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COMPLETENESS OF HAMILTONIAN VECTOR FIELDS IN JACOBI AND CONTACT GEOMETRY

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Lucrarea prezintă condiţii suficiente de completitudine a câmpurilor vectoriale Hamiltoniene utilizând o proprietate topologică a Hamiltonianului corespunzător. În particular se studiază cazul geometriilor Poisson, contact şi cosimplectic precum şi cel al varietăţilor Nambu-Poisson. Ca aplicaţii, se discută completitudinea a două câmpuri vectoriale Hamiltoniene de tip contact ce apar în geometrizarea termodinamicii.

The completeness of the Hamiltonian vector fields in the Jacobi manifolds is studied here providing a sufficient condition in terms of the topological properness for a function assuring a sublinear growth along the flow. In particular, the settings of Poisson, contact and cosymplectic geometries are presented while for similarities with the Poisson case, the Nambu-Poisson structures are included too. As applications, the completeness of contact-Hamiltonian vector fields arising in the geometrization of thermodynamics is discussed with examples.

Keywords: complete vector field, Jacobi structure, Hamiltonian vector field, first integral, proper function, Poisson (Nambu-Poisson) structure, contact structure, Reeb vector field, thermodynamics.
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1. Introduction

This paper is dedicated to a study of the completeness of the Hamiltonian vector fields in a special type of structures namely Jacobi manifolds which together with Poisson manifolds are introduced exactly thirty years ago by André Lichnerowicz in [21] and [22]. Since then, these structures become a main tool in several studies regarding the geometrization of mechanics; for a good picture the reader is invited to browse the papers of Lichnerowicz and his co-workers from our bibliography: [8], [13], [23]-[28], [29], as well as some surveys like [19] and [37]. Recently, in addition to the well-known relation of the Jacobi structures with the classical mechanics, the quantization of these mathematical objects was discussed in [17]. A constant interest is in the connection between the Jacobi structures and the theory of Lie algebroids (and generalizations) as appears for example in [15], as well as the computation of a suitable cohomology called Lichnerowicz-Jacobi cohomology, [18].

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The present paper is devoted to another subject namely the completeness of the Hamiltonian vector fields which appear in the Jacobi geometry on a manifold M. This question can be of main interest in some applications of the Jacobi structures to concrete mathematical or physical theories. More precisely, from a mathematical point of view, such a complete flow induces an action of Lie group $\mathbb{R}$ and the symmetries of this action can provide useful information about our structure; for example, if the orbit space $M/\mathbb{R}$ is again a manifold, a natural question is about some versions of the well-known *Marsden-Weinstein-Raţiu reduction theory* like in [1] and [30]. Also, very important geometrical objects on a manifold are the Riemannian metrics and the great importance of completeness in the Riemannian geometry is pointed out in Chapter 7 of classical by now [7]; a class of Riemannian metric naturally associated to the contact structures is added also in our study through an appendix. From a physical point of view completeness corresponds to well-defined dynamics persisting eternally but as is point out in [4, p. 60] "in some circumstances (shock waves in fluids and solids, singularities in general relativity) one has to live with incompleteness".

The contents of the paper is as follows. The first section begins by reviewing the general notions regarding the completeness and a sufficient condition is recalled after [1, p. 71]; see [11] and [38] for related results. The completeness of the gradient vector fields in Riemannian geometry and Euler-Lagrange vector fields of the classical mechanics is discussed, the last case in connection with the celebrated Poincaré Recurrence Theorem.

In the next section the Jacobi setting is studied in details including local expressions for the main geometrical objects. A generalization of the notion of first integral is introduced toward study the completeness of the associated Hamiltonian vector field and a connection with the theory of complete Poisson maps introduced in [6] is pointed out via proper maps. The case of Nambu-Poisson brackets, although does not belongs to Jacobi structures (but to [14]), ends this section since there exists a strong similarity with the Poisson case, namely the properness of a Hamiltonian.

The next section is devoted to the contact and cosymplectic manifolds. Using an adapted atlas of Darboux type we consider the class of functions previously introduced and then the completeness of the Hamiltonian-contact and Hamiltonian-cosymplectic vector fields for these functions, particularly the Reeb vector field, is discussed including two examples connected with the symplectic geometry and geometric theory for PDEs respectively. An important example of this section is the Reeb vector field of a contact manifold admitting a Legendre foliation in which the charts obtained by Paulette Libermann are used instead of the canonical Darboux charts.

Since the contact structure is a main tool in the geometrization of the thermodynamics we end this paper with a connection of our results with this physical theory. For our examples, inspired by [31], we add the expression of the flow, appearing also in the cited paper, in order to verify the completeness.

Two open problems are rising: one concerning the notion of regular first integral and the second regarding the universal completition of a Jacobi manifold. An appendix discussing the completeness in terms of a complete associated Riemannian metric (to a contact structure) ends the paper.

