A STUDY ON HYPERSURFACES OF META-GOLDEN RIEMANNIAN MANIFOLDS

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This paper focuses on hypersurfaces within meta-golden Riemannian manifolds, investigating the structures induced by the ambient manifold's meta-golden Riemannian structure. The study provides characterizations of invariant and non-invariant hypersurfaces and explores their geometric properties. Three distinct examples are also presented to illustrate the theoretical findings.

Keywords: Meta-golden Riemannian manifold, hypersurfaces, invariant structures, differential geometry.

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1. Introduction

In the fields of art, architecture, and aesthetics, the golden ratio is a standard. This circumstance prompted worldwide search for a variety of items that satisfy the golden ratio. Among these was the idea that the golden ratio can also be found in a logarithmic spiral. However, Barlett [4] recently demonstrated that this assertion is wrong. However, he demonstrated that the meta-golden chi ratio was perfect for a significant class of logarithmic spirals. Rectangles with golden ratio proportions are known to have a shorter side of 1 to a longer side of $\Phi = \frac{1+\sqrt{5}}{2} \approx 1,618$. Infinitely, this Φ rectangle can be separated into a square and another rectangle with a golden ratio. Using the meta-golden ratio Chi $\chi = \frac{1+\sqrt{4\Phi+5}}{2} \approx 1.355674$ a similar structure was constructed by Barlett [4] (see also Huylebrouck [2]).

Rich differential geometric structures, including almost complex and almost product structures, have yielded significant insights into the intrinsic and extrinsic geometry of submanifolds. Hypersurfaces play a critical role in this context, offering a window into higher-dimensional spaces underlying geometry. This study extends the framework of meta-golden Riemannian manifolds to include hypersurfaces, exploring their invariant and non-invariant properties.

There are multiple reasons why studying hypersurfaces is crucial in a variety of scientific and academic fields. In differential geometry, hypersurfaces are basic objects that give mathematicians the opportunity to study and comprehend the intrinsic and extrinsic geometry of higher-dimensional spaces. In this context, concepts like normals, tangents, and curvature are essential. With wide-ranging applications in computer science, physics, engineering, and many other scientific domains, hypersurface studies are essential to mathematics. Understanding hypersurfaces helps us build mathematical tools for real-world problem solving and advances our knowledge of the basic structures that underlie the physical world.

Manifolds with a polynomial structure defined by constant coefficients enable the formulation of several conclusions in classical algebra and geometry. Tensor fields, 1-forms, and

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reduced structures are some of the tools these manifolds provide to facilitate computations and proofs.Yano [5] defined an f-structure on a manifold, which is a generalization of complex and contact manifolds. Later, Goldberg and Yano [6] expanded on this structure and introduced the concept of polynomial structure on a manifold.
Based on the studies discussed earlier, Şahin and Şahin [1] proposed the concept of meta-golden Riemannian manifolds. These manifolds represent a novel category inspired by golden manifolds and the meta-golden ratio, offering a more expansive and versatile framework compared to the geometry of traditional golden manifolds.
In Riemannian and semi-Riemannian manifolds, various geometric structures lead to significant results when exploring the differential and geometric characteristics of submanifolds. Numerous researchers have examined manifolds with these differential geometric structures, [3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Şahin and Şahin [1] proposed a new type of manifold, called the meta-golden Riemannian manifold. This manifold was constructed using the concept of golden manifolds and the meta-golden ratio. This article is organized into fourth sections. In Section 2, the meta-golden structure and meta-golden manifolds were defined with the help of the golden ratio. In Section 3, some definitions and properties of meta-golden manifolds were provided and in finally section, presents a thorough analysis of hypersurfaces in meta-golden Riemannian manifolds. In particular, it provides characterizations for both invariant and non-invariant hypersurfaces in these manifolds. Lastly, three distinct examples are presented.

2. **Preliminaries**

Assume two positive integers, p and q. The family of metallic ratios includes the positive solution of equation $x^2 - px - q = 0$. The solution set is given by

$$\tilde{\phi} = \frac{p \mp \sqrt{p^2 + 4q}}{2}$$

and the elements of this set are known as (p, q)- metallic numbers. When $p = 1$ and $q = 1$ are substituted into the expression for the positive root of the metallic ratio, the result simplifies to $\tilde{\phi} = \frac{1+\sqrt{5}}{2}$, commonly recognized as the golden ratio. It was asserted that the logarithmic spiral met the golden ratio until 2019. Bartlett, however, showed that this argument was false and that the meta-golden-Chi ratio is properly satisfied by a significant class of logarithmic spirals. Based on this, Şahin and Şahin [1] used this ratio to describe a new class of manifolds. Şahin and Şahin [1] acquired

$$\dot{\chi} = \frac{1}{\tilde{\phi}} + \frac{1}{\dot{\chi}}$$

which implies that $\dot{\chi}^2 - \frac{1}{\tilde{\phi}}\dot{\chi} - 1 = 0$. Thus, the roots are given by

$$\frac{\frac{1}{\tilde{\phi}} \mp \sqrt{4 + \frac{1}{\tilde{\phi}^2}}}{2}.$$

The relationship between continuing fractions and the meta-golden Chi ratio $\dot{\chi}$ was established by Hylebrouck [2] in 2014. If we denote the positive and negative roots as,

$$\dot{\chi} = \frac{\frac{1}{\tilde{\phi}} + \sqrt{4 + \frac{1}{\tilde{\phi}^2}}}{2}$$

and

$$\ddot{\chi} = \frac{\frac{1}{\tilde{\phi}} - \sqrt{4 + \frac{1}{\tilde{\phi}^2}}}{2}$$

then,

$$\ddot{\chi} = \frac{1}{\tilde{\phi}} - \dot{\chi}.$$

It is clear that

$$\tilde{\phi}\dot{\chi}^2 = \tilde{\phi} + \dot{\chi}$$

and

$$\tilde{\phi}\ddot{\chi}^2 = \tilde{\phi} + \ddot{\chi}.$$

Inspired by the golden ratio, Hreţcanu and Crâşmăreanu [3] established the concept of almost golden manifolds in the following way. Given a differentiable real manifold $\tilde{\mathcal{M}}$, let $\tilde{\mathfrak{J}}$ be an endomorphism on it. If the following equality is true for any $\mathcal{X}$

$$\tilde{\mathfrak{J}}^2\mathcal{X} = \tilde{\mathfrak{J}}\mathcal{X} + \mathcal{X}, \tag{1}$$

then $\tilde{\mathfrak{J}}$ is a golden structure. Since $\tilde{g}$ is the Riemannian metric on $\tilde{\mathcal{M}}$, for any $\mathcal{X}, \mathcal{Y} \in \mathfrak{X}(\tilde{\mathcal{M}})$

$$\tilde{g}(\tilde{\mathfrak{J}}\mathcal{X}, \mathcal{Y}) = \tilde{g}(\mathcal{X}, \tilde{\mathfrak{J}}\mathcal{Y}), \tag{2}$$

then $(\tilde{g}, \tilde{\mathfrak{J}})$ is a golden Riemannian structure. $(\tilde{\mathcal{M}}, \tilde{g}, \tilde{\mathfrak{J}})$ is referred to as a golden Riemannian manifold. From (2) , we obtain

$$\tilde{g}(\tilde{\mathfrak{J}}\mathcal{X}, \tilde{\mathfrak{J}}\mathcal{Y}) = \tilde{g}(\tilde{\mathfrak{J}}\mathcal{X}, \mathcal{Y}) + \tilde{g}(\mathcal{X}, \mathcal{Y}). \tag{3}$$

3. Meta-Golden Manifolds

A novel class of manifolds inspired by meta-golden manifolds will be introduced in this section and will give basic definitions of these manifolds.

Definition 3.1. *Let $(\tilde{\mathcal{M}}, \tilde{\mathfrak{J}})$ be a golden manifold. If $\tilde{\mathcal{M}}$ have a endomorphism $\tilde{\beta}$ on $\tilde{\mathcal{M}}$ such that*

$$\tilde{\mathfrak{J}}\tilde{\beta}^2\mathcal{X} = \tilde{\mathfrak{J}}\mathcal{X} + \tilde{\beta}\mathcal{X}, \tag{4}$$

for all $\mathcal{X} \in \mathfrak{X}(\tilde{\mathcal{M}})$, $\tilde{\beta}$ is referred to an almost meta-golden structure and $(\tilde{\mathcal{M}}, \tilde{\mathfrak{J}})$ is a almost meta-golden manifold [1].

