N, A),$$

where $N \cap K \subseteq F(x)$ for all $x \in supp(F, A)$.

Proof. It is clear that $(F/(N \cap K), A)$ and $((N + F)/N, A)$ are soft Γ-hypermodules over $(K/(N \cap K)$ and $(N + K)/N$, respectively.

Define $f : K \to (N + K)/N$ by $f(x) = N^*[x]$ for all $x \in K$. It is easy to check that f is a Γ-hypermodule homomorphism.

For any $N^*[x] \in (N + K)/N$, where $x \in N + K$, that is, there exist $a \in N$ and $b \in K$ such that $x \in a+b$, we have $N^*[x] = N+x = N+a+b = N+b = N^*[b] = f(b)$. Thus, f is a Γ-hypermodule epimorphism.

Define $g : A \to A$ by $g(x) = x$ for all $x \in A$. Then g is bijective.

We know $\{N^*[a] \mid a \in F(x)\} \subseteq (N + F(x))/N$. On the other hand, for any $N^*[b] \in (N + F(x))/N$, where $b \in N + F(x)$, there exist $n \in N$ and $k \in F(x)$ such that $b \in n + k$. We have $N^*[b] = N + b = N + n + k = N + k = N^*[k] \in$

$\{N^*[a] \mid a \in F(x)\}$, which implies, $(N + F(x))/N \subseteq \{N^*[a] \mid a \in F(x)\}$, and so $\{N^*[a] \mid a \in F(x)\} = (N + F(x))/N$. For all $x \in A$, we have $f(F(x)) = \{N^*[a] \mid a \in F(x)\} = (N + F(x))/N = (N + F(g(x)))/N$.

Therefore, (f, g) is a soft Γ-hypermodule epimorphism from (F, A) to $((N + F)/N, A)$.

Since $N \cap K$ is a Γ-hypermodule of K, we have $kerf = N \cap K$. In fact, for any $x \in K$, $x \in kerf \Leftrightarrow f(x) = N^*[0] = N \Leftrightarrow N^*[x] = N + x = N \Leftrightarrow x \in N$ (since $x \in K$)$\Leftrightarrow x \in N \cap K$. Hence $kerf = N \cap K$.

Therefore, it follows from Theorem 3.1 that $(F/(N \cap K), A) \cong ((N+F)/N, A)$. □

Theorem 3.3. *(Third Isomorphism Theorem). Let N and K be two normal Γ-subhypermodules of M such that $N \subseteq K$. If (F, A) is a soft Γ-hypermodule over M, and $K \subseteq F(x)$ for all $x \in supp(F, A)$, then we have*

$$((F/N)/(K/N), A) \simeq (F/K, A).$$

Proof. Since K and N are normal Γ-subhypermodules of M, and $N \subseteq K$, we know that K/N is a Γ-hypermodule of M/N, and so $(M/N)/(K/N)$ is well-defined.

Furthermore, we can deduce easily that $(F/N, A)$, $(F/K, A)$ and $((F/N)/(K/N), A)$ are soft Γ-hypermodules over R/N, R/K and $(M/N)/(K/N)$, respectively.

Define $f : M/N \to M/K$ by $f(N^*[x]) = K^*[x]$. It is clear that f is a Γ-hypermodule epimorphism.

We define $g : A \to A$ by $g(x) = x$ for all $x \in A$, then g is bijective.

Furthermore, for all $x \in A$, $f(F(x)/N) = F(x)/K = F(g(x))/K$.

Consequently, (f, g) is a soft Γ-hypermodule epimorphism from $(F/N, A)$ to $(F/K, A)$.

We show that $kerf = K/N$. In fact, for any $N^*[x] \in M/N$, $N^*[x] \in kerf \Leftrightarrow f(N^*[x]) = K^*[0] = K \Leftrightarrow K^*[x] = K + x = K \Leftrightarrow x \in K \Leftrightarrow N^*[x] \in K/N$. Thus, we have $kerf = K/N$.

Therefore, it follows from Theorem 3.1 that $((F/N)/(K/N), A) \simeq (F/K, A)$. □

4. **Fuzzy Isomorphism Theorems**

Let μ be a normal fuzzy Γ-subhypermodule of M. Define the relation on M:

$$x \equiv y (mod\ \mu)$$

if and only if there exists $r \in x - y$ such that $\mu(r) = \mu(0)$, denoted by $x\mu^*y$. The relation μ^* is an equivalence relation. If $x\mu^*y$, then $\mu(x) = \mu(y)$.

