SELF-ADAPTIVE ALGORITHMS FOR SOLVING FIXED POINT PROBLEMS OF PSEUDOCONTRACTIVE OPERATORS

Yang Chao[1], Yeong-Cheng Liou[2], Lu Zheng[3]

In this paper, fixed point problems of pseudocontractive operators are investigated. We present a self-adaptive algorithm for finding a fixed point of a Lipschitz pseudocontractive operator in a real Hilbert space. Our algorithm has no need to know a priori the Lipschitz constant of pseudocontractive operators. Strong convergence result is obtained under some additional assumptions.

Keywords: fixed point, pseudocontractive operator, self-adaptive technique, strong convergence.

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1. Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonlinear mapping. In this paper, we devote to solve the following interesting problem of finding $x \in C$ such that

$$x = Tx. \tag{1}$$

Now, we know that problem (1) is an important topic because many nonlinear problems can be reformulated as fixed point equation (1). Many methods have been proposed for solving (1), see [1, 3, 10, 13, 15, 20, 23, 30]. There are four common methods ([7–9, 12, 14]) for approximating a fixed point of T:

$$\text{Picard: } x_1 \in C,\ x_{n+1} = Tx_n,\ n \geq 1, \tag{2}$$

$$\text{Krasnoselskii-Mann: } x_1 \in C,\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 1, \tag{3}$$

where $\alpha_n \in (0,1)$,

$$\text{Ishikawa: } x_1 \in C, y_n = \alpha_n x_n + (1-\alpha_n Tx_n, x_{n+1} = \beta_n x_n + (1-\beta_n)Ty_n, n \geq 1, \tag{4}$$

where $\alpha_n \in (0,1)$ and $\beta_n \in (0,1)$ for all $n \geq 1$ and

$$\text{Halpern: } x_1 \in C, x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, n \geq 1, \tag{5}$$

where $u \in C$ is a fixed point and $\alpha_n \in (0,1)$.

Remark 1.1. *(i) If T is contractive, then the sequence $\{x_n\}$ generated by Picard's method* (2) *converges to the unique fixed point of T. (ii) If T is nonexpansive (not contractive), then the sequence $\{x_n\}$ generated by* (2) *does not converge and the sequence $\{x_n\}$ generated by Krasnoselskii-Mann's method* (3) *converges weakly to a fixed point of T. However, Krasnoselskii-Mann's method does not converge in the strong topology. (iii) The sequence*

[1]Business School, Concordia University Chicago, River Forest, IL 60305, United State, e-mail: 441954050@qq.com

[2]Corresponding author. Department of Healthcare Administration and Medical Informatics, and Research Center of Nonlinear Analysis and Optimization, Kaohsiung Medical University, and Department of Medical Research, Kaohsiung Medical University Hospital, Kaohsiung 807, Taiwan, e-mail: simplex_liou@hotmail.com

[3]Corresponding author. Graduate School, CheongJu University, South Korea, e-mail: Loriluz@163.com

$\{x_n\}$ generated by Halpern's method (5) *converges strongly to a fixed point of a nonexpansive mapping T.*

The importance of pseudocontractive operators depends on their relation with monotone operators. Browder and Petryshyn ([2]) studied weak convergence of Krasnoselskii-Mann's method for strictly pseudocontractive operators. However, Krasnoselskii-Mann's method fails to converge for pseudocontractive operators ([4]). Consequently, Ishikawa's method introduced in [1] is more attractive than that of Krasnoselskii-Mann's method and which converges to a fixed point of a pseudocontractive operator. Unfortunately, strong convergence of Ishikawa's method has not been obtained without compactness assumption on C or T. Construction of iterative algorithms for finding fixed points of nonlinear operators is still an interesting work and has attracted so much attention, see [5, 6, 16–19, 24–29, 31, 33, 34].