Theorem 3.1. *Suppose that $(\tilde{\mathcal{M}}, \tilde{\mathfrak{J}})$ is a golden manifold and $\tilde{\beta}$ is an endomorphism on $\tilde{\mathcal{M}}$ in this situation $\tilde{\beta}$ is almost meta-golden structure if and only if*

$$\tilde{\beta}^2 = \tilde{\mathfrak{J}}\tilde{\beta} - \tilde{\beta} + I, \tag{5}$$

where I stands for the map of identities, [1].

Definition 3.2. *Let $(\tilde{\mathcal{M}}, \tilde{\mathfrak{J}}, \tilde{g})$ be a golden Riemannian manifold and if the meta-golden structure $\tilde{\beta}$ on $\tilde{\mathcal{M}}$ aligns with the metric $\tilde{g}$,*

$$\tilde{g}(\tilde{\beta}\mathcal{X}, \mathcal{Y}) = \tilde{g}(\mathcal{X}, \tilde{\beta}\mathcal{Y}), \tag{6}$$

for all $\mathcal{X}, \mathcal{Y} \in \Gamma(T\tilde{\mathcal{M}})$, $(\tilde{\mathcal{M}}, \tilde{\mathfrak{J}}, \tilde{\beta}, \tilde{g})$ is referred to as a almost meta-golden Riemannian manifold, [1].
Also, we observe that $\tilde{g}$ is compatible with the golden structure $\tilde{\mathfrak{J}}$. Consequently, we obtain

$$\tilde{g}(\tilde{\beta}\mathcal{X}, \tilde{\beta}\mathcal{Y}) = \tilde{g}(\mathcal{X}, \tilde{\beta}^2\mathcal{Y}),$$

Using (5) *and* (2), *we have*

$$\tilde{g}(\tilde{\beta}\mathcal{X}, \tilde{\beta}\mathcal{Y}) = \tilde{g}(\tilde{\mathfrak{J}}\mathcal{X}, \tilde{\beta}\mathcal{Y}) - \tilde{g}(\mathcal{X}, \tilde{\beta}\mathcal{Y}) + \tilde{g}(\mathcal{X}, \mathcal{Y}). \tag{7}$$

Theorem 3.2. *Let $(\tilde{\mathcal{M}}, \tilde{\mathfrak{J}}, \tilde{\beta}, \tilde{g})$ be a Riemannian manifold that is almost meta-golden. If the Codazzi-like equation $(\nabla_{\tilde{\beta}\mathcal{X}}\tilde{\beta})\mathcal{Y} - \tilde{\beta}(\nabla_{\mathcal{X}}\tilde{\beta})\mathcal{Y} = 0$ holds true for all $\mathcal{X}, \mathcal{Y} \in \Gamma(T\tilde{\mathcal{M}})$, then $\tilde{\beta}$ is integrable* [1].

Theorem 3.3. *Let $(\tilde{\mathcal{M}}, \tilde{\mathfrak{J}}, \tilde{\beta}, \tilde{g})$ be an almost meta-golden Riemannian manifold. When $\nabla\tilde{\beta} = 0$, $\nabla\tilde{\mathfrak{J}} = 0$, the definition of the Nijenhuis tensor field of $\tilde{\beta}$ is as follows:*

$$\tilde{N}_{\tilde{\beta}}(\mathcal{X}, \mathcal{Y}) = \tilde{\beta}^2[\mathcal{X}, \mathcal{Y}] + [\tilde{\beta}\mathcal{X}, \tilde{\beta}\mathcal{Y}] - \tilde{\beta}[\mathcal{X}, \tilde{\beta}\mathcal{Y}] - \tilde{\beta}[\tilde{\beta}\mathcal{X}, \mathcal{Y}],$$

for any $\mathcal{X}, \mathcal{Y} \in \Gamma(T\tilde{\mathcal{M}})$. $\tilde{\beta}$ is integrable if the Nijenhuis tensor field $\tilde{N}_{\tilde{\beta}}$ disappears. In this situation, a meta-golden Riemannian manifold is defined as $(\tilde{\mathcal{M}}, \tilde{\mathfrak{J}}, \tilde{\beta}, \tilde{g})$ [1].

Corollary 3.1. *Consider the Riemannian manifold $(\tilde{\mathcal{M}}, \tilde{\mathfrak{J}}, \tilde{\beta}, \tilde{g})$ to be almost meta-golden. A meta-golden manifold is $(\tilde{\mathcal{M}}, \tilde{\mathfrak{J}}, \tilde{\beta}, \tilde{g})$ if $\nabla\tilde{\beta} = 0$ and the meta-golden structure is integrable* [1].

Proposition 3.1. *At every point p on $\tilde{\mathcal{M}}$, the almost meta-golden structure $\tilde{\beta}$ defines an isomorphism on its tangent space* [1].

Proposition 3.2. *Let $(\tilde{\mathcal{M}}, \tilde{\mathfrak{J}}, \tilde{\beta}, \tilde{g})$ be an almost meta-golden Riemannian manifold. Then,*

- *for the eigen value $\tilde{\phi}$ of the golden structure $\tilde{\mathfrak{J}}$, the eigen values of the meta-golden structure $\tilde{\beta}$ are $\dot{\chi}$ and $\ddot{\chi}$.*
- *for the eigen value $1 - \tilde{\phi}$ of the golden structure $\tilde{\mathfrak{J}}$, the eigen values of the meta-golden structure $\tilde{\beta}$ are*

$$\tilde{G}_m = \frac{\frac{1}{1-\tilde{\phi}} + \sqrt{4 + \frac{1}{(1-\tilde{\phi})^2}}}{2}$$

and

$$\tilde{G}_{\bar{m}} = \frac{\frac{1}{1-\tilde{\phi}} - \sqrt{4 + \frac{1}{(1-\tilde{\phi})^2}}}{2}.$$

[1].

4. Hypersurfaces of Meta-golden Riemannian Manifolds

Let $(\tilde{\mathcal{M}}, \tilde{\mathfrak{J}}, \tilde{\beta}, \tilde{g})$ represent an almost meta-golden Riemannian manifold and let $\mathcal{M}$ denote a hypersurface of $\tilde{\mathcal{M}}$. For any $\mathcal{X} \in \Gamma(T\mathcal{M})$, 1- forms $\mathbf{u}, \mathbf{v}$ and local unit normal vector field $\mathrm{N} \in \Gamma(T\mathcal{M}^{\perp})$, we compose

$$\tilde{\mathfrak{J}}\mathcal{X} = \mathfrak{J}\mathcal{X} + \mathbf{v}(\mathcal{X})\mathrm{N}, \tag{8}$$

$$\tilde{\mathfrak{J}}\mathrm{N} = \mathfrak{E} + b\mathrm{N}, \qquad b \in C^{\infty}(\tilde{\mathcal{M}}, \mathbb{R}), \tag{9}$$

Here, $\mathfrak{J}\mathcal{X} \in \Gamma(T\mathcal{M})$ and $\mathfrak{E}$ represents the tangential component of $(\tilde{\mathfrak{J}}\mathrm{N})$. Similarly,

$$\tilde{\beta}\mathcal{X} = \beta\mathcal{X} + \mathbf{u}(\mathcal{X})\mathrm{N}, \tag{10}$$

$$\tilde{\beta}\mathrm{N} = \mathcal{V} + c\mathrm{N}, \qquad c \in C^{\infty}(\tilde{\mathcal{M}}, \mathbb{R}) \tag{11}$$

Here, $\beta\mathcal{X} \in \Gamma(T\mathcal{M})$ and $\mathcal{V}$ represents the tangential component of $(\tilde{\beta}\mathrm{N})$.