2. **Completeness of general and Euler-Lagrange vector fields**

Let M be a smooth, real, n-dimensional manifold. Let us denote by:
· $C^\infty(M)$ the ring of smooth real functions on M,
· $\mathcal{X}(M)$ the $C^\infty(M)$-module of vector fields on M,
· $\mathcal{X}^k(M)$ the $C^\infty(M)$-module of k-multivector fields on M; in particular $\mathcal{X}^1(M)$ is exactly $\mathcal{X}(M)$,
· $\Omega^k(M)$ the $C^\infty(M)$-module of differential k-forms on M; in particular $\Omega^0(M)$ is exactly $C^\infty(M)$,
· if $\Lambda \in \mathcal{X}^2(M)$ is a bivector field on M then we associate the map
$\Lambda^\sharp : \Omega^1(M) \to \mathcal{X}(M), \alpha \to \Lambda(\alpha, \cdot)$; so $\Lambda^\sharp(\alpha)(f) = \Lambda(\alpha, df)$ for $f \in C^\infty(M)$.

Definition 2.1 i) $X \in \mathcal{X}(M)$ is a *complete vector field* if for every $x_0 \in M$ the maximal interval of existence (t_-, t_+) for the solution of the flow equation of X with initial condition $x(0) = x_0$ is given by $t_\pm = \pm\infty$.
ii) $f \in C^\infty(M)$ is a *first integral* of $X \in \mathcal{X}(M)$ if $X(f) \equiv 0$.
iii) $f \in C^\infty(M)$ is a *proper function if* f^{-1}(compact) =compact.

Let us remark that Proposition 5.11. from [10, p. 25] assures that on every manifold M there exist proper functions. Also, in [16] it is proved that for any manifold with a vector field there exists an universal completion to a manifold with complete vector field. A sufficient condition of completeness is provided by [1, p. 71]:

Theorem 2.2 *Let $X \in \mathcal{X}(M)$. If there exist $f \in C^\infty(M)$ with f proper and $A, B \in \mathbb{R}_+$ such that for each $x \in M$ we have*:

$$|X(f)(x)| \leq A|f(x)| + B, \tag{2.1}$$

then X is complete.

This has the following consequence for $A = 0$:

Corollary 2.3 *If $X(f)$ is a bounded function with f proper then X is complete. In particular, if $X \in \mathcal{X}(M)$ has a proper first integral then X is complete.*

Example 2.4 Let (M, g) be a Riemannian manifold and fix $h \in C^\infty(M)$. If follows that the existence of a proper function f such that one of the following conditions holds:
· $|g(\nabla f, \nabla h)(x)| \leq A|f(x)| + B$ for every $x \in M$,
· $g(\nabla f, \nabla h)$ is a bounded function, in particular ∇f is g-orthogonal to ∇h i.e. f is a first integral of ∇h,
implies the completeness of the gradient vector field ∇h.

In particular for $f = h$ we derive, by using the Gordon completeness criterion [12] (see also the second part of Theorem 7.3. of [35, p. 25]) or the Appendix of the present paper:

Corollary 2.5 *If h is a proper smooth function on the Riemannian manifold (M, g) with bounded gradient then the gradient of h and the Riemannian metric g are complete. The last item means that the geodesic spray of g is a complete vector field on TM.*

Since we are placed in a physical oriented framework let us add to this section a discussion of the completeness of Euler-Lagrange vector fields.

Denote with TM and T^*M the tangent and cotangent bundle respectively. If $L : TM \to \mathbb{R}$ is a smooth function, usually called *Lagrangian*, let $FL : TM \to T^*M$ be *the fiber derivative* of L [30, p. 26]:

$$FL(v) \cdot w = \frac{d}{d\varepsilon} |_{\varepsilon=0} L(v + \varepsilon w) \tag{2.2}$$

for $v, w \in T_pM, p \in M$. If Ω denotes the canonical symplectic structure of T^*M let $\Omega_L = (FL)^* \Omega$ be the pullback on TM.

Definition 2.6 ([30]) (i) The Lagrangian L is called *regular* if Ω_L is a symplectic structure on TM.
(ii) *The energy* of L is $\mathcal{E}(L) : TM \to \mathbb{R}$ given by:

$$\mathcal{E}(L)(v) = FL(v) \cdot v - L(v). \tag{2.3}$$

Sometimes the energy appears under the name of *Hamiltonian* but in our setting being a function on the tangent bundle not on the cotangent bundle we prefer this name. If L is a regular Lagrangian by using the non-degeneracy of the symplectic form Ω_L of TM it result that there exists a unique vector field $S_L \in \mathcal{X}(TM)$ such that:

$$i_{S_L}\Omega_L = -d\mathcal{E}(L) \tag{2.4}$$

where i_Z denotes the interior product with respect to the vector field Z. S_L is called the *Euler-Lagrange vector field* of L since (2.4) is the global expression of the well-known *Euler-Lagrange equations* of L.