Let $\mu^*[x]$ be the equivalence class containing the element $x \in M$, and M/μ be the set of all equivalence classes, i.e., $M/\mu = \{\mu^*[x] \mid x \in M\}$. Define the following two operations in M/μ:

$$\mu^*[x] \boxplus \mu^*[y] = \{\mu^*[z] \mid z \in \mu^*[x] \oplus \mu^*[y]\}, \quad r \odot_\alpha \mu^*[x] = \mu^*[r\alpha x].$$

Then we can easily obtain the following result:

Theorem 4.1. *$(M/\mu, \boxplus, \odot_\alpha)$ is a Γ-hypermodule.*

The following propositions are straightforward.

Proposition 4.1. *Let N be a normal Γ-subhypermodule of M, and μ be a normal fuzzy Γ-subhypermodule of M. If μ is restricted to N, then μ is a normal fuzzy Γ-subhypermodule of N, and N/μ is a normal Γ-subhypermodule of M/μ.*

Proposition 4.2. *If μ and ν are normal fuzzy Γ-subhypermodules of M, then so is $\mu \cap \nu$.*

If X and Y are two non-empty sets, $f : X \to Y$ is a mapping, and μ and ν are the fuzzy sets of X and Y, respectively, then the image $f(\mu)$ of μ is the fuzzy subset of Y defined by

$$f(\mu)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \{\mu(x)\} & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

for all $y \in Y$. The inverse image $f^{-1}(\nu)$ of ν is the fuzzy subset of X defined by $f^{-1}(\nu)(x) = \nu(f(x))$ for all $x \in X$.

Proposition 4.3. *Let M_1 and M_2 be two Γ-hypermodules, and $f : M_1 \to M_2$ be a Γ-hypermodule homomorphism. If μ and ν are normal fuzzy Γ-subhypermodules of M_1 and M_2, respectively, then*

(1) *$f(\mu)$ is a fuzzy Γ-subhypermodule of M_2;*
(2) *$f^{-1}(\nu)$ is a normal fuzzy Γ-subhypermodule of M_1;*
(3) *if f is a Γ-hypermodule epimorphism, then $f(f^{-1}(\nu)) = \nu$;*
(4) *if μ is a constant on $kerf$, then $f^{-1}(f(\mu)) = \mu$.*

Proposition 4.4. *Let μ be a normal fuzzy Γ-subhypermodule of M, then $M_\mu = \{x \in M \mid \mu(x) = \mu(0)\}$ is a normal Γ-subhypermodule of M.*

Theorem 4.2. *(First Fuzzy Isomorphism Theorem). Let M_1 and M_2 be two Γ-hypermodules, and (F, A) and (G, B) be soft Γ-hypermodules over M_1 and M_2, respectively. If (f, g) is a soft Γ-hypermodule epimorphism from (F, A) to (G, B) and μ is a normal fuzzy Γ-subhypermodule of M_1 with $(M_1)_\mu \supseteq kerf$, then*

(1) $(F/\mu, A) \simeq (f(F)/f(\mu), A)$, where $(F/\mu)(x) = F(x)/\mu$ for all $x \in A$;
(2) if g is bijective, then $(F/\mu, A) \simeq (G/f(\mu), B)$.

Proof. (1) Since (F, A) is a soft Γ-hypermodule over M_1, and μ is a normal fuzzy Γ-subhypermodule of M_1, $(F/\mu, A)$ is a soft Γ-hypermodule over M_1/μ. For all $x \in \mathrm{supp}(F, A)$, $f(F(x)) = G(g(x)) \neq \emptyset$ is a Γ-subhypermodule of M_2. It follows that $(f(F)/f(\mu), A)$ is a soft Γ-hypermodule over $M_2/f(\mu)$.

Define $\overline{f} : M_1/\mu \to M_2/f(\mu)$ by $\overline{f}(\mu^*[x]) = f(\mu)^*[f(x)]$, for all $x \in M_1$. If $\mu^*[x] = \mu^*[y]$, then $\mu(x) = \mu(y)$. Since $(M_1)_\mu \supseteq kerf$, μ is a constant on $kerf$. So we have $f^{-1}(f(\mu)) = \mu$. It follows that $f^{-1}(f(\mu))(x) = f^{-1}(f(\mu))(y)$, i.e.,

$f(\mu)(f(x)) = f(\mu)(f(y))$. Thus, $f(\mu)^*[(f(x))] = f(\mu)^*[(f(y))]$. So $\overline{f}$ is well-defined. Furthermore, we have