The main purpose of this paper is to construct iterative algorithm for approximating fixed points of pseudocontractive operators. We present a self-adaptive algorithm for finding a fixed point of a Lipschitz pseudocontractive operator in a real Hilbert space. Our algorithm has no need to know a priori the Lipschitz constant of pseudocontractive operators. Strong convergence result is obtained under some additional assumptions.

2. **Preliminaries**

In this section, we collect some definitions and lemmas. Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Throughout, the symbols " $\rightharpoonup$ " and " $\rightarrow$ " denote weak convergence and strong convergence, respectively. Use $Fix(T)$ to denote the set of fixed points of a mapping T.

Definition 2.1. *A mapping $T : C \to C$ is said to be L-Lipschitz if there is a nonnegative constant L such that*

$$\|Tx - Ty\| \le L\|x - y\|,\ \forall x, y \in C.$$

T is said to be nonexpansive when $L = 1$ and T is said to be contractive if $L < 1$.

Definition 2.2. *A mapping $T : C \to C$ is said to be pseudocontractive if*

$$\langle Tx - Ty, x - y\rangle \le \|x - y\|^2,\ \forall x, y \in C. \tag{6}$$

Recall that the metric projection $P_C : H \to C$ is defined by

$$P_C(x) := \arg\min_{y\in C} \|x - y\|,\ \forall x \in H.$$

It is well known that P_C has the following property: for given $x \in H$,

$$\langle x - P_C(x), P_C(x) - y\rangle \ge 0,\ \forall y \in C.$$

The following results are well known:

$$\|\tau x + (1-\tau)y\|^2 = \tau\|x\|^2 + (1-\tau)\|y\|^2 - \tau(1-\tau)\|x - y\|^2,\ \forall x, y \in H, \tau \in \mathbb{R}, \tag{7}$$

and

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle,\ \forall x, y \in H. \tag{8}$$

Lemma 2.1 ([32])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. If $T : C \to C$ is a continuous pseudocontractive operator, then $I - T$ is demiclosed at the origin.*

Lemma 2.2 ([11])**.** *Let $\{\omega_n\}$ be a real number sequence. Suppose that there is a subsequence $\{\omega_{n_i}\} \subset \{\omega_n\}$ such that $\omega_{n_i} \le \omega_{n_i+1}, \forall i \ge 0$. For every $n \ge n_0$, set $\varsigma(n) = \max\{n_0 \le i \le n : \omega_{n_i} < \omega_{n_i+1}\}$. Then $\lim_{n\to\infty} \varsigma(n) = \infty$ and $\max\{\omega_{\varsigma(n)}, \omega_n\} \le \omega_{\varsigma(n)+1}, \forall n \ge n_0$.*

Lemma 2.3 ([22])**.** *Let $\{a_n\}$ be a real number sequence. If $a_n \geq 0$ and $a_{n+1} \leq (1-\gamma_n)a_n + b_n$, where γ_n and b_n satisfy (i) $0 < \gamma_n < 1$, (ii) $\sum_{n=1}^{\infty} \gamma_n = \infty$ and (iii) $\limsup_{n\to\infty} \frac{b_n}{\gamma_n} \leq 0$, then $\lim_{n\to\infty} a_n = 0$.*

3. **Main results**

In this section, we present a self-adaptive algorithm for finding a fixed point of a Lipschitz pseudocontractive operator. Assume that: (i) H is a real Hilbert space and $C \subset H$ is a nonempty closed convex set; (ii) $T : C \to C$ is an L-Lipschitz pseudocontractve mapping with $Fix(T) \neq \emptyset$ and $\varphi : C \to C$ is a σ-contractive mapping. Let τ and λ be two constants in $(0, 1)$ and $\mu \in (0, \frac{1-\lambda^2}{2})$. Suppose that $\{\gamma_n\}$, $\{\zeta_n\}$ and $\{\delta_n\}$ are three real number sequences in $[0, 1]$ satisfying the following assumptions

(c1): $\gamma_n + \delta_n + \zeta_n \leq 1$, $\lim_{n\to\infty} \gamma_n = 0$, and $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(c2): $\delta_n \geq 1 - \min\{\mu, \frac{\sqrt{\mu}\tau}{L}\}$ and $0 < \liminf_{n\to\infty} \zeta_n \leq \limsup_{n\to\infty} \zeta_n < 1$;

(c3): $\lim_{n\to\infty} \frac{1-\gamma_n-\delta_n-\zeta_n}{\gamma_n} = 0$.