The completeness of S_L is provided by the first part of the *Poincaré Recurrence Theorem* as it appears in [5, p. 87]:

Proposition 2.7 *If the energy $\mathcal{E}_L$ is a proper function on TM then the Euler-Lagrange vector field S_L is complete.*

Example 2.8 If L is a *natural Lagrangian* i.e. the difference:

$$L = K(g) - V,$$

with $K(g)$ the energy of the Riemannian metric g and $V = V(x)$ a *potential*, i.e. a smooth function on M, then, according to [9], the Euler-Lagrange vector field S_L is complete if g is complete and the potential V is bounded below.

Returning to the general case of Corollary 2.3 remark that the Definition 7.3.7 from [1, p. 533] introduce the notion of *regular first integral* of X as a proper first integral which is not constant on any open subset of M and X has *property (G5)* if X has no regular first integral. An important result of the cited book is that property $(G5)$ is C^1 generic and then every vector field can be approximated as closely as we wish by one without regular first integrals.

Open problem 1 If we define a weak-regular first integral by give up to the condition of no-constancy on open subsets then a similar result with respect to a weak-(G5) property as been generic holds?

3. Completeness in Jacobi, Poisson and Nambu-Poisson geometry

Definition 3.1 i) A *Jacobi structure* on M is a pair $(\Lambda, E) \in \mathcal{X}^2(M) \times \mathcal{X}(M)$ such that the following *Jacobi equations* hold:

$$\begin{cases} [\Lambda, \Lambda] = 2\Lambda \wedge E \\ [E, \Lambda] = \mathcal{L}_E \Lambda = 0 \end{cases} \tag{3.1}$$

where $[,]$ is the *Schouten bracket* on multivectors, $\wedge$ is the Grassmann *wedge product* and $\mathcal{L}_E$ is the *Lie derivative* with respect to the vector field E. The triple (M, Λ, E) is a *Jacobi manifold.* A Jacobi manifold with $E = 0$ is a *Poisson manifold.* Let us call Λ and E the *structural bivector* and *vector field* respectively.
ii) Let (M, Λ, E) be a Jacobi manifold and $f \in C^\infty(M)$. The *Hamiltonian vector field* X_f associated to f is:

$$X_f = \Lambda^\sharp(df) + fE. \tag{3.2}$$

Then f is called the *Hamiltonian* of X_f.

In order to handle concrete examples let us provide the above setting with local coordinates. So, let $\left(x^i\right)_{1\leq i\leq n}$ be a local chart on M in which the geometrical objects defining the Jacobi structure has the expressions: $\Lambda = \frac{1}{2}\Lambda^{ij}\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}, E = E^i\frac{\partial}{\partial x^i}$. Then, the Jacobi equations become:

$$\begin{cases} \Lambda^{im}\frac{\partial \Lambda^{jk}}{\partial x^m} + \Lambda^{jm}\frac{\partial \Lambda^{ki}}{\partial x^m} + \Lambda^{km}\frac{\partial \Lambda^{ij}}{\partial x^m} + \Lambda^{ij}E^k + \Lambda^{jk}E^i + \Lambda^{ki}E^j = 0 \\ E^k\frac{\partial \Lambda^{ij}}{\partial x^k} - \Lambda^{ik}\frac{\partial E^j}{\partial x^k} + \Lambda^{jk}\frac{\partial E^i}{\partial x^k} = 0 \end{cases} \tag{3.3}$$

while the Jacobi bracket is:

$$\{f, g\} = \Lambda^{ij}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j} + fE^i\frac{\partial g}{\partial x^i} - gE^i\frac{\partial f}{\partial x^i}. \tag{3.4}$$

The Hamiltonian vector field X_f has the expression:

$$X_f = \Lambda^{ij}\frac{\partial f}{\partial x^j}\frac{\partial}{\partial x^i} + fE^i\frac{\partial}{\partial x^i}. \tag{3.5}$$

Example 3.2 $E = X_1$. In fact, every manifold M with a fixed vector field E is a Jacobi manifold with $\Lambda = 0$.

Open problem 2 Let (M, Λ, E) be a Jacobi manifold with E non-complete. The universal completion of M in the sense of [16] admits a "lifted" Jacobi structure?

The fixed Jacobi structure yields a *Jacobi bracket* $\{,\}$ on $C^\infty(M)$:

$$\{f, g\} = \Lambda(df, dg) + f \cdot E(g) - g \cdot E(f) \tag{3.6}$$

which is a *local Lie algebra* structure in the Kirillov sense; in the Poisson case we get a global Lie algebra structure on $C^\infty(M)$. This permits us to introduce:

Definition 3.3 $C \in C^\infty(M)$ is a *Casimir* of the Jacobi structure if $\{f, C\} = 0$ for every $f \in C^\infty(M)$.