(i) $\overline{f}(\mu^*[x] \boxplus \mu^*[y]) = \overline{f}(\{\mu^*[z] \mid z \in \mu^*[x] \oplus \mu^*[y]\}) = \{f(\mu)^*[f(z)] \mid z \in \mu^*[x] \oplus \mu^*[y]\} = f(\mu)^*(f(\mu^*[x])) \oplus f(\mu)^*(f(\mu^*[y])) = \overline{f}(\mu^*[x]) \oplus \overline{f}(\mu^*[y])$;

(ii) $\overline{f}(r \odot_\alpha \mu^*[x]) = \overline{f}(\mu^*[r\alpha x]) = f(\mu)^*(f(r\alpha x)) = f(\mu)^*(r\alpha f(x)) = r\alpha f(\mu)^* ([f(x)]) = r\alpha\, \overline{f}(\mu^*[x])$.

Hence, $\overline{f}$ is a Γ-hypermodule homomorphism. Clearly, $\overline{f}$ is a Γ-hypermodule epimorphism. Now, we show that $\overline{f}$ is a Γ-hypermodule monomorphism. Let $f(\mu)^*[f(x)] = f(\mu)^*[f(y)]$, then we have $f(\mu)(f(x)) = f(\mu)(f(y))$, i.e.,

$(f^{-1}(f(\mu)))(x) = (f^{-1}(f(\mu)))(y)$, and so $\mu(x) = \mu(y)$. Furthermore, we have $\mu^*[x] = \mu^*[y]$. Therefore, $\overline{f}$ is a Γ-hypermodule isomorphism.

Define $\overline{g} : A \to A$ by $\overline{g}(x) = x$ for all $x \in A$, then $\overline{g}$ is a bijective mapping. Furthermore, for all $x \in A$, we have $\overline{f}(F(x)/\mu) = \{f(\mu)^*[a] \mid a \in f(F(x))\} = f(F(x))/f(\mu) = f(F(\overline{g}(x)))/f(\mu)$. Consequently, $(\overline{f}, \overline{g})$ is a soft Γ-hypermodule isomorphism. So we have $(F/\mu, A) \cong (f(F)/f(\mu), A)$.

(2) Since $\overline{f}$ is a Γ-hypermodule isomorphism, g is bijective and for all $x \in A$, $\overline{f}(F(x)/\mu) = \{f(\mu)^*[a] \mid a \in f(F(x))\} = f(F(x))/f(\mu) = G(g(x))/f(\mu)$. Hence, $(\overline{f}, g)$ is a soft Γ-hypermodule isomorphism. Furthermore, we have $(F/\mu, A) \cong (G/f(\mu), B)$. □

Corollary 4.1. *Let M_1 and M_2 be two Γ-hypermodules, and (F, A) and (G, B) be soft Γ-hypermodules over M_1 and M_2 respectively. If (f, g) is a soft Γ-hypermodule epimorphism from (F, A) to (G, B) and ν is a normal fuzzy Γ-subhypermodule of M_2, then we have*

(1) $(F/f^{-1}(\nu), A) \cong (f(F)/\nu, A)$;

(2) if g is bijective, then $(F/f^{-1}(\nu), A) \cong (G/\nu, B)$.

Now, we give the Second Fuzzy and Third Fuzzy Isomorphism Theorems.

Theorem 4.3. *(Second Fuzzy Isomorphism Theorem). Let (F, A) be a soft Γ-hypermodule over M. If μ and ν are two normal fuzzy Γ-subhypermodules with $\mu(0) = \nu(0)$, then we have*

$$(M_\mu/(\mu \cap \nu), A) \simeq ((M_\mu + M_\nu)/\nu, A).$$

Proof. We know ν and $\mu \cap \nu$ are two normal fuzzy Γ-subhypermodules of $M_\mu + M_\nu$ and M_μ, respectively. Thus $(M_\mu + M_\nu)/\nu$ and $M_\mu/(\mu \cap \nu)$ are both Γ-hypermodules.

Since (F, A) is a soft Γ-hypermodule over M, we can deduce that $(M_\mu/(\mu \cap \nu), A)$ and $((M_\mu + M_\nu)/\nu, A)$ are soft Γ-hypermodules over $M_\mu/(\mu \cap \nu)$ and $(M_\mu + M_\nu)/\nu$, respectively.

Define $f : M_\mu \to (M_\mu + M_\nu)/\nu$ by $f(x) = \nu^*[x]$, for all $x \in M_\mu$. It is easy to see that f is a Γ-hypermodule epimorphism. We check that $kerf = \mu \cap \nu$.