From (8), (9) and (2),

$$v(\mathcal{X}) = \tilde{g}(\tilde{\mathfrak{J}}\mathcal{X}, \mathrm{N}) = \tilde{g}(\mathcal{X}, \tilde{\mathfrak{J}}\mathrm{N}) = g(\mathcal{X}, \mathfrak{E}).$$

Here, g denotes the metric induced on $\mathcal{M}$ given by $g = \tilde{g}|_{\mathcal{M}}$ When $\tilde{\mathfrak{J}}$ is applied to both sides (8), we get

$$\begin{aligned} \tilde{\mathfrak{J}}^2\mathcal{X} &= \tilde{\mathfrak{J}}\mathfrak{J}\mathcal{X} + \mathbf{v}(\mathcal{X})\tilde{\mathfrak{J}}\mathrm{N}, \\ \mathfrak{J}\mathcal{X} + \mathbf{v}(\mathcal{X})\mathrm{N} + \mathcal{X} &= \mathfrak{J}^2\mathcal{X} + \mathbf{v}(\mathfrak{J}\mathcal{X})\mathrm{N} + \mathbf{v}(\mathcal{X})\mathfrak{E} + b\mathbf{v}(\mathcal{X})\mathrm{N}. \end{aligned}$$

The tangential and normal components of the previous equation are equated, we have

$$\mathfrak{J}^2\mathcal{X} = \mathfrak{J}\mathcal{X} + \mathcal{X} - \mathbf{v}(\mathcal{X})\mathfrak{E}$$

and

$$\mathbf{v}(\mathfrak{J}\mathcal{X}) = (1-b)\mathbf{v}(\mathcal{X}).$$

In a similar manner, applying $\tilde{\mathfrak{J}}$ to both sides of equation (9),

$$\begin{aligned}\tilde{\mathfrak{J}}^2\mathrm{N} &= \tilde{\mathfrak{J}}\mathfrak{E} + b\tilde{\mathfrak{J}}\mathrm{N},\\ \mathfrak{E} + b\mathrm{N} + \mathrm{N} &= \mathfrak{J}\mathfrak{E} + \mathbf{v}(\mathfrak{E})\mathrm{N} + b\mathfrak{E} + b^2\mathrm{N},\end{aligned}$$

which implies

$$\mathfrak{J}\mathfrak{E} = (1-b)\mathfrak{E}$$

and

$$\mathbf{v}(\mathfrak{E}) = 1 + b - b^2.$$

Using equations (2), (3) and (8), we obtain the following,

$$g(\mathfrak{J}\mathcal{X},\mathcal{Y}) + g(\mathcal{X},\mathcal{Y}) = g(\mathfrak{J}\mathcal{X},\mathfrak{J}\mathcal{Y}) + \mathbf{v}(\mathcal{X})\mathbf{v}(\mathcal{Y}),$$

which implies

$$g(\mathfrak{J}\mathcal{X},\mathfrak{J}\mathcal{Y}) = g(\mathfrak{J}\mathcal{X},\mathcal{Y}) + g(\mathcal{X},\mathcal{Y}) - \mathbf{v}(\mathcal{X})\mathbf{v}(\mathcal{Y}).$$

Additionally, we have

$$\tilde{g}(\tilde{\mathfrak{J}}\mathcal{X},\mathcal{Y}) = g(\mathfrak{J}\mathcal{X},\mathcal{Y})$$

and

$$\tilde{g}(\mathcal{X},\tilde{\mathfrak{J}}\mathcal{Y}) = g(\mathcal{X},\mathfrak{J}\mathcal{Y})$$

This leads to $g(\mathfrak{J}X,Y) = g(X,\mathfrak{J}Y)$ by using equation (2).
Alternatively, by using $\tilde{\beta}$ on both sides of equation (10), we obtain

$$\begin{aligned}\tilde{\beta}^2\mathcal{X} &= \tilde{\beta}\beta\mathcal{X} + \mathbf{u}(\mathcal{X})\tilde{\beta}\mathrm{N},\\ \tilde{\mathfrak{J}}\tilde{\beta}\mathcal{X} - \tilde{\beta}\mathcal{X} + \mathcal{X} &= \beta^2\mathcal{X} + \mathbf{u}(\beta\mathcal{X})\mathrm{N} + \mathbf{u}(\mathcal{X})\mathcal{V} + c\mathbf{u}(\mathcal{X})\mathrm{N},\end{aligned}$$

which implies

$$\begin{aligned}&\mathfrak{J}\beta\mathcal{X} + \mathbf{v}(\beta\mathcal{X})\mathrm{N} + \mathbf{u}(\mathcal{X})\mathfrak{E} + b\mathbf{u}(\mathcal{X})\mathrm{N} - \beta\mathcal{X} - \mathbf{u}(\mathcal{X})\mathrm{N} + \mathcal{X}\\ &\qquad = \beta^2\mathcal{X} + \mathbf{u}(\beta)\mathrm{N} + \mathbf{u}(\mathcal{X})\mathcal{V} + c\mathbf{u}(\mathcal{X})\mathrm{N}\end{aligned} \tag{12}$$

through (8) and (9). The tangential and normal components of (12) can be equated to produce the following equalities;

$$\begin{aligned}\beta^2\mathcal{X} &= \mathfrak{J}\beta\mathcal{X} - \beta\mathcal{X} + \mathcal{X} + \mathbf{u}(\mathcal{X})\mathfrak{E} - \mathbf{u}(\mathcal{X})\mathcal{V},\\ \mathbf{v}(\beta\mathcal{X}) &= (1-b+c)\mathbf{u}(\mathcal{X}) + \mathbf{u}(\beta\mathcal{X}).\end{aligned}$$

When we apply $\tilde{\beta}$ to both sides of (11), we obtain

$$\begin{aligned}\tilde{\beta}^2\mathrm{N} &= \tilde{\beta}\mathcal{V} + c\tilde{\beta}\mathrm{N},\\ \mathfrak{J}\mathcal{V} + \mathbf{v}(\mathcal{V})\mathrm{N} + c\mathfrak{E} + bc\mathrm{N} - \mathcal{V} - c\mathrm{N} + \mathrm{N} &= \beta\mathcal{V} + \mathbf{u}(\mathcal{V})\mathrm{N} + c\mathcal{V} + c^2\mathrm{N}\end{aligned}$$

which provides

$$\begin{aligned}\beta\mathcal{V} &= \mathfrak{J}\mathcal{V} - \mathcal{V} + c\mathfrak{E} - c\mathcal{V},\\ \mathbf{u}(\mathcal{V}) &= \mathbf{v}(\mathcal{V}) + (bc - c + 1 - c^2),\end{aligned}$$

through the tangential and normal components being equalized.
And by using (6), (10), and (11)

$$\begin{aligned}\tilde{g}(\beta\mathcal{X} + \mathbf{u}(\mathcal{X})\mathrm{N},\mathcal{Y}) &= \tilde{g}(\mathcal{X},\beta\mathcal{Y} + \mathbf{u}(\mathcal{Y})\mathrm{N}),\\ g(\beta\mathcal{X},\mathcal{Y}) &= g(\mathcal{X},\beta\mathcal{Y})\end{aligned}$$

and

$$\begin{aligned} \tilde{g}(\beta\mathcal{X}+\mathbf{u}(\mathcal{X})\mathrm{N},\mathrm{N}) &= \tilde{g}(\mathcal{X},\mathcal{V}+c\mathrm{N}), \\ \mathbf{u}(\mathcal{X}) &= g(\mathcal{X},\mathcal{V}), \end{aligned}$$

are acquired. Then, utilizing (7), we obtain

$$\tilde{g}(\tilde{\beta}\mathcal{X},\tilde{\beta}\mathcal{Y}) = \tilde{g}(\tilde{\mathfrak{J}}\mathcal{X},\tilde{\beta}\mathcal{Y}) - \tilde{g}(\mathcal{X},\tilde{\beta}\mathcal{Y}) + \tilde{g}(\mathcal{X},\mathcal{Y}),$$

$$\begin{aligned} \tilde{g}(\beta\mathcal{X}+\mathbf{u}(\mathcal{X})\mathrm{N},\beta\mathcal{Y}+\mathbf{u}(\mathcal{Y})\mathrm{N}) &= \tilde{g}(\mathfrak{J}\mathcal{X}+\mathbf{v}(\mathcal{X})\mathrm{N},\beta\mathcal{Y}+\mathbf{u}(\mathcal{Y})\mathrm{N}) \\ &\quad -\tilde{g}(\mathcal{X},\beta\mathcal{Y}+\mathbf{u}(\mathcal{Y})\mathrm{N}) + \tilde{g}(\mathcal{X},\mathcal{Y}), \end{aligned}$$

$$g(\beta\mathcal{X},\beta\mathcal{Y}) + \mathbf{u}(\mathcal{X})\mathbf{u}(\mathcal{Y}) = g(\mathfrak{J}\mathcal{X},\beta\mathcal{Y}) + \mathbf{v}(\mathcal{X})\mathbf{u}(\mathcal{Y}) - g(\mathcal{X},\beta\mathcal{Y}) + g(\mathcal{X},\mathcal{Y}),$$

which gives

$$g(\beta\mathcal{X},\beta\mathcal{Y}) = g(\mathfrak{J}\mathcal{X},\beta\mathcal{Y}) - g(\mathcal{X},\beta\mathcal{Y}) + g(\mathcal{X},\mathcal{Y}) - \mathbf{u}(\mathcal{X})\mathbf{u}(\mathcal{Y}) + \mathbf{v}(\mathcal{X})\mathbf{u}(\mathcal{Y}).$$