Algorithm 3.1. *For any initial point $x_0 \in C$, let $\{x_n\}$ be a sequence generalized by*

$$y_n = (1 - \eta_n)x_n + \eta_n T x_n, \tag{9}$$

where $\eta_n = \mu\tau^m$ and $m = \min\{0, 1, 2, \cdots\}$ such that

$$\eta_n \|Ty_n - Tx_n\| \leq \lambda \|y_n - x_n\|, \tag{10}$$

and

$$x_{n+1} = \gamma_n \varphi(x_n) + \delta_n x_n + \zeta_n T y_n,\ n \geq 0. \tag{11}$$

Remark 3.1. *The search rule* (10) *is well-defined and $\mu \geq \eta_n \geq \min\{\mu, \frac{\sqrt{\mu}\tau}{L}\}, n \geq 0$.*

Next, we show the convergence of the sequence defined by Algorithm 3.1.

Theorem 3.1. *The sequence $\{x_n\}$ generalized by (11) converges strongly to $p = P_{Fix(T)}\varphi(p)$.*

Proof. Let $p^* \in Fix(T)$. From (11), we have

$$\begin{aligned} \|x_{n+1} - p^*\| &= \|\gamma_n(\varphi(x_n) - p^*) + \delta_n(x_n - p^*) + \zeta_n(Ty_n - p^*) - (1 - \gamma_n - \delta_n - \zeta_n)p^*\| \\ &\leq \gamma_n \|\varphi(x_n) - p^*\| + \|\delta_n(x_n - p^*) + \zeta_n(Ty_n - p^*)\| \\ &\quad + (1 - \gamma_n - \delta_n - \zeta_n)\|p^*\|. \end{aligned} \tag{12}$$

According to equality (7), we have

$$\begin{aligned} \|\delta_n(x_n - p^*) + \zeta_n(Ty_n - p^*)\|^2 &= \delta_n(\delta_n + \zeta_n)\|x_n - p^*\|^2 + \zeta_n(\delta_n + \zeta_n)\|Ty_n - p^*\|^2 \\ &\quad - \delta_n\zeta_n\|x_n - Ty_n\|^2. \end{aligned} \tag{13}$$

Since T is pseudocontractive and $p^* \in Fix(T)$, we obtain

$$\|Ty_n - p^*\|^2 \leq \|y_n - p^*\|^2 + \|y_n - Ty_n\|^2, \tag{14}$$

and

$$\|Tx_n - p^*\|^2 \leq \|x_n - p^*\|^2 + \|x_n - Tx_n\|^2. \tag{15}$$

From (7), (9) and (14), we have

$$\begin{aligned} \|y_n - p^*\|^2 &= \|(1 - \eta_n)(x_n - p^*) + \eta_n(Tx_n - p^*)\|^2 \\ &= (1 - \eta_n)\|x_n - p^*\|^2 + \eta_n\|Tx_n - p^*\|^2 - \eta_n(1 - \eta_n)\|x_n - Tx_n\|^2 \\ &\leq (1 - \eta_n)\|x_n - p^*\|^2 + \eta_n(\|x_n - p^*\|^2 + \|x_n - Tx_n\|^2) \\ &\quad - \eta_n(1 - \eta_n)\|x_n - Tx_n\|^2 \\ &= \|x_n - p^*\|^2 + \eta_n^2\|x_n - Tx_n\|^2. \end{aligned} \tag{16}$$