$kerf = \{x \in M_\mu | \ f(x) = \nu^*[0]\} = \{x \in M_\mu | \ \nu^*[x] = \nu^*[0]\} = \{x \in M_\mu | \ \nu(x) = \nu(0)\} = \{x \in M_\mu | \ \mu(x) = \mu(0) = \nu(0) = \nu(x)\} = \{x \in M_\mu | \ x \in M_\nu\} = \mu \cap \nu$.

This implies that f is a Γ-hypermodule isomorphism.

Define $g : A \to A$ by $g(x) = x$ for all $x \in A$, then g is bijective.

We show that $M_\mu(x)/\nu = (M_\mu + M_\nu)(x)/\nu$. In fact, clearly, $M_\mu(x)/\nu \subseteq (M_\mu + M_\nu)(x)/\nu$. SAt $\nu^*[a] \in (M_\mu + M_\nu)(x)/\nu$, where $a \in (M_\mu + M_\nu)(x)$, which implies that there exist $m \in M_\mu(x)$ and $n \in M_\nu(x)$ such that $a \in m + n$, there is $\alpha \in a - m \subseteq m + n - m \subseteq M_\nu(x)$, i.e., $\nu(\alpha) = \nu(0)$, and so we have $\nu^*[a] = \nu^*[m] \in M_\mu(x)/\nu$. Hence, for all $x \in A$, $f(M_\mu(x)/(\mu \cap \nu)) = M_\mu(x)/\nu = (M_\mu + M_\nu)(x)/\nu = (M_\mu + M_\nu)(g(x))/\nu$.

Therefore, (f, g) is a soft Γ-hypermodule epimorphism and $(M_\mu/\mu \cap \nu, A) \cong ((M_\mu + M_\nu)/\nu, A)$. □

Theorem 4.4. *(Third Fuzzy Isomorphism Theorem). Let (F, A) be a soft Γ-hypermodule over M. If μ and ν are two normal fuzzy Γ-subhypermodules with $\nu \leq \mu$, $\mu(0) = \nu(0)$ and $F_\mu(x) = M_\mu$ for all $x \in$Supp(F, A), then we have*

$$((F/\nu)/(F_\mu/\nu), A) \simeq (F/\mu, A).$$

Proof. We know that M_μ/ν is a normal Γ-subhypermodule of M/ν. Since (F, A) is a soft Γ-hypermodule over M, it follows that $(F/\nu, A)$, $((F/\nu)/(F_\mu/\nu), A)$ and $(F/\mu, A)$ are soft Γ-hypermodules over M/ν, $(M/\nu)/(M_\mu/\nu)$ and M/μ, respectively.

Define $f : M/\nu \to M/\mu$ by $f(\nu^*[x]) = \mu^*[x]$, for all $x \in M$. If $\nu^*[x] = \nu^*[y]$, for all $x, y \in M$, then there exists $r \in x - y$, such that $\nu(r) = \nu(0)$. Since $\nu \leq \mu$ and $\mu(0) = \nu(0)$, we have $\mu(r) \geq \nu(r) = \nu(0) = \mu(0)$, which implies that $\mu(r) = \mu(0)$, and so $\mu^*(x) = \mu^*(y)$. Hence, f is well-defined.

Moreover, we have

(i)$f(\nu^*[x] \boxplus \nu^*[y]) = f(\{\nu^*[z] \mid z \in \nu^*[x] \oplus \nu^*[y]\}) = \{\mu^*[z\,] | \; z \in \nu^*[x] \oplus \nu^*[y]\}$ $= \mu^*[\nu^*[x]] \boxplus \mu^*[\nu^*[y]] = \mu^*[x] \boxplus \mu^*[y] = f(\nu^*[x]) \boxplus f(\nu^*[y])$;

(ii)$f(r \odot_\alpha \nu^*[x]) = f(\nu^*[r\alpha x]) = \mu^*[r\alpha x] = r \odot_\alpha \mu^*[x] = r \odot_\alpha f(\nu^*[x])$.

Hence, f is a Γ-hypermodule homomorphism. Clearly, f is a Γ-hyperring epimorphism. Next, we show that $kerf = M_\mu/\nu$. In fact,

$kerf = \{\nu^*[x] \in M/\nu \mid f(\nu^*[x]) = \mu^*[0]\} = \{\nu^*[x] \in M/\nu \mid \mu^*[x] = \mu^*[0]\}$ $= \{\nu^*[x] \in M/\nu \mid \mu(x) = \mu(0)\} = \{\nu^*[x] \in M/\nu \mid x \in M_\mu\} = M_\mu/\nu$.