We are able to derive one of the main results of the paper:

Proposition 3.4 *Fix* $f \in C^{\infty}(M)$.
i) *If there exist* $g \in C^{\infty}(M)$ *with* g *proper and* $A, B \in \mathbb{R}_+$ *such that*:

$$|\Lambda(df, dg)(x) + f(x) \cdot E(g)(x)| \leq A|g(x)| + B, \quad \forall x \in M \tag{3.7}$$

then X_f *is complete.*
ii) *If* f *is proper and there exist* $A, B \in \mathbb{R}_+$ *such that*:

$$|f(x) \cdot E(f)(x)| \leq A|f(x)| + B, \quad \forall x \in M \tag{3.8}$$

then X_f *is complete. In particular, if* f *is proper and* $E(f)$ *is a bounded function then* X_f *is complete.*
iii) *If* f *is proper and first integral of* E *then* X_f *and* E *are complete vector fields.*

Proof i) is a reformulation of the Theorem 2.2, ii) derives from i) since $\Lambda(\alpha, \alpha) = 0$, iii) is a direct consequence of ii). □

The above result can be reformulated in terms of the Jacobi bracket:

Proposition 3.5 *Fix* $f \in C^{\infty}(M)$.
i) *If there exist* $g \in C^{\infty}(M)$ *with* g *proper and* $A, B \in \mathbb{R}_+$ *such that*:

$$|\{f, g\}(x) + g(x) \cdot E(f)(x)| \leq A|g(x)| + B, \quad \forall x \in M \tag{3.9}$$

then X_f *is complete. In particular, if* $E(f)$ *is bounded and there exists* $g \in C^{\infty}(M)$ *with* g *proper and* $\{f, g\}$ *bounded then* X_f *is complete.*
ii) *If* f *is a first integral of* E *and there exist* $g \in C^{\infty}(M)$ *with* g *proper and* $A, B \in \mathbb{R}_+$ *such that for each* $x \in M$ *we have:*

$$|\{f, g\}(x)| \leq A|g(x)| + B, \tag{3.10}$$

then X_f *is complete. In particular, if* f *is a first integral of* E *and there exists a proper* $g \in C^{\infty}(M)$ *such that* $\{f, g\}$ *is bounded (or zero) then* X_f *is complete.*
iii) *Suppose that the given Jacobi structure admits a Casimir which is a proper function. If* f *is a first integral of* E *then* X_f *is complete.*

Example 3.6 Suppose that the given Jacobi structure admits a Casimir which is a proper function. Then, $E = X_1$ is a complete vector field since the constant functions are first integrals of every vector field.

A slight generalization of the notion of first integral provides a new example. Namely, inspired by the Lichnerowicz's papers let us introduce:

Definition 3.7 Let $c \in \mathbb{R}$ and $\tau \in C^{\infty}(M)$. Then, τ is a *c-time function* if $E(\tau) \equiv c$.

Example 3.8 Suppose that the Jacobi manifold (M, Λ, E) admits a c-time function which is proper. Then, using Proposition 3.4 ii) with $A = c$ and $B = 0$ it results that X_τ is a complete vector field.

Let us turn to the Poisson setting. From Propositions 3.4 and 3.5 we get:

Corollary 3.9 *Let* (M, Λ) *be a Poisson manifold and fix* $f \in C^{\infty}(M)$.
i) *If there exist* $g \in C^{\infty}(M)$ *with* g *proper and* $A, B \in \mathbb{R}_+$ *such that*:

$$|\Lambda(df, dg)(x)| = |\{f, g\}(x)| \leq A|g(x)| + B, \quad \forall x \in M \tag{3.11}$$

then X_f *is complete. In particular, if there exists a proper function* g *such that* $\{f, g\}$ *is bounded or zero then* X_f *is complete.*
ii) ([33]) *If the Hamiltonian* f *is a proper function then* X_f *is complete.*

Let us apply the last item of the previous result to complete Poisson maps. Recall, after [6, p. 31], that a Poisson map $\varphi : M \to N$ between two Poisson manifolds (i.e. φ preserves the Poisson brackets) is *complete* if, for each $h \in C^\infty(N)$, X_h being a complete vector field implies that $X_{\varphi^* h}$ is also complete. A justification of terminology is provided by Proposition 6.2 of [6, p. 32] that a Poisson map $\varphi : M \to \mathbb{R}$ is complete if and only if X_φ is a complete vector field. It results that every Poisson function from a compact Poisson manifold is complete but we derive a sufficient condition of completeness for Poisson functions from $C^\infty(M)$ with non-compact M:

Corollary 3.10 i) *If there exists* $\psi : M \to \mathbb{R}$ *such that* $\{\varphi, \psi\}$ *is bounded or zero then* φ *is complete.*
ii) *If the Poisson bracket admits a proper Casimir then every Poisson map* $\varphi : M \to \mathbb{R}$ *is complete.*
iii)*Let* $\varphi : M \to \mathbb{R}$ *be a proper Poisson map. Then* φ *is complete.*

Although the last framework presented in this section does not belongs to Jacobi structures we add it here for the similarities with the Poisson case. For more details about Nambu-Poisson structures we refer to [32, 34, 36] and the references therein.