Proposition 4.1. *Let $(\tilde{\mathcal{M}},\tilde{\mathfrak{J}},\tilde{\beta},\tilde{g})$ be an almost meta-golden Riemannian manifold and $\mathcal{M}$ be a hypersurface of $\tilde{\mathcal{M}}$. Then, there exists a structure $\tilde{\Pi} = (\mathfrak{J},\beta,\mathfrak{E},\mathcal{V},\mathbf{v},\mathbf{u},b,c,g)$ on $\mathcal{M}$ which fulfills the following properties;*

1. $\mathfrak{J}^2\mathcal{X} = \mathfrak{J}\mathcal{X} + \mathcal{X} - \mathbf{v}(\mathcal{X})\mathfrak{E}$,
2. $\mathbf{v}(\mathfrak{J}\mathcal{X}) = (1-b)\mathbf{v}(\mathcal{X})$,
3. $\mathfrak{J}\mathfrak{E} = (1-b)\mathfrak{E}$,
4. $\mathbf{v}(\mathfrak{E}) = b + 1 - b^2$,
5. $g(\mathfrak{J}\mathcal{X},\mathcal{Y}) = g(\mathcal{X},\mathfrak{J}\mathcal{Y}) \quad \mathbf{v}(\mathcal{X}) = g(\mathcal{X},\mathfrak{E})$,
6. $g(\mathfrak{J}\mathcal{X},\mathfrak{J}\mathcal{Y}) = g(\mathfrak{J}\mathcal{X},\mathcal{Y}) + g(\mathcal{X},\mathcal{Y}) - \mathbf{v}(\mathcal{X})\mathbf{v}(\mathcal{Y})$,
7. $\beta^2\mathcal{X} = \mathfrak{J}\beta\mathcal{X} - \beta\mathcal{X} + \mathcal{X} + \mathbf{u}(\mathcal{X})(\mathfrak{E} - \mathcal{V})$,
8. $\mathbf{v}(\beta\mathcal{X}) = (1-b+c)\mathbf{u}(\mathcal{X}) + \mathbf{u}(\beta\mathcal{X})$,
9. $\beta\mathcal{V} = \mathfrak{J}\mathcal{V} - \mathcal{V} + c(\mathfrak{E} - \mathcal{V})$,
10. $\mathbf{u}(\mathcal{V}) = 1 + c(b-1-c) + \mathbf{v}(\mathcal{V})$,
11. $\mathbf{u}(\mathcal{X}) = g(\mathcal{X},\mathcal{V})$,
12. $g(\beta\mathcal{X},\mathcal{Y}) = g(\mathcal{X},\beta\mathcal{Y})$,
13. $g(\beta\mathcal{X},\beta\mathcal{Y}) = g(\mathfrak{J}\mathcal{X},\beta\mathcal{Y}) - g(\mathcal{X},\beta\mathcal{Y}) + g(\mathcal{X},\mathcal{Y}) - \mathbf{u}(\mathcal{X})\mathbf{u}(\mathcal{Y}) + \mathbf{v}(\mathcal{X})\mathbf{u}(\mathcal{Y})$,

Here, $\mathcal{X},\mathcal{Y} \in \Gamma(T\mathcal{M})$, and g represent the restriction of $\tilde{g}$ to M, i.e., $g = \tilde{g}|_{\mathcal{M}}$.

Let $\mathcal{M}$ be a hypersurface of $(\tilde{\mathcal{M}},\tilde{\mathfrak{J}},\tilde{\beta},\tilde{g})$. The Gauss and Weingarten equations can be expressed as:

$$\bar{\nabla}_{\mathcal{X}}\mathcal{Y} = \nabla_{\mathcal{X}}\mathcal{Y} + \mathbf{h}(\mathcal{X},\mathcal{Y})\mathrm{N} \quad \nabla_{\mathcal{X}}\mathcal{Y} = -\mathcal{A}_{\mathrm{N}}\mathcal{X}$$

where $\mathcal{X} \in \Gamma(T\mathcal{M})$ and $\mathrm{N} \in \Gamma(T\mathcal{M}^{\perp})$. In this context, $\mathbf{h}$ represents the second fundamental form of $\mathcal{M}$; The shape operator $\mathcal{A}_{\mathrm{N}}$ associated with the normal direction N satisfies $g(\mathcal{A}_{\mathrm{N}}\mathcal{X},\mathcal{Y}) = \mathbf{h}(\mathcal{X},\mathcal{Y})$, and ∇ refers to the connection induced on the hypersurface $\mathcal{M}$. Suppose now that $\mathcal{M}$ is a hypersurface within a meta-golden Riemannian manifold $(\tilde{\mathcal{M}},\tilde{\mathfrak{J}},\tilde{\beta},\tilde{g})$. It is a well-established fact that $\bar{\nabla}\tilde{\beta} = 0$. Using equations (10) and (11), we proceed the followings;

$$\begin{aligned} \bar{\nabla}_{\mathcal{X}}\tilde{\beta}\mathcal{Y} &= \tilde{\beta}\bar{\nabla}_{\mathcal{X}}\mathcal{Y}, \\ \begin{pmatrix} \nabla_{\mathcal{X}}\beta\mathcal{Y} + \mathbf{h}(\mathcal{X},\beta\mathcal{Y})\mathrm{N} \\ +\mathcal{X}(\mathbf{u}(\mathcal{Y}))\mathrm{N} - \mathbf{u}(\mathcal{Y})\mathcal{A}_{\mathrm{N}}\mathcal{X} \end{pmatrix} &= \begin{pmatrix} \beta\nabla_{\mathcal{X}}\mathcal{Y} + \mathbf{u}(\nabla_{\mathcal{X}}\mathcal{Y})\mathrm{N} \\ +\mathbf{h}(\mathcal{X},\mathcal{Y})\mathcal{V} + c\mathbf{h}(\mathcal{X},\mathcal{Y})\mathrm{N}, \end{pmatrix} \end{aligned} \tag{13}$$

for all $\mathcal{X},\mathcal{Y} \in \Gamma(T\mathcal{M})$, the result is derived. If the tangential and normal components of equation (13) are equal, it follows that, $(\nabla_{\mathcal{X}}\mathbf{u})\mathcal{Y} = c\mathbf{h}(\mathcal{X},\mathcal{Y}) - \mathbf{h}(\mathcal{X},\beta\mathcal{Y})$, and $(\nabla_{\mathcal{X}}\beta)\mathcal{Y} = g(\mathcal{A}_{\mathrm{N}}\mathcal{X},\mathcal{Y})\mathcal{V} + \mathbf{u}(\mathcal{Y})\mathcal{A}_{\mathrm{N}}\mathcal{X}$.

Moreover, because $(\bar{\nabla}_{\mathcal{X}}\tilde{\beta})\mathrm{N} = 0$, it can be concluded that

$$\begin{aligned} \bar{\nabla}_{\mathcal{X}}\tilde{\beta}\mathrm{N} &= \tilde{\beta}\bar{\nabla}_{\mathcal{X}}\mathrm{N}, \\ \nabla_{\mathcal{X}}\mathcal{V} + \mathbf{h}(\mathcal{X},\mathcal{V})\mathrm{N} + \mathcal{X}(c)\mathrm{N} - c\mathcal{A}_{\mathrm{N}}\mathcal{X} &= -\beta\mathcal{A}_{\mathrm{N}}\mathcal{X} - \mathbf{u}(\mathcal{A}_{\mathrm{N}}\mathcal{X})\mathrm{N}, \end{aligned}$$

through the use of equations (10), (11) and the Gauss-Weingarten formulas. By comparing the tangential and normal parts of the final equation, we arrive at $\nabla_{\mathcal{X}}\mathcal{V} = c\mathcal{A}_{\mathrm{N}}\mathcal{X} - \beta\mathcal{A}_{\mathrm{N}}\mathcal{X}$ and $\mathcal{X}(c) = -\mathbf{u}(\mathcal{A}_{\mathrm{N}}\mathcal{X}) - \mathbf{h}(\mathcal{X},\mathcal{V})$.