This implies that f is a Γ-hypermodule isomorphism.

Define $g : A \to A$ by $g(x) = x$ for all $x \in A$, then g is bijective. For all $x \in A$, $f(F(x)/\nu) = F(x)/\mu = F(g(x))/\mu$. Thus, (f, g) is a soft Γ-hypermodule isomorphism from $(F/\nu, A)$ to $(F/\mu, A)$.

Therefore, from Theorem 4.2 it follows that $((F/\nu)/(F_\mu/\nu), A) \simeq (F/\mu, A)$. □

5. **Conclusions**

In this paper, we investigate three isomorphism theorems and three fuzzy isomorphism theorems in the context of soft Γ-hypermodules.

In our future study of fuzzy structures of Γ-hypermodules, the following topics could be considered:

(1) To consider roughness of soft Γ-hypermodules;

(2) To establish probability soft Γ-hypermodules.
(3) To describe the fuzzy soft Γ-hypermodules and their applications.

Acknowledgements

This research is partially supported by a grant of National Natural Science Foundation of China (61175055), a grant of Natural Innovation Term of Higher Education of Hubei Province, China (T201109), and the Natural Science Foundation of Education Committee of Hubei Province, China (B20122904) and Innovation Term of Hubei University for Nationalities(MY2012T002).

REFERENCES

[1] *U. Acar, F. Koyuncu and B. Tanay*, Soft sets and soft rings, Comput. Math. Appl. **59** (2010), 3458-3463.
[2] *M. I. Ali, F. Feng, X. Liu, W.K. Min, M. Shabir*, On some new operations in soft set theory, Comput. Math. Appl. **57 (9)** (2009), 1547-1553.
[3] *R. Ameri, H. Hedayati, A. Molaee*, On fuzzy hyperideals of Γ-hyperrings, Iran J. Fuzzy Systems, **6**(2009), 47-60.
[4] *R. Ameri, R. Sadeghi*, Gamma modules, Ratio Math. **20**(2010), 127-147.
[5] *S.M. Anvariyeh, B. Davvaz*, Strongly transitive geometric spaces associated to hypermodules, Journal of Algebra **322** (2009), 1340-1359.
[6] *W.E. Barbes*, On the Γ-rings of Nobusawa, Pacific J. Math. **18**(1966), 411-422.
[7] *S. K. Bhakat, P. Das*, $(\in, \in \vee q)$-fuzzy subgroups, Fuzzy Sets and Systems **80**(1996), 359–368.
[8] *N. Cagman*, Serdar Enginoglu, Soft matrix theory and its decision making, Comput. Math. Appl. **59**(2010) ,3308-3314.
[9] *D. Chen, E.C.C. Tsang, D.S. Yeung, X. Wang*, The parameterization reduction of soft sets and its applications, Comput. Math. Appl. **49** (2005), 757-763.
[10] *P. Corsini*, Prolegomena of Hypergroup Theory, Aviani Editore, second edition, 1993.
[11] *P. Corsini, V. Leoreanu*, Applications of hyperstructure theory, Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, 2003.
[12] *I. Cristea, B. Davvaz*, Atanassov's intuitionistic fuzzy grade of hypergroups, Inform. Sci. **180**(2010), 1506-1517.
[13] *B. Davvaz, P. Corsini*, Redefined fuzzy H_v-submodules and many valued implications, Inform. Sci. **177**(2007), 865-875.
[14] *B. Davvaz, V. Leoreanu-Fotea*, Hyperrings Theory and Applications, Int. Acad. Press, USA, 2007.
[15] *B. Davvaz, J. Zhan, K.H. Kim*, Fuzzy Γ-hypernear-rings, Comput. Math. Appl. **59**(2010), 2846-2853.
[16] *B. Davvaz, J. Zhan, Y. Yin*, Fuzzy H_v-ideals in Γ-H_v-rings, Comput. Math. Appl. **61**(2011), 690-698.
[17] *T.K. Dutta, T. Chanda*, Structures of fuzzy ideals of Γ-rings, Bull. Malays. Math. Sci. Soc.**28(1)** (2005), 9-15.
[18] *F. Feng, Y.B. Jun, X. Liu, L. Li*, An adjustable approach to fuzzy soft set based decision making, J. Comput. Appl. Math. **234 (1)** (2010), 10-20.
[19] *Y.B. Jun*, On fuzzy prime ideals of Γ-rings, Soochow J. Math. **21(1)**(1995), 41-48.
[20] *Y.B. Jun*, Soft BCK/BCI-algebras, Comput. Math. Appl. **56** (2008), 1408-1413.