Definition 3.11 A *Nambu-Poisson bracket* or *structure* of order m, $2 \leq m \leq n$ is an internal m-ary operation on $C^\infty(M)$, denoted by $\{\,\}$, which satisfies the following axioms:
(i) $\{\,\}$ is $\mathbb{R}$-multilinear and totally skew-symmetric
(ii) the *Leibniz rule:*

$$\{f_1, \ldots, f_{m-1}, gh\} = \{f_1, \ldots, f_{m-1}, g\}h + g\{f_1, \ldots, f_{m-1}, h\}$$

(iii) the *fundamental identity:*

$$\{f_1, \ldots, f_{m-1}, \{g_1, \ldots, g_m\}\} = \sum_{k=1}^{m} \{g_1, \ldots, \{f_1, \ldots, f_{m-1}, g_k\}, \ldots, g_m\}.$$

The Lie brackets associated to the Poisson structures correspond to the case $m = 2$ in the above definition.

By (ii), $\{\,\}$ acts on each factor as a vector field, hence it must be of the form:

$$\{f_1, \ldots, f_m\} = \Lambda(df_1, \ldots, df_m),$$

where Λ is a field of m-vectors on M. If such a field defines a Nambu-Poisson bracket, it is called a *Nambu-Poisson tensor field.* Λ defines a bundle mapping:

$$\sharp_\Lambda : \Omega^{m-1}(M) \to \mathcal{X}(M)$$

given by:

$$< \beta, \sharp_\Lambda(\alpha_1, \ldots, \alpha_{n-1}) >= \Lambda(\alpha_1, \ldots, \alpha_{n-1}, \beta)$$

where all the arguments are 1-forms.

The next notion which extends the similar one from Poisson geometry is that of the Λ-*Hamiltonian vector field* of $(m-1)$ functions defined by:

$$X_{F_1,\dots,F_{m-1}} = \sharp_\Lambda(dF_1,\dots,dF_{m-1}).$$

Since F_i is a first integral of $X_{F_1,\dots,F_{m-1}}$ the Corollary 2.3 yields a natural generalization of the Corollary 3.9 ii):

Corollary 3.12 *Let $X_{F_1\dots F_{m-1}}$ be a Nambu-Poisson Hamiltonian vector field. If there exists $i \in \{1,\dots,m-1\}$ such that the Hamiltonian F_i is proper then $X_{F_1\dots F_{n-1}}$ is a complete vector field.*

4. Completeness in contact and cosymplectic geometry

The contact geometry is a very important particular case of Jacobi geometry living only in odd dimensions. In the following suppose that $n = 2m+1$.

Definition 4.1 A 1-form $\theta \in \Omega^1(M)$ is a *contact form* on M if it is non-degenerated i.e. the n-form $V := \theta \wedge (d\theta)^m$ is a volume form on M. The pair (M,θ) is a *contact* manifold.

On a contact manifold there exists a remarkable global vector field:

Proposition 4.2 (Reeb) *On (M,θ) lives $E \in \mathcal{X}(M)$ uniquely determined by:*

$$\begin{cases} i_E\theta = 1 \\ i_E d\theta = 0 \end{cases}. \tag{4.1}$$

Definition 4.3 i) E is called the *Reeb* (or sometimes the *characteristic*) *vector field* of the contact manifold (M,θ).
ii) For $f \in C^\infty(M)$ the *contact Hamiltonian vector field* X_f is uniquely determined by:

$$\begin{cases} i_{X_f}\theta = f \\ i_{X_f}d\theta = E(f)\cdot\theta - df \end{cases}. \tag{4.2}$$

f is called again the *Hamiltonian* of X_f.

Example 4.4 The unit sphere $S^3 \subset \mathbb{R}^4$ has a standard contact form: $\theta = \frac{1}{2}\left(x^1dx^2 - x^2dx^1 + x^3dx^4 - x^4dx^3\right)$ with the associated Reeb vector field the unit tangent field to the well-known *Hopf fibration* $S^3 \overset{S^1}{\to} S^2$. Since $M = S^3$ is a compact manifold it results that all vector fields on M, in particular the contact vector fields and the Reeb vector field, are complete.