It is a well-established fact that when $\bar{\nabla}\tilde{\beta} = 0$, it follows that $\bar{\nabla}\tilde{\mathfrak{J}} = 0$. Therefore, using equations (8) and (9) for a hypersurface $\mathcal{M}$ of a meta-golden Riemannian manifold $(\tilde{\mathcal{M}}, \tilde{\mathfrak{J}}, \tilde{\beta}, \tilde{g})$, we obtain $\bar{\nabla}_{\mathcal{X}}\tilde{\mathfrak{J}}\mathcal{Y} = \tilde{\mathfrak{J}}\bar{\nabla}_{\mathcal{X}}\mathcal{Y}$,

$$\begin{pmatrix} \nabla_{\mathcal{X}}\mathfrak{J}\mathcal{Y} + \mathbf{h}(\mathcal{X},\mathfrak{J}\mathcal{Y})\mathrm{N} \\ +\mathcal{X}(\mathbf{v}(\mathcal{Y}))\mathrm{N} - \mathbf{v}(\mathcal{Y})\mathcal{A}_{\mathrm{N}}\mathcal{X} \end{pmatrix} = \begin{pmatrix} \mathfrak{J}\nabla_{\mathcal{X}}\mathcal{Y} + \mathbf{v}(\nabla_{\mathcal{X}}\mathcal{Y})\mathrm{N} \\ +h(\mathcal{X},\mathcal{Y})\mathfrak{E} + b\mathbf{h}(\mathcal{X},\mathcal{Y})\mathrm{N}, \end{pmatrix} \tag{14}$$

which implies $(\nabla_{\mathcal{X}}\mathfrak{J})\mathcal{Y} = \mathbf{v}(\mathcal{Y})\mathcal{A}_{\mathrm{N}}\mathcal{X} + \mathbf{h}(\mathcal{X},\mathcal{Y})\mathfrak{E}$ and $(\nabla_{\mathcal{X}}\mathbf{v})\mathcal{Y} = -\mathbf{h}(\mathcal{X},\mathfrak{J}\mathcal{Y}) + b\mathbf{h}(\mathcal{X},\mathcal{Y})$, through the matching of equation (14)'s tangential and normal components. Alternatively, by applying equations(8), (9), and $\bar{\nabla}\tilde{\mathfrak{J}} = 0$, we obtain

$$\bar{\nabla}_{\mathcal{X}}\tilde{\mathfrak{J}}\mathrm{N} = \tilde{\mathfrak{J}}\bar{\nabla}_{\mathcal{X}}\mathrm{N},$$

$$\nabla_{\mathcal{X}}\mathfrak{E} + \mathbf{h}(\mathcal{X},\mathfrak{E})\mathrm{N} + \mathcal{X}(b)\mathrm{N} - b\mathcal{A}_{\mathrm{N}}\mathcal{X} = -\mathfrak{J}\mathcal{A}_{\mathrm{N}}\mathcal{X} - \mathbf{v}(\mathcal{A}_{\mathrm{N}}\mathcal{X})\mathrm{N},$$

which provides

$$\nabla_{\mathcal{X}}\mathfrak{E} = -\mathfrak{J}\mathcal{A}_{\mathrm{N}}\mathcal{X} + b\mathcal{A}_{\mathrm{N}}\mathcal{X},$$

and

$$\mathcal{X}(b) = -\mathbf{h}(\mathcal{X},\mathfrak{E}) - \mathbf{v}(\mathcal{A}_{\mathrm{N}}\mathbb{X}).$$

Consequently, we state:

Proposition 4.2. *Let $\mathcal{M}$ be a hypersurface embedded in a meta-golden Riemannian manifold $(\tilde{\mathcal{M}}, \tilde{\mathfrak{J}}, \tilde{\beta}, \tilde{g})$. The structure*

$$\tilde{\Pi} = (\mathfrak{J}, \beta, \mathfrak{E}, \mathcal{V}, \mathbf{v}, \mathbf{u}, b, c, g)$$

induced on $\mathcal{M}$ fulfills the following requirements:

1. $(\nabla_{\mathcal{X}}\beta)\mathcal{Y} = g(\mathcal{A}_{\mathrm{N}}\mathcal{X},\mathcal{Y})\mathcal{V} + \mathbf{u}(\mathcal{Y})\mathcal{A}_{\mathrm{N}}\mathcal{X}$,
2. $(\nabla_{\mathcal{X}}\mathbf{u})\mathcal{Y} = cg(\mathcal{A}_{\mathrm{N}}\mathcal{X},\mathcal{Y}) - g(\mathcal{A}_{\mathrm{N}}\mathcal{X},\beta\mathcal{Y})$,
3. $\nabla_{\mathcal{X}}\mathcal{V} = (c-\beta)\mathcal{A}_{\mathrm{N}}\mathcal{X}$,
4. $\mathcal{X}(c) = -2g(\mathcal{A}_{\mathrm{N}}\mathcal{X},\mathcal{V})$,
5. $(\nabla_{\mathcal{X}}\mathfrak{J})\mathcal{Y} = g(\mathcal{A}_{\mathrm{N}}\mathcal{X},\mathcal{Y})\mathfrak{E} + \mathbf{v}(\mathcal{Y})\mathcal{A}_{\mathrm{N}}\mathcal{X}$,
6. $(\nabla_{\mathcal{X}}\mathbf{v})\mathcal{Y} = bg(\mathcal{A}_{\mathrm{N}}\mathcal{X},\mathcal{Y}) - g(\mathcal{A}_{\mathrm{N}}\mathcal{X},\mathfrak{J}\mathcal{Y})$,
7. $\nabla_{\mathcal{X}}\mathfrak{E} = (b-\mathfrak{J})\mathcal{A}_{\mathrm{N}}\mathcal{X}$,
8. $\mathcal{X}(b) = -2g(\mathcal{A}_{\mathrm{N}}\mathcal{X},\mathfrak{E})$,

in this case, $\mathcal{X},\mathcal{Y} \in \Gamma(T\mathcal{M})$, and ∇ represents the Levi-Civita connection induced on $\mathcal{M}$.

Definition 4.1. *Consider $\mathcal{M}$ as a hypersurface within an almost meta-golden Riemannian manifold $(\tilde{\mathcal{M}}, \tilde{\mathfrak{J}}, \tilde{\beta}, \tilde{g})$. At any point $p \in \mathcal{M}$,*

1. *if $\tilde{\mathfrak{J}}(T_p\mathcal{M}) \subset T_p\mathcal{M}$ and $\tilde{\beta}(T_p\mathcal{M}) \subset T_p\mathcal{M}$, leading to $\tilde{\mathfrak{J}}\tilde{\beta}(T_p\mathcal{M}) \subset T_p\mathcal{M}$, then $\mathcal{M}$ referred to as an invariant hypersurface*
2. *if $\tilde{\mathfrak{J}}\tilde{\beta}(T_p\mathcal{M}) \not\subset T_p\mathcal{M}$, the hypersurface $\mathcal{M}$ is described as non-invariant.*

Let $\mathcal{M}$ be a hypersurface of an almost meta-golden Riemannian manifold $(\tilde{\mathcal{M}}, \tilde{\mathfrak{J}}, \tilde{\beta}, \tilde{g})$. Under these conditions, using equations $(8)-(11)$, it follows that $\mathbf{v} = 0$ (equivalently, $\mathfrak{E} = 0$) and $\mathbf{u} = 0$ (equivalently, $\mathcal{V} = 0$). Consequently, the following theorem is stated.