Similarly, we have

$$\begin{aligned}\|y_n - Ty_n\|^2 &= \|(1-\eta_n)(x_n - Ty_n) + \eta_n(Tx_n - Ty_n)\|^2 \\ &= (1-\eta_n)\|x_n - Ty_n\|^2 + \eta_n\|Tx_n - Ty_n\|^2 - \eta_n(1-\eta_n)\|x_n - Tx_n\|^2.\end{aligned} \tag{17}$$

Note that $\eta_n\|Ty_n - Tx_n\| \le \lambda\|y_n - x_n\|$ and $\|x_n - y_n\| = \eta_n\|x_n - Tx_n\|$. This together with (17) implies that

$$\|y_n - Ty_n\|^2 \le (1-\eta_n)\|x_n - Ty_n\|^2 + \eta_n\lambda^2\|x_n - Tx_n\|^2 - \eta_n(1-\eta_n)\|x_n - Tx_n\|^2. \tag{18}$$

Combining (14), (16) and (18), we have

$$\begin{aligned}\|Ty_n - p^*\|^2 &\le \|x_n - p^*\|^2 + \eta_n^2\|x_n - Tx_n\|^2 + (1-\eta_n)\|x_n - Ty_n\|^2 \\ &\quad + \eta_n\lambda^2\|x_n - Tx_n\|^2 - \eta_n(1-\eta_n)\|x_n - Tx_n\|^2 \\ &= \|x_n - p^*\|^2 + (1-\eta_n)\|x_n - Ty_n\|^2 - \eta_n(1-\lambda^2-2\eta_n)\|x_n - Tx_n\|^2.\end{aligned} \tag{19}$$

Since $\eta_n \le \mu$, $1-\lambda^2-2\eta_n \ge 1-\lambda^2-2\mu > 0$. It follows from (19) that

$$\|Ty_n - p^*\|^2 \le \|x_n - p^*\|^2 + (1-\eta_n)\|x_n - Ty_n\|^2. \tag{20}$$

Since $\delta_n \ge 1-\eta_n$ and $0 < \delta_n + \zeta_n < 1$, $(1-\eta_n)(\delta_n+\zeta_n) - \delta_n < 0$. Substituting (20) to (13), we have

$$\begin{aligned}\|\delta_n(x_n - p^*) + \zeta_n(Ty_n - p^*)\|^2 &\le \delta_n(\delta_n+\zeta_n)\|x_n - p^*\|^2 + \zeta_n(\delta_n+\zeta_n)(\|x_n - p^*\|^2 \\ &\quad + (1-\eta_n)\|x_n - Ty_n\|^2) - \delta_n\zeta_n\|x_n - Ty_n\|^2 \\ &= (\delta_n+\zeta_n)^2\|x_n - p^*\|^2 + \zeta_n[(1-\eta_n)(\delta_n+\zeta_n) - \delta_n]\|x_n - Ty_n\|^2 \\ &\le (\delta_n+\zeta_n)^2\|x_n - p^*\|^2.\end{aligned} \tag{21}$$

It follows from (12) and (21) that

$$\begin{aligned}\|x_{n+1} - p^*\| &\le \gamma_n\|\varphi(x_n) - \varphi(p^*)\| + \gamma_n\|\varphi(p^*) - p^*\| + (\delta_n+\zeta_n)\|x_n - p^*\| \\ &\quad + (1-\gamma_n-\delta_n-\zeta_n)\|p^*\| \\ &\le (\gamma_n\sigma + \delta_n + \zeta_n)\|x_n - p^*\| + \gamma_n\|\varphi(p^*) - p^*\| + (1-\gamma_n-\delta_n-\zeta_n)\|p^*\| \\ &\le [1-(1-\sigma)(1-\delta_n-\zeta_n)]\|x_n - p^*\| \\ &\quad + (1-\delta_n-\zeta_n)\max\{\|\varphi(p^*) - p^*\|, \|p^*\|\} \\ &\le \max\{\|x_n - p^*\|, \frac{\max\{\|\varphi(p^*) - p^*\|, \|p^*\|\}}{1-\sigma}\} \\ &\le \max\{\|x_0 - p^*\|, \frac{\max\{\|\varphi(p^*) - p^*\|, \|p^*\|\}}{1-\sigma}\}.\end{aligned}$$

This implies that the sequence $\{x_n\}$ is bounded.