Properties of the Hamiltonian vector fields, [31, p. 39]:

$$\begin{cases} X_c = cE, \quad c \in \mathbb{R}, \quad X_{-f} = -X_f \\ X_{f+g} = X_f + X_g, \quad X_{fg} = fX_g + fX_g - fgE \\ X_f(f) = f\cdot E(f), \quad X_f\left(f^k\right) = kf^k\cdot E(f) \end{cases}. \tag{4.3}$$

Proposition 4.5([13]) *The bivector Λ given by: $\Lambda(df,dg) := d\theta(X_f,X_g)$, together with E yields a Jacobi structure on (M,θ).*

Therefore the results of above section apply to this framework. Particularly, the Proposition 3.4 i) becomes:

Proposition 4.6 *Fix* $f \in C^{\infty}(M)$. *If there exist* $g \in C^{\infty}(M)$ *with* g *proper and* $A, B \in \mathbb{R}_+$ *such that*:

$$|d\theta(X_f, X_g)(x) + f(x) \cdot E(g)(x)| \leq A|g(x)| + B, \quad \forall x \in M \tag{4.4}$$

then X_f *is complete.*

Also the Example 3.6 becomes:

Corollary 4.7 *Suppose that the Jacobi structure associated to the contact manifold* (M, θ) *admits a Casimir which is a proper function. Then the Reeb vector field* E *is complete.*

On a contact manifold there exists an adapted atlas with local coordinates $(z, q^a, p_a)_{1 \leq a \leq m}$ such that θ has the *canonical* or *Darboux form*:

$$\theta = dz - \sum_{a=1}^{m} p_a dq^a. \tag{4.5}$$

In this *canonical atlas* we have, [19, p. 325]:

$$\begin{cases} E = \frac{\partial}{\partial z}, \qquad \Lambda = \sum_a \frac{\partial}{\partial q^a} \wedge \frac{\partial}{\partial p_a} + \sum_a p_a \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial p_a} \\ \{f, g\} = \left(f - \sum_a p_a \frac{\partial f}{\partial p_a}\right) \frac{\partial g}{\partial z} - \left(g - \sum_a p_a \frac{\partial g}{\partial p_a}\right) \frac{\partial f}{\partial z} + \sum_a \left(\frac{\partial f}{\partial q^a} \frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a} \frac{\partial g}{\partial q^a}\right) \\ X_f = \left(f - \sum_a p_a \frac{\partial f}{\partial p_a}\right) \frac{\partial}{\partial z} - \sum_a \frac{\partial f}{\partial p_a} \frac{\partial}{\partial q^a} + \sum_a \left(\frac{\partial f}{\partial q^a} + p_a \frac{\partial f}{\partial z}\right) \frac{\partial}{\partial p_a} \end{cases} \tag{4.6}$$

with the obvious simplifications if the function f, respectively g, is 1-homogeneous in (p_a).

It follows that a c-time function on a contact manifold has the form: $\tau = cz + F(q^a, p_a)$ and then:

$$X_\tau = \left(cz + F - \sum_a p_a \frac{\partial F}{\partial p_a}\right) \frac{\partial}{\partial z} - \sum_a \frac{\partial F}{\partial p_a} \frac{\partial}{\partial q^a} + \sum_a \left(\frac{\partial F}{\partial q^a} + cp_a\right) \frac{\partial}{\partial p_a}. \tag{4.7}$$

In particular, if the function F is 1-homogeneous with respect to the variables (p_a) it results, via the Euler theorem, that:

$$X_\tau = cz \frac{\partial}{\partial z} - \sum_a \frac{\partial F}{\partial p_a} \frac{\partial}{\partial q^a} + \sum_a \left(\frac{\partial F}{\partial q^a} + cp_a\right) \frac{\partial}{\partial p_a}. \tag{4.8}$$

It results that the Example 3.8 implies:

Corollary 4.8 *Let* $\tau \in C^{\infty}(M)$ *be an c-time function on the contact manifold* (M, θ) *of expression above, in particular* $\tau = \tau(q^a, p_a)$. *If* τ *is a proper function too then* X_τ *given by* (4.7), *particularly by* (4.8) *if* F *is* 1*-homogeneous with respect to* (p_a), *is a complete vector field.*

Examples 4.9 i) Let $(P, d\alpha)$ be an exact symplectic $2m$-dimensional manifold. Inspired by Example 1 from [19, p. 294] we consider the manifold $M = I \times P$ with $I = (a, b)$ a bounded real interval and $\theta = dt - \alpha$ where t is the canonical coordinate in I. Then (M, θ) is a contact manifold with the Reeb vector field $E = \frac{\partial}{\partial t}$. A c-time function on M having the expression $\tau = cz + F$ with $F \in C^{\infty}(P)$ is proper