Theorem 4.1. *A hypersurface $\mathcal{M}$ within an almost meta-golden Riemannian manifold $(\tilde{\mathcal{M}}, \tilde{\mathfrak{J}}, \tilde{\beta}, \tilde{g})$ is classified as invariant if and only if the normal vector field of $\mathcal{M}$ serves as an eigenvector of $\tilde{\mathfrak{J}}$ and $\tilde{\beta}$, corresponding to the eigenvalues b and c, respectively.*

Theorem 4.2. *Consider $\mathcal{M}$ as a hypersurface of an almost meta-golden Riemannian manifold $(\tilde{\mathcal{M}}, \tilde{\mathfrak{J}}, \tilde{\beta}, \tilde{g})$. The hypersurface $\mathcal{M}$ is said to be invariant iff*

$$b = \frac{1 \pm \sqrt{5}}{2} \tag{15}$$

and

$$c = \frac{b - 1 \pm \sqrt{1 - 2(b+2) + b^2}}{2}. \tag{16}$$

Proof. Based on statements 4 and 5 Proposition 4.1, if $\mathcal{M}$ is an invariant hypersurface, it follows that

$$b + 1 - b^2 = 0$$

and

$$1 + c(b - 1 - c) = 0$$

which leads to

$$b = \frac{1 \pm \sqrt{5}}{2}$$

and

$$c = \frac{b - 1 \pm \sqrt{1 - 2(b+2) + b^2}}{2}.$$

In contrast, let's say that b and c are provided above. Therefore, using statements 4 and 5 of Proposition 3, we obtain $\mathbf{v} = 0$ and $\mathbf{u} = 0$, which demonstrates that $\mathcal{M}$ is an invariant hypersurface. □

Corollary 4.1. $(\tilde{\mathcal{M}}, \tilde{\mathfrak{J}}, \tilde{\beta}, \tilde{g})$ *be an almost meta-golden Riemannian manifold and $\mathcal{M}$ is a hypersurface of this manifold. As a result, $\mathfrak{J}$ and β are parallel relative to ∇.*

Theorem 4.3. *Consider $\mathcal{M}$ as a non-invariant hypersurface of a meta-golden Riemannian manifold $(\tilde{\mathcal{M}}, \tilde{\mathfrak{J}}, \tilde{\beta}, \tilde{g})$ with the induced structure*
$\tilde{\Pi} = (\mathfrak{J}, \beta, \mathfrak{E}, \mathcal{V}, \mathbf{v}, \mathbf{u}, b, c, g)$. *$\mathcal{M}$ is totally geodesic if and only if $\mathfrak{J}$ and β are parallel along $\mathcal{M}$ with respect to ∇.*

Proof. The evidence is clear from points 1 and 5 of Proposition 4.2. □

Consider $\mathcal{M}$ as a totally umbilical hypersurface of a meta-golden Riemannian manifold $(\tilde{\mathcal{M}}, \tilde{\mathfrak{J}}, \tilde{\beta}, \tilde{g})$, endowed with the induced structure
$\tilde{\Pi} = (\mathfrak{J}, \beta, \mathfrak{E}, \mathcal{V}, \mathbf{v}, \mathbf{u}, b, c, g)$. Given that $\mathcal{A} = \tilde{\lambda} I$, Proposition 4 gives us

$$\left.\begin{aligned}
(\nabla_{\mathcal{X}}\beta)\mathcal{Y} &= \tilde{\lambda}(g(\mathcal{X},\mathcal{Y})\mathcal{V} + \mathbf{u}(\mathcal{Y})\mathcal{X}),\\
(\nabla_{\mathcal{X}}\mathbf{u})\mathcal{Y} &= \tilde{\lambda}(cg(\mathcal{X},\mathcal{Y}) - g(\mathcal{X},\beta\mathcal{Y})),\\
\nabla_{\mathcal{X}}\mathcal{V} &= \tilde{\lambda}(c - \beta)\mathcal{X},\\
\mathcal{X}(c) &= -2\tilde{\lambda} g(\mathcal{X},\mathcal{V}),\\
(\nabla_{\mathcal{X}}\mathfrak{J})\mathcal{Y} &= \tilde{\lambda}(g(\mathcal{X},\mathcal{Y})\mathfrak{E} + \mathcal{V}(\mathcal{Y})\mathcal{X}),\\
(\nabla_{\mathcal{X}}\mathbf{v})\mathcal{Y} &= \tilde{\lambda}(bg(\mathcal{X},\mathcal{Y}) - g(\mathcal{X},\mathfrak{J}\mathcal{Y})),\\
\nabla_{\mathcal{X}}\mathfrak{E} &= \tilde{\lambda}(b - \mathfrak{J})\mathcal{X},\\
\mathcal{X}(b) &= -2\tilde{\lambda} g(\mathcal{X},\mathfrak{E}),
\end{aligned}\right\} \tag{17}$$

for all $\mathcal{X}, \mathcal{Y} \in \Gamma(T\mathcal{M})$.

Theorem 4.4. *Consider $\mathcal{M}$ as a totally umbilical hypersurface of a meta-golden Riemannian manifold $(\tilde{\mathcal{M}}, \tilde{\mathfrak{J}}, \tilde{\beta}, \tilde{g})$, endowed with the induced structure $\tilde{\Pi} = (\mathfrak{J}, \beta, \mathfrak{E}, \mathcal{V}, \mathbf{v}, \mathbf{u}, b, c, g)$. Under these conditions, $\mathfrak{J} = bI$ and $\beta = cI$, with b and c being constant functions.*

Proof. From the third, fourth and the last two equations in (17), the proof is finished.

On the other hand, let $\tilde{\mathcal{M}}$ be a hypersurface of a meta-golden Riemannian manifold such that $\tilde{\mathfrak{J}} = bI$ and $\beta = cI$. Then, it is clear from (17) that $\nabla_{\mathcal{X}}\mathcal{Y} = 0$ and $\nabla_{\mathcal{X}}\mathfrak{E} = 0$. □

Example 4.1. *Consider the Euclidean space $\mathbb{E}^5$, which has a almost golden structure $\tilde{\mathfrak{J}}$, as provided by*

$$\tilde{\mathfrak{J}} : \mathbb{E}^5 \rightarrow \mathbb{E}^5$$
$$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{y}_1, \mathbf{y}_2) \rightarrow (\tilde{\phi}\mathbf{x}_1, \tilde{\phi}\mathbf{x}_2, \tilde{\phi}\mathbf{x}_3, (1-\tilde{\phi})\mathbf{y}_1, (1-\tilde{\phi})\mathbf{y}_2).$$

We introduce a $(1,1)$ -tensor field $\tilde{\beta}$ on $(\mathbb{E}^5, \tilde{\mathfrak{J}})$ as follows.

$$\tilde{\beta} : \mathbb{E}^5 \rightarrow \mathbb{E}^5$$
$$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{y}_1, \mathbf{y}_2) \rightarrow (\dot{\chi}\mathbf{x}_1, \dot{\chi}\mathbf{x}_1, \dot{\chi}\mathbf{x}_1, -\hat{\chi}\mathbf{y}_1, -\hat{\chi}\mathbf{y}_2),$$

where $\hat{\chi} = \frac{\tilde{\phi}+\sqrt{4+\tilde{\phi}^2}}{2}$.
One can readily confirm that $\dot{\beta}$ constitutes a meta-metallic structure on $\mathbb{E}^5$. Consequently, the tuple $(\mathbb{E}^5, \tilde{\mathfrak{J}}, \tilde{\beta}, <,>)$ represents an almost meta-golden Riemannian manifold, where $<,>$ denotes the usual Euclidean metric on $\mathbb{E}^5$.

Now, let $\mathcal{M}$ be a hypersurface of $\mathbb{E}^5$ defined by $\mathbf{x}_1 = \mathbf{x}_2$. The tangent bundle $(T\mathcal{M})$ is generated by

$$\tilde{\mathcal{Z}}_1 = \frac{\partial}{\partial \mathbf{x}_1} + \frac{\partial}{\partial \mathbf{x}_2}, \quad \tilde{\mathcal{Z}}_2 = \frac{\partial}{\partial \mathbf{x}_3}, \quad \tilde{\mathcal{Z}}_3 = \frac{\partial}{\partial \mathbf{y}_1}, \quad \tilde{\mathcal{Z}}_4 = \frac{\partial}{\partial \mathbf{y}_2}.$$

In this situation, it is clear that $\tilde{\mathfrak{J}}(T\mathcal{M}) \subset (T\mathcal{M})$ and $\beta(T\mathcal{M}) \subset T\mathcal{M}$, suggesting that $\mathcal{M}$ is an invariant hypersurface within $\mathbb{E}^5$.