Take into account of (8) and (11), we have

$$\begin{aligned}
\|x_{n+1}-p^*\|^2 &= \|\gamma_n(\varphi(x_n)-p^*)+\delta_n(x_n-p^*)+\zeta_n(Ty_n-p^*)\\
&\quad -(1-\gamma_n-\delta_n-\zeta_n)p^*\|^2\\
&\le \|\delta_n(x_n-p^*)+\zeta_n(Ty_n-p^*)\|^2+2\gamma_n\langle\varphi(x_n)-p^*,x_{n+1}-p^*\rangle\\
&\quad -2(1-\gamma_n-\delta_n-\zeta_n)\langle p^*,x_{n+1}-p^*\rangle\\
&\le (\delta_n+\zeta_n)^2\|x_n-p^*\|^2-\zeta_n[\delta_n\eta_n-(1-\eta_n)\zeta_n]\|x_n-Ty_n\|^2\\
&\quad +2\gamma_n\sigma\|x_n-p^*\|\|x_{n+1}-p^*\|+2\gamma_n\langle\varphi(p^*)-p^*,x_{n+1}-p^*\rangle\\
&\quad -2(1-\gamma_n-\delta_n-\zeta_n)\langle p^*,x_{n+1}-p^*\rangle\\
&\le (1-\gamma_n)^2\|x_n-p^*\|^2-\zeta_n[\delta_n\eta_n-(1-\eta_n)\zeta_n]\|x_n-Ty_n\|^2\\
&\quad +\gamma_n\sigma\|x_n-p^*\|^2+\gamma_n\sigma\|x_{n+1}-p^*\|^2\\
&\quad +2\gamma_n\langle\varphi(p^*)-p^*,x_{n+1}-p^*\rangle\\
&\quad -2(1-\gamma_n-\delta_n-\zeta_n)\langle p^*,x_{n+1}-p^*\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|x_{n+1}-p^*\|^2 &\le [1-\frac{2(1-\sigma)}{1-\sigma\gamma_n}\gamma_n]\|x_n-p^*\|^2+\frac{\gamma_n^2}{1-\sigma\gamma_n}\|x_n-p^*\|^2\\
&\quad -\frac{\zeta_n}{1-\sigma\gamma_n}[\delta_n\eta_n-(1-\eta_n)\zeta_n]\|x_n-Ty_n\|^2\\
&\quad +\frac{2}{1-\sigma\gamma_n}\gamma_n\langle\varphi(p^*)-p^*,x_{n+1}-p^*\rangle\\
&\quad -\frac{2}{1-\sigma\gamma_n}(1-\gamma_n-\delta_n-\zeta_n)\langle p^*,x_{n+1}-p^*\rangle \qquad (22)\\
&\le [1-\frac{2(1-\sigma)}{1-\sigma\gamma_n}\gamma_n]\|x_n-p^*\|^2+\frac{2}{1-\sigma\gamma_n}(1-\gamma_n-\delta_n-\zeta_n)M\\
&\quad -\frac{\zeta_n}{1-\sigma\gamma_n}[\delta_n\eta_n-(1-\eta_n)\zeta_n]\|x_n-Ty_n\|^2+\frac{\gamma_n^2}{1-\sigma\gamma_n}M\\
&\quad +\frac{2}{1-\sigma\gamma_n}\gamma_n\langle\varphi(p^*)-p^*,x_{n+1}-p^*\rangle,
\end{aligned}$$

where $M\ge\sup_n\{\|x_n-u\|^2,\|p^*\|\|x_{n+1}-p^*\|\}$.