provided the function F is proper. It follows that the previous result applies in this framework.
ii) Functions $\tau = \tau(q^a, p_a)$ rise naturally in the geometric theory of first order PDE (Partial Differential Equations). Let M be the first jet bundle of $\mathbb{R}^m$; locally such space can be described by $x \in \mathbb{R}^m$ and a germ of a function in x considered up to its gradient. To each $m \in J^1(\mathbb{R}^m)$ we associate $(x, z(x), \nabla z)$ with z a germ of a smooth function in $\mathbb{R}^m$. The manifold $J^1(\mathbb{R}^m)$ is a contact manifold with contact 1-form $\theta = dz - y_a dx^a$ which vanishes on any germ z such that $\frac{\partial z}{\partial x^a} = y_a$. This jet space is the natural place to study the geometry of first order PDEs; more precisely, any PDE can be understood as a submanifold of $J^1(\mathbb{R}^m)$. Let us suppose that we have a smooth function $\tau : J^1(\mathbb{R}^m) \to \mathbb{R}$ with $\frac{\partial \tau}{\partial z} = 0$. Then, after [2], the set $\tau = 0$ defines a manifold in $J^1(\mathbb{R}^m)$ which corresponds to the PDE: $\tau((x, z(x), \nabla z)(x)) = 0$.

Let us study the case of a contact structure endowed with a Legendre foliation. On (M, θ) the distribution $H(M) = Ker(\theta)$ is called the *contact distribution* and is not integrable. A codimension $m+1$ foliation $\mathcal{F}$ on M is said to be a *Legendre foliation* if $T(\mathcal{F})$ is a m-subbundle of the $2m$-distribution $H(M)$. In other words, $\mathcal{F}$ is a foliation of (M, θ) by m-dimensional integral manifolds of the contact distribution $H(M)$.

Proposition 4.10 ([20]) *Let $\mathcal{F}$ be a Legendre foliation on the contact manifold (M, θ). Then, for any $x \in M$ there exists an open neighbourhood U of x which admits local coordinates $(x^a, p_a, t)_{1 \leq a \leq m}$ such that: $\theta = \sum_a p_a dx^a - H dt$ with $H \in C^\infty(M)$ satisfying the condition; the function:*

$$A = \sum_a p_a \frac{\partial H}{\partial p_a} - H$$

has no zero. By means of these coordinates the Reeb vector field is expressed by:

$$E = \frac{1}{A} \left(\frac{\partial}{\partial t} + \left(\sum_a \frac{\partial H}{\partial p_a} \frac{\partial}{\partial x^a} - \frac{\partial H}{\partial x^a} \frac{\partial}{\partial p_a} \right) \right). \tag{4.9}$$

Since $E(H) = \frac{1}{A} \frac{\partial H}{\partial t}$ from Theorem 2.2 we get:

Corollary 4.11 *Suppose that the contact manifold (M, θ) admits a Legendre foliation as above with H a proper function. If there exist $U, V \in \mathbb{R}_+$ such that for each $x \in M$ we have:*

$$\left| \frac{1}{A(x)} \cdot \frac{\partial H}{\partial t}(x) \right| \leq U |H(x)| + V, \tag{4.10}$$

in particular H is time-independent, then the Reeb vector field E is complete.

The last abstract framework included here is the cosymplectic geometry. A cosymplectic manifold is a triple (M, Ω, η) with $\Omega \in \Omega^2(M)$ and $\eta \in \Omega^1(M)$ such that $V := \eta \wedge \Omega^m$ is a volume form. Exactly as in the contact geometry there exists a global *Reeb vector field* R uniquely determined by:

$$\begin{cases} i_R \eta = 1 \\ i_R \Omega = 0 \end{cases} \tag{4.11}$$

and for every $f \in C^\infty(M)$ there is associated a *Hamiltonian vector field* X_f uniquely determined by:

$$\begin{cases} i_{X_f}\eta = 0 \\ i_{X_f}\Omega = df - R(f)\cdot\eta \end{cases}. \tag{4.12}$$

One can prove that in a neighbourhood of each point of a cosymplectic manifold one can define *canonical coordinates* (q^a, p_a, z) such that:

$$\begin{cases} \Omega = dq^a \wedge dp_a \\ \eta = dz, \quad R = \frac{\partial}{\partial z} \end{cases} \tag{4.13}$$

and in these coordinates:

$$X_f = \sum_a \frac{\partial f}{\partial p_a}\frac{\partial}{\partial q^a} - \sum_a \frac{\partial f}{\partial q^a}\frac{\partial}{\partial p_a} \tag{4.14}$$

Since f is a first integral of X_f we apply the Corollary 2.3: if f is a proper function then the cosymplectic Hamiltonian vector field X_f is a complete vector field.