Example 4.2. *Let $\mathbb{R}^{n+\ell}$ represent the real space of dimension $n+\ell$, where the coordinates are given by $(\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}, \ldots, \mathbf{x}_{n+\ell})$. We stipulate*

$$\tilde{\mathfrak{J}}(\mathbf{x}_1, ..., \mathbf{x}_n, \mathbf{x}_{n+1}, ..., \mathbf{x}_{n+\ell}) = (\tilde{\phi}\mathbf{x}_1, ..., \tilde{\phi}\mathbf{x}_n, (1-\tilde{\phi})\mathbf{x}_{n+1}, ..., (1-\tilde{\phi})\mathbf{x}_{n+\ell})$$

and

$$\tilde{\beta}(\mathbf{x}_1, ..., \mathbf{x}_n, \mathbf{x}_{n+1}, ..., \mathbf{x}_{n+\ell}) = (\dot{\chi}\mathbf{x}_1, \dot{\chi}\mathbf{x}_2, ..., \dot{\chi}\mathbf{x}_n, -\hat{\chi}\mathbf{x}_{n+1}, -\dot{\chi}\mathbf{x}_{n+2}, ..., -\dot{\chi}\mathbf{x}_{n+\ell})$$

where $\tilde{\phi}$ denotes the golden ratio,

$$\dot{\chi} = \frac{\frac{1}{\tilde{\phi}} + \sqrt{4 + \frac{1}{\tilde{\phi}^2}}}{2}$$

and $\hat{\chi} = \dfrac{\tilde{\phi} \mp \sqrt{\tilde{\phi}^2 + 4}}{2}$. So we can write

$$\tilde{\mathfrak{J}}\tilde{\beta}^2(\mathbf{x}_1, ..., \mathbf{x}_n, \mathbf{x}_{n+1}, ..., \mathbf{x}_{n+\ell}) = (\dot{\chi}^2\tilde{\phi}\mathbf{x}_1, ..., \dot{\chi}^2\tilde{\phi}\mathbf{x}_n, \hat{\chi}^2(1-\tilde{\phi})\mathbf{x}_{n+1}, ..., \hat{\chi}^2(1-\tilde{\phi})\mathbf{x}_{n+\ell}),$$

therefore it is evident that

$$\tilde{\mathfrak{J}}\tilde{\beta}^2 = \tilde{\mathfrak{J}} + \tilde{\beta},$$

which indicate that $(\mathbb{R}^{n+\ell}, \tilde{\mathfrak{J}}, \tilde{\beta})$ is a manifold that is almost meta-golden. Furthermore, the standard inner product $<,>$ on $\mathbb{R}^{n+\ell}$ satisfies equation (2) (or equivalently (3)), which

implies that $(\mathbb{R}^{n+\ell}, \tilde{\mathfrak{J}}, \tilde{\beta}, <,>)$ *is an almost meta-golden Riemannian manifold.*
Let $\mathcal{S}^{n+\ell-1}(r)$ *denote the hypersurface of* $\mathbb{R}^{n+\ell}$*, characterized by*

$$\mathcal{S}^{n+\ell-1}(r) = \left\{ (\mathbf{x}_1, ..., \mathbf{x}_n, \mathbf{x}_{n+1}, ..., \mathbf{x}_{n+\ell}) : \sum_{i=1}^{n} \mathbf{x}_i^2 + \sum_{j=n+1}^{n+\ell} \mathbf{x}_j^2 = r^2 \right\} \subset \mathbb{R}^{n+\ell},$$

and the normal vector field of $\mathcal{S}^{n+\ell-1}(r)$ *at any point* $(\mathbf{x}_1, ..., \mathbf{x}_n, \mathbf{x}_{n+1}, ..., \mathbf{x}_{n+\ell}) \in \mathcal{S}^{n+\ell-1}(r)$ *is described by*

$$\tilde{\mathcal{N}} = \frac{1}{r}(\mathbf{x}_1, ..., \mathbf{x}_n, \mathbf{x}_{n+1}, ..., \mathbf{x}_{n+\ell}). \tag{18}$$

For each point $p = (x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k}) \in S^{n+k-1}(r)$*, a tangent vector* $(X_1, \dots, X_n, X_{n+1}, \dots, X_{n+k})$ *exists on the hypersphere. There is a tangent vector* $(\dot{\mathbb{X}}_1, ..., \dot{\mathbb{X}}_n, \dot{\mathbb{X}}_{n+1}, ..., \dot{\mathbb{X}}_{n+\ell})$ *on the hypersphere for each point* $\mathbf{p} = (\mathbf{x}_1, ..., \mathbf{x}_n, \mathbf{x}_{n+1}, ..., \mathbf{x}_{n+\ell}) \in \mathcal{S}^{n+\ell-1}(r)$ *iff*

$$\sum_{i=1}^{n} \mathbf{x}_i \dot{\mathbb{X}}_i + \sum_{j=n+1}^{n+\ell} \mathbf{x}_j \dot{\mathbb{X}}_j = 0. \tag{19}$$

Then, we write using (8) *and* (9).

$$\begin{aligned} \tilde{\mathfrak{J}}(\dot{\mathbb{X}}_i, \dot{\mathbb{X}}_j) &= \mathfrak{J}(\dot{\mathbb{X}}_i, \dot{\mathbb{X}}_j) + \mathbf{v}(\dot{\mathbb{X}}_i, \dot{\mathbb{X}}_j)\mathbb{N}, \\ \tilde{\mathfrak{J}}\tilde{\mathcal{N}} &= \mathfrak{E} + b\tilde{\mathcal{N}}, \end{aligned} \tag{20}$$

where $(\dot{\mathbb{X}}_i, \dot{\mathbb{X}}_j) = (\dot{\mathbb{X}}_1, ..., \dot{\mathbb{X}}_n, \dot{\mathbb{X}}_{n+1}, ..., \dot{\mathbb{X}}_{n+\ell}) \in T_p\mathcal{S}^{n+\ell-1}(r)$.
Since

$$\tilde{\mathfrak{J}}\tilde{\mathcal{N}} = \frac{1}{r}(\tilde{\phi}\mathbf{x}_1, ..., \tilde{\phi}\mathbf{x}_n, (1-\tilde{\phi})\mathbf{x}_{n+1}, ..., (1-\tilde{\phi})\mathbf{x}_{n+\ell}) \tag{21}$$

and $b =< \tilde{\mathfrak{J}}\tilde{\mathcal{N}}, \tilde{\mathcal{N}} >$*, then we get*

$$b = \frac{1}{r^2}(\tilde{\phi}\sum_{i=1}^{n} \mathbf{x}_i^2 + (1-\tilde{\phi})\sum_{j=n+1}^{n+\ell} \mathbf{x}_j^2). \tag{22}$$

From the previous equation, $\mathfrak{E} = \tilde{\mathfrak{J}}\tilde{\mathcal{N}} - b\tilde{\mathcal{N}}$*, we derive*

$$\mathfrak{E} = \frac{2\tilde{\phi}-1}{r^3}\big((\sum_{j=n+1}^{n+\ell} \mathbf{x}_j^2)\mathbf{x}_1, ..., (\sum_{j=n+1}^{n+\ell} \mathbf{x}_j^2)\mathbf{x}_n, -(\sum_{i=1}^{n} \mathbf{x}_i^2)\mathbf{x}_{n+1}, ..., -(\sum_{i=1}^{n} \mathbf{x}_i^2)\mathbf{x}_{n+\ell}\big), \tag{23}$$

it suggest that

$$\mathbf{v}(\dot{\mathbb{X}}_i, \dot{\mathbb{X}}_j) =< (\dot{\mathbb{X}}_i, \dot{\mathbb{X}}_j), \mathfrak{E} >= \frac{2\tilde{\phi}-1}{r^3}((\sum_{j=n+1}^{n+\ell} \mathbf{x}_j^2)(\sum_{i=1}^{n} \mathbf{x}_i \dot{\mathbb{X}}_i) - (\sum_{i=1}^{n} \mathbf{x}_i^2)(\sum_{j=n+1}^{n+\ell} \mathbf{x}_j \dot{\mathbb{X}}_j)).$$

If $\sum_{i=1}^{n} \mathbf{x}_i \dot{\mathbb{X}}_i = -\sum_{j=n+1}^{n+\ell} \mathbf{x}_j \dot{\mathbb{X}}_j = \tilde{\theta}$*, then the final equation can be expressed as follows*

$$\mathbf{v}(\dot{\mathbb{X}}_i, \dot{\mathbb{X}}_j) = \frac{2\tilde{\phi}-1}{r}\tilde{\theta}. \tag{24}$$

Utilizing (20) *and* (24)*, we obtain*

$$\mathfrak{J}(\dot{\mathbb{X}}_i, \dot{\mathbb{X}}_j) = (\tilde{\phi}\dot{\mathbb{X}}_i - \frac{2\tilde{\phi}-1}{r}\tilde{\theta}\mathbf{x}_i, (1-\tilde{\phi})\dot{\mathbb{X}}_j - \frac{2\tilde{\phi}-1}{r}\tilde{\theta}\mathbf{x}_j). \tag{25}$$