If there is some integer $m>0$ such that $\{\|x_n-p^*\|,\|\varphi(p^*)-p^*\|\|x_{n+1}-p^*\|\}$ is decreasing for all $n\ge m$, then $\lim_{n\to\infty}\|x_n-p^*\|$ exists. Thanks to (22), we have

$$\begin{aligned}
&\frac{\zeta_n}{1-\sigma\gamma_n}[\delta_n\eta_n-(1-\eta_n)\zeta_n]\|x_n-Ty_n\|^2\\
&\quad\le [1-\frac{2(1-\sigma)}{1-\sigma\gamma_n}\gamma_n]\|x_n-p^*\|^2-\|x_{n+1}-p^*\|^2+\frac{\gamma_n^2}{1-\sigma\gamma_n}M \qquad (23)\\
&\qquad +\frac{2}{1-\sigma\gamma_n}\gamma_nM+\frac{2}{1-\sigma\gamma_n}(1-\gamma_n-\delta_n-\zeta_n)M
\end{aligned}$$

Observe that $\liminf_{n\to\infty}\frac{\zeta_n}{1-\sigma\gamma_n}[\delta_n\eta_n-(1-\eta_n)\zeta_n]>0$. This together with (23) implies that

$$\lim_{n\to\infty}\|x_n-Ty_n\|=0. \qquad (24)$$

By (8), (9) and (10), we have

$$\begin{aligned}\|y_n - Ty_n\|^2 &= \|(1-\eta_n)(x_n - Ty_n) + \eta_n(Tx_n - Ty_n)\|^2 \\ &= (1-\eta_n)\|x_n - Ty_n\|^2 + \eta_n\|Tx_n - Ty_n\|^2 - \eta_n(1-\eta_n)\|x_n - Tx_n\|^2 \\ &\le (1-\eta_n)\|x_n - Ty_n\|^2 + \eta_n\lambda^2\|x_n - Tx_n\|^2 - \eta_n(1-\eta_n)\|x_n - Tx_n\|^2 \\ &= (1-\eta_n)\|x_n - Ty_n\|^2 - \eta_n(1-\eta_n-\lambda^2)\|x_n - Tx_n\|^2 \\ &\le (1-\eta_n)\|x_n - Ty_n\|^2,\end{aligned} \tag{25}$$

which together with (24) yields $\|y_n - Ty_n\| \to 0$. By (25), we conclude that

$$\eta_n(1-\eta_n-\lambda^2)\|x_n - Tx_n\|^2 \le 1-\eta_n)\|x_n - Ty_n\|^2 - \|y_n - Ty_n\|^2 \to 0,$$

and hence

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{26}$$

Since $\{x_n\}$ is bounded, there is $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup \hat{p}$ and

$$\limsup_{n\to\infty}\langle\varphi(p) - p, x_n - p\rangle = \lim_{i\to\infty}\langle\varphi(p) - p, x_{n_i} - p\rangle, \tag{27}$$

where $p = P_{Fix(T)}\varphi(p)$.

From (26), we have $\|x_{n_i} - Tx_{n_i}\| \to 0$. Using Lemma 2.1, we deduce $\hat{p} \in Fix(T)$. It follows from (27) that

$$\limsup_{n\to\infty}\langle\varphi(p) - p, x_n - p\rangle = \lim_{i\to\infty}\langle\varphi(p) - p, x_{n_i} - p\rangle = \langle\varphi(p) - p, \hat{p} - p\rangle \le 0.$$

Note that $\|x_{n+1} - x_n\| \le \gamma_n\|\varphi(x_n) - x_n\| + \zeta_n\|x_n - Ty_n\| \to 0$. Thus,

$$\limsup_{n\to\infty}\langle\varphi(p) - p, x_{n+1} - p\rangle \le 0. \tag{28}$$

Thanks to (22), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le [1 - \frac{2(1-\sigma)}{1-\sigma\gamma_n}\gamma_n]\|x_n - p\|^2 + \frac{\gamma_n^2}{1-\sigma\gamma_n}M \\ &\quad + \frac{2}{1-\sigma\gamma_n}\gamma_n\langle\varphi(p) - p, x_{n+1} - p\rangle + \frac{2}{1-\sigma\gamma_n}(1-\gamma_n-\delta_n-\zeta_n)M,\end{aligned} \tag{29}$$

According to Lemma 2.3, (28) and (29), we conclude that $x_n \to p$.