5. Examples in thermodynamics

Consider after the classical theory of thermodynamics a material in an enclosure of volume V, pressure P, temperature T, entropy S and internal energy U. The first law of thermodynamics says that infinitesimal changes of these thermodynamic variables must satisfy:

$$\theta := dU + PdV - TdS \equiv 0. \tag{5.1}$$

Therefore, we can attach the contact manifold M an open subset of $(\mathbb{R}^5, \theta)$, called *thermodynamical phase space* with the canonical coordinates $(z; q^1, q^2, p_1, p_2) = (U; V, S, -P, T)$ and Reeb vector field $E = \frac{\partial}{\partial U}$. Using the formula (4.6_4) it results that a function $f = f(U; V, S, -P, T)$ has the Hamiltonian vector field:

$$X_f = \left(f - P\frac{\partial f}{\partial P} - T\frac{\partial f}{\partial T}\right)\frac{\partial}{\partial U} - \frac{\partial f}{\partial T}\frac{\partial}{\partial S} + \frac{\partial f}{\partial P}\frac{\partial}{\partial V} - \left(\frac{\partial f}{\partial V} - P\frac{\partial f}{\partial U}\right)\frac{\partial}{\partial P} + \\ + \left(\frac{\partial f}{\partial S} + T\frac{\partial f}{\partial U}\right)\frac{\partial}{\partial T}. \tag{5.2}$$

Applying the Proposition 3.4 ii) one has:

Proposition 5.1 *Let $f \in C^\infty(M)$ proper with:*

$$|f \cdot \frac{\partial f}{\partial U}(U, V, S, -P, T)| \leq A|f(U, V, S, -P, T)| + B \tag{5.3}$$

for every $(U, V, S, -P, T) \in M$, in particular f does not depends on the internal energy U. Then X_f is a complete vector field.

Example 5.2 ([31, p. 43]) Let us consider an ideal gas describing an isothermal process with the constant internal energy U and the particle number N. It follows that: $f = PV - NRT$ with R the so-called *gas constant*. Then, from (5.2):

$$X_f = NR\frac{\partial}{\partial S} + V\frac{\partial}{\partial V} - P\frac{\partial}{\partial P} \tag{5.4}$$

with the flow:

$$U = U_0, V = V_0e^t, S = S_0 + NRt, P = P_0e^{-t}, T = T_0 \tag{5.5}$$

with $(U_0, V_0, S_0, P_0, T_0)$ an initial Cauchy data. Locking at the expression of f we see that it is a c-time function with $c = 0$ (i.e. first integral of the Reeb vector field E) which is also 1-homogeneous in $(-P, T)$. The vector field X_f is complete conform Proposition 5.1 although $f : \mathbb{R}^5 \to \mathbb{R}$ is not a proper function; let us remark that the solutions (5.5) confirm the completeness of X_f.

Example 5.3 ([31, p. 43]) Consider $f = U - \frac{3}{2}PV$ a Hamiltonian depending of the internal energy U. Then, from (5.2):

$$X_f = U\frac{\partial}{\partial U} - \frac{3}{2}V\frac{\partial}{\partial V} + \frac{5}{2}P\frac{\partial}{\partial P} + T\frac{\partial}{\partial T} \tag{5.6}$$

with the flow:

$$U = U_0e^t, V = V_0e^{-3t/2}, S = S_0, P = P_0e^{5t/2}, T = T_0e^t. \tag{5.7}$$

The Hamiltonian f is again a c-time function with $c = 1$ and the same conclusion as in the previous example it results about the completeness of X_f.

Appendix: Completeness in terms of a metric

In [1, p. 71] a criterion of completeness in terms of a Riemannian metric is proved:

Proposition A1 *Let (M, g) be a complete Riemannian manifold and X a C^k vector field, $k \geq 1$, such that for any integral curve σ the norm $\|X(\sigma(t))\|_{\sigma(t)}$ is bounded on finite t-intervals. Then X is a complete vector field.*

It results immediately:

Corollary A2 *On a complete Riemannian manifold an unitary vector field is complete.*

As examples of such vector fields we have the Reeb vector field. More precisely, on a contact manifold (M, θ, E) there exists a nonunique Riemannian metric g such that $g(X, E) = \theta(X)$ for every $X \in \mathcal{X}(M)$; these metrics are called *associated* and their class provides important types of contact structures (e.g. *K-contact manifolds* when E is Killing with respect to g) as appears in [3]. But the previous relation means that the Reeb vector field E is unitary with respect to g.

Corollary A3 *Let (M, θ, E) be a contact manifold such that one associated Riemannian metric g is complete. Then E is a complete vector field.*

Example A4 Let (M, g) be a Riemannian manifold and $K(g)$ the energy of g which is a regular Lagrangian; we use the notations and notions of the first section. The canonical symplectic structure Ω of T^*M is the differential of *Liouville 1-form* λ. Let $\lambda_{K(g)} = (FK(g))^* \lambda$ be the pullback on TM. Then the restriction of $\lambda_{K(g)}$ to the unit tangent bundle T_1M is a contact structure with the Reeb vector field twice the geodesic flow, [3]. Then, the completeness of the initial Riemannian metric g yields the completeness of the Reeb vector field of the associated contact structure on T_1M.

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