Moreover, we write utilizing (10) *and* (11)

$$\begin{aligned} \tilde{\beta}(\dot{\mathbb{X}}_i, \dot{\mathbb{X}}_j) &= \beta(\dot{\mathbb{X}}_i, \dot{\mathbb{X}}_j) + \mathbf{u}(\dot{\mathbb{X}}_i, \dot{\mathbb{X}}_j)\tilde{\mathcal{N}}, \\ \tilde{\beta}\tilde{\mathcal{N}} &= \mathcal{V} + c\tilde{\mathcal{N}}, \end{aligned} \tag{26}$$

since

$$\tilde{\beta}\tilde{\mathcal{N}} = \frac{1}{r}(\dot{\chi}\mathbf{x}_1, ..., \dot{\chi}\mathbf{x}_n, -\hat{\chi}\mathbf{x}_{n+1}, ..., -\hat{\chi}\mathbf{x}_{n+\ell}) \tag{27}$$

and $c =< \tilde{\beta}\tilde{\mathcal{N}}, \tilde{\mathcal{N}} >$, *we compute*

$$c = \frac{1}{r^2}(\dot{\chi}(\sum_{i=1}^{n}\mathbf{x}_i^2) - \hat{\chi}(\sum_{j=n+1}^{n+\ell}\mathbf{x}_j^2)). \tag{28}$$

By utilizing $\mathcal{V} = \tilde{\beta}\tilde{\mathcal{N}} - c\tilde{\mathcal{N}}$, *we acquire*

$$\mathcal{V} = \frac{\dot{\chi}+\hat{\chi}}{r^3}((\sum_{j=n+1}^{n+\ell}\mathbf{x}_j^2)\mathbf{x}_1, ..., (\sum_{j=n+1}^{n+\ell}\mathbf{x}_j^2)\mathbf{x}_n, -(\sum_{i=1}^{n}\mathbf{x}_i^2)\mathbf{x}_{n+1}, ..., -(\sum_{i=1}^{n}\mathbf{x}_i^2)\mathbf{x}_{n+\ell}), \tag{29}$$

it suggests that

$$\mathbf{u}(\dot{\mathbb{X}}_i, \dot{\mathbb{X}}_j) =< (\dot{\mathbb{X}}_i, \dot{\mathbb{X}}_j), \mathcal{V} >= \frac{\dot{\chi}+\hat{\chi}}{r^3}((\sum_{j=n+1}^{n+\ell}\mathbf{x}_j^2)(\sum_{i=1}^{n}\mathbf{x}_i\dot{\mathbb{X}}_i) - (\sum_{i=1}^{n}\mathbf{x}_i^2)(\sum_{j=n+1}^{n+\ell}\mathbf{x}_j\dot{\mathbb{X}}_j)).$$

Then, we obtain

$$\mathbf{u}(\dot{\mathbb{X}}_i, \dot{\mathbb{X}}_j) = \frac{\dot{\chi}+\hat{\chi}}{r}\tilde{\theta}. \tag{30}$$

With the help of (26) *and* (30), *we get*

$$\beta(\dot{\mathbb{X}}_i, \dot{\mathbb{X}}_j) = (\dot{\chi}\dot{\mathbb{X}}_i - \frac{\dot{\chi}+\hat{\chi}}{r^2}\tilde{\theta}\mathbf{x}_i, -\hat{\chi}\dot{\mathbb{X}}_j - \frac{\dot{\chi}+\hat{\chi}}{r^2}\tilde{\theta}\mathbf{x}_j). \tag{31}$$

Because of this, $\mathcal{S}^{n+\ell-1}$ *is a non-invariant hypersurface of the nearly meta-golden Riemannian manifold* $(\mathbb{R}^{n+\ell}, \tilde{\mathfrak{J}}, \tilde{\beta}, <,>)$ *that has the induced structure* $(\mathfrak{J}, \beta, \mathfrak{E}, \mathcal{V}, \mathbf{v}, \mathbf{u}, b, c)$ *provided by* $(23)-(25)$ *and* $(29)-(31)$.

Example 4.3. *Consider the Euclidean space* $\mathbb{E}^5$, *which has a structure that is almost golden.* $\tilde{\mathfrak{J}}$ *provided by*

$$\tilde{\mathfrak{J}} : \mathbb{E}^5 \to \mathbb{E}^5$$

$$(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}) \to \tilde{\mathfrak{J}}(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}) = \big(\frac{1}{2}\mathbf{x}_i + \frac{\sqrt{5}}{2}\mathbf{y}_i, \quad \frac{1}{2}\mathbf{y}_i + \frac{\sqrt{5}}{2}\mathbf{x}_i, \quad (1-\tilde{\phi})\mathbf{z}\big),$$

The golden ratio is $\tilde{\phi}$, *and the coordinate system on* E^5 *for i=1,2 is* $(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z})$. *On* $(\mathbb{E}^5, \tilde{\mathfrak{J}})$, *we define a* $(1,1)$ *tensor field* $\tilde{\beta}$ *by*

$$\tilde{\beta} : \mathbb{E}^5 \to \mathbb{E}^5$$

$$(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}) \to \tilde{\beta}(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}) = (-\frac{\hat{\chi}+\hat{\hat{\chi}}}{2}\mathbf{x}_i + \frac{\hat{\chi}-\hat{\hat{\chi}}}{2}\mathbf{y}_i, \quad \frac{\hat{\chi}-\hat{\hat{\chi}}}{2}\mathbf{x}_i - \frac{\hat{\chi}+\hat{\hat{\chi}}}{2}\mathbf{y}_i, \quad -\hat{\chi}\mathbf{z}),$$

where $\hat{\chi} = \frac{\tilde{\phi}+\sqrt{4+\tilde{\phi}^2}}{2}$ *and* $\hat{\hat{\chi}} = \frac{-(1-\tilde{\phi})\mp\sqrt{(1-\tilde{\phi})^2+4}}{2}$.
The meta-golden structure of $\tilde{\beta}$ *in* $\mathbb{E}^5$ *is readily apparent and therefore* $(\mathbb{E}^5, \tilde{\mathfrak{J}}, \tilde{\beta}, <,>)$ *is an essentially meta-golden Riemannian manifold, and the standard Euclid metric on* $\mathbb{E}^5$ *is* $<,>$.
We now examine a hypersurface $\mathcal{M}$ *of* $\mathbb{E}^5$, *which is found using the formula* $\mathbf{y}_1 = \mathbf{y}_2{}^2$. *Then* $T\mathcal{M}$ *is spanned by*

$$\tilde{\mathcal{Z}}_1 = \frac{\partial}{\partial \mathbf{x}_1}, \quad \tilde{\mathcal{Z}}_2 = \frac{\partial}{\partial \mathbf{x}_2}, \quad \tilde{\mathcal{Z}}_3 = 2\mathbf{y}_2\frac{\partial}{\partial \mathbf{y}_1} + \frac{\partial}{\partial \mathbf{y}_2}, \quad \tilde{\mathcal{Z}}_4 = \frac{\partial}{\partial \mathbf{z}}$$

and $T\mathcal{M}^{\perp}$ *is spanned by* $\tilde{N} = \frac{\partial}{\partial \mathbf{y}_1} - 2\mathbf{y}_2\frac{\partial}{\partial \mathbf{y}_2}$.
$\tilde{\mathfrak{J}}\tilde{\beta}(T\mathcal{M}) \not\subset T\mathcal{M}$ *is clearly visible in this instance, suggesting that* $\mathcal{M}$ *is a non-invariant hypersurface of* $\mathbb{E}^3$.

5. **Conclusion**

Meta-Golden structures offer a novel framework for reinterpreting the Einstein field equations under alternative geometric settings. In particular, they facilitate the construction of new classes of symmetric solutions in space-time models defined with either torsion-free or torsion-rich connections, such as those appearing in modified theories of gravity.

Moreover, in contemporary applied domains such as information geometry, Riemannian optimization, and manifold learning, meta-Golden structures provide a rich and alternative geometric modeling framework for data spaces. Within this context, aligning the Riemannian curvature tensor with the meta-Golden structural tensor may yield significant results, especially in geometries where these tensors evolve compatibly, such as in Einstein-like manifolds.

Such compatibility conditions can serve as powerful tools for characterizing whether a given manifold belongs to specific distinguished classes, including Einstein, Sasakian, or K-contact manifolds.

In conclusion, meta-Golden Riemannian manifolds hold substantial research potential not only within the theoretical scope of differential geometric structure theory but also across various applied fields such as physics, data science, and computational geometry.

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