[21] *Y.B. Jun, C.Y. Lee*, Fuzzy Γ-rings, Pusan Kyongnam Math. J.(now, East Asian Math. J.) **8(2)**(1992), 163-170.

[22] *M. Krasner*, A class of hyperring and hyperfields, Int. J. Math. Math. Sci. **2**(1983), 307-312.

[23] *V. Leoreanu-Fotea*, Fuzzy hypermodules, Comput. Math. Appl. **57**(2009), 466-475.

[24] *V. Leoreanu-Fotea, P. Corsini*, Soft hypergroups, Critical Review. A publication of Society for Mathematics of Uncertainty, Creighton University - USA, July 2010, vol. IV, 81-97;

[25] *X. Ma, J. Zhan*, Fuzzy h-ideals in h-hemiregular and h-semisimple Γ-hemirings, Neural Comput. Appl. **19**(2010), 477-485.

[26] *X. Ma, J. Zhan, Y.B. Jun*, Some characterizations of regular and semisimple Γ-rings, Kyungpook Math. J. **50**(2010), 411-417.

[27] *X. Ma, J. Zhan, V. Leoreanu-Fotea*, On (fuzzy) isomorphism theorems of Γ-hyperrings, Comput. Math. Appl. **60**(2010), 2594-2600.

[28] *P.K. Maji, R. Biswas, A.R. Roy*, Fuzzy soft sets, J. Fuzzy Math. **9 (3)** (2001), 589-602.

[29] *P.K. Maji, R. Biswas, A.R. Roy*, Soft set theory, Comput. Math. Appl. **45 (4-5)** (2003), 555-562.

[30] *P.K. Maji, A.R. Roy, R. Biswas*, An application of soft sets in a decision making problem, Comput. Math. Appl. **44** (2002) ,1077-1083.

[31] *P. Majumdar, S.K. Samanta*, Generalized fuzzy soft sets, Comput. Math. Appl. **59 (4)** (2010), 1425-1432.

[32] *F. Marty*, Sur une generalization de la notion de groupe, in: 8th Congress Math. Scandianaves, Stockholm, 1934, pp. 45-49.

[33] *C. G. Massouros*, Free and cyclic hypermodules, Ann. Mat. Pure Appl. **150(5)** (1998), 153-166.

[34] *D. Molodtsov*, Soft set theory-First results, Comput. Math. Appl. **37 (4-5)** (1999), 19-31.

[35] *T. Vougiouklis*, Hyperstructures and their Representations, Hadronic Press Inc., Palm Harbor, USA, 1994.

[36] *J. Wang et al.*, Some new results on soft hypermodules, Ars combinatoria, **104**(2012), in press.

[37] *Y. Yin, B. Davvaz, J. Zhan*, A fuzzy view of Γ-hyperrings, Neural Comput & Applic, 2011, Doi. 10. 1007/s00521-010-0509-Y.

[38] *Y. Yin, J. Zhan, D. Xu, J. Wang*, The L-fuzzy hypermodules, Comput. Math. Appl. **59**(2010), 953-963.

[39] *L. A. Zadeh*, Fuzzy sets, Inform. Control **8** (1965), 338-353.

[40] *J. Zhan, I. Cristea*, Γ-hypermodules: Isomorphism theorems and Regular relations, U.P.B. Sci. Bull., Series A **73**(2011) 71-78.

[41] *J. Zhan, B. Davvaz, K.P. Shum*, Isomorphism theorems of hypermodules, Acta Math. Sinica (Chinese Ser.) **50**(2007), 909-914.

[42] *J. Zhan, B. Davvaz, K.P. Shum*, On fuzzy isomorphism theorems of hypermodules, Soft Comput. **11**(2007), 1053-1057.

[43] *J. Zhan, B. Davvaz, K. P. Shum*, A new view of fuzzy hypermodules, Acta Math. Sinica (English Ser.) **23**(2007), 1345-1356.

[44] *J. Zhan, B. Davvaz, K. P. Shum*, On probabilistic n-ary hypergroups, Inform. Sci. **180**(2010), 1159-1166.

[45] *J. Zhan, Y.B. Jun*, Soft BL-algebras based on fuzzy sets, Comput. Math. Appl. **59**(2010), 2037-2046.