Suppose that there is an integer k such that $\|x_k - p\| \le \|x_{k+1} - p\|$. Let $\omega_n = \{\|x_n - p\|^2\}$. Hence, $\omega_k \le \omega_{k+1}$. For all $n \ge k$, set $\varsigma(n) = \max\{l \in \mathbb{N} | k \le l \le n, \omega_l \le \omega_{l+1}\}$. Then, $\varsigma(n)$ is non-decreasing, $\lim_{n\to\infty}\varsigma(n) = \infty$ and $\omega_{\varsigma(n)} \le \omega_{\varsigma(n)+1}$ for all $n \ge k$.

Similarly, based on (26), we have $\lim_{n\to\infty}\|x_{\varsigma(n)} - Tx_{\varsigma(n)}\| = 0$ which results in that $\omega_w(x_{\varsigma(n)}) \subset Fix(T)$. So,

$$\limsup_{n\to\infty}\langle\varphi(p) - p, x_{\varsigma(n)+1} - p\rangle \le 0. \tag{30}$$

By (29), we have

$$\begin{aligned}\omega_{\varsigma(n)+1} &\le [1 - \frac{2(1-\sigma)}{1-\sigma\gamma_{\varsigma(n)}}\gamma_{\varsigma(n)}]\omega_{\varsigma(n)} + \frac{2}{1-\sigma\gamma_{\varsigma(n)}}\gamma_{\varsigma(n)}\langle\varphi(p) - p, x_{\varsigma(n)+1} - p\rangle \\ &\quad + \frac{2}{1-\sigma\gamma_{\varsigma(n)}}(1-\gamma_{\varsigma(n)}-\delta_{\varsigma(n)}-\zeta_{\varsigma(n)})M + \frac{\gamma_{\varsigma(n)}^2}{1-\sigma\gamma_{\varsigma(n)}}M,\end{aligned} \tag{31}$$

This together with (30) leads to

$$\limsup_{n\to\infty}\omega_{\varsigma(n)+1} \le \limsup_{n\to\infty}\omega_{\varsigma(n)}. \tag{32}$$

Since $\omega_{\varsigma(n)} \leq \omega_{\varsigma(n)+1}$, it follows from (31) that

$$\begin{aligned}\omega_{\varsigma(n)} \leq \frac{\gamma_{\varsigma(n)}}{2(1-\sigma)} M + \frac{1}{1-\sigma}\langle \varphi(p) - p, x_{\varsigma(n)+1} - p\rangle \\ + \frac{1}{1-\sigma}(1 - \gamma_{\varsigma(n)} - \delta_{\varsigma(n)} - \zeta_{\varsigma(n)})M,\end{aligned} \tag{33}$$

Combining (30) and (33), we have $\limsup_{n\to\infty} \omega_{\varsigma(n)} \leq 0$ and hence $\lim_{n\to\infty} \omega_{\varsigma(n)} = 0$ which together with (32) implies that $\lim_{n\to\infty} \omega_{\varsigma(n)+1} = 0$. Applying Lemma 2.2 to get $0 \leq \omega_n \leq \max\{\omega_{\varsigma(n)}, \omega_{\varsigma(n)+1}\}$ which indicates $\omega_n \to 0$, i.e., $x_n \to p$. The proof is completed. □

4. **Conclusion**

Krasnoselskii-Mann's method fails to converge for a pseudocontractive operator T. At the same time, strong convergence of Ishikawa's method has not been obtained without compactness assumption on C or T. In this paper, we construct a self-adaptive algorithm for finding a fixed point of a Lipschitz pseudocontractive operator. Our algorithm has no need to know a priori the Lipschitz constant of pseudocontractive operators. Strong convergence result is obtained under some standard conditions.

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