ON THE APPROXIMATE DUALITY OF G-FRAMES AND FUSION FRAMES

Morteza Mirzaee Azandaryani[1]

In this paper we obtain some new results for the approximate duality of frames and g-frames in Hilbert spaces; especially we consider approximate duals of Riesz bases and g-Riesz bases. We also introduce a new kind of approximate duals for g-frames and fusion frames and generalize some of the results obtained for duals and approximate duals. Moreover, we introduce θ and $(\theta, \|\theta\|)-$approximate g-duals, where θ is a bounded operator on a separable Hilbert space and we show that in this case approximate duals share many useful properties with those introduced for frames, g-frames and fusion frames.

Keywords: Frame, g-frame, fusion frame, approximate duality

MSC2010: 41A 58, 42C 15

1. Introduction

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [11] in 1952 to study some problems in nonharmonic Fourier series, reintroduced in 1986 by Daubechies, Grossmann and Meyer [10].

Let H be a separable Hilbert space and let I be a finite or countable index set. A family $\mathcal{F} = \{f_i\}_{i\in I} \subseteq H$ is a *frame* for H, if there exist two positive numbers A and B such that

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2,$$

for each $f \in H$. A and B are the *lower* and *upper* frame bounds, respectively. If $A = B$, $\mathcal{F}$ is called an $A-$*tight frame.* If $A = B = 1$, it is called a *Parseval* frame. If only the second inequality is required, $\mathcal{F}$ is a *B-Bessel* sequence. If $\mathcal{F}$ is a Bessel sequence, then the *synthesis operator* $T_{\mathcal{F}} : \ell^2(I) \longrightarrow H$ which is defined by $T_{\mathcal{F}}(\{c_i\}_{i\in I}) = \sum_{i\in I} c_i f_i$ is bounded. Its adjoint operator $T^*_{\mathcal{F}}(f) = \{\langle f, f_i\rangle\}_{i\in I}$ is called the *analysis operator* of $\mathcal{F}$. The operator $S_{\mathcal{F}}(f) = T_{\mathcal{F}}T^*_{\mathcal{F}}(f) = \sum_{i\in I}\langle f, f_i\rangle f_i$ is bounded and positive. If $\mathcal{F}$ is a frame, we call $S_{\mathcal{F}}$ the frame operator of $\mathcal{F}$ which is invertible. In this case $\{S^{-1}_{\mathcal{F}} f_i\}_{i\in I}$ is also a frame and if $\tilde{f}_i = S^{-1}_{\mathcal{F}} f_i$, then each $f \in H$ can be reconstructed as

$$\sum_{i\in I}\langle f, \tilde{f}_i\rangle f_i = f = \sum_{i\in I}\langle f, f_i\rangle \tilde{f}_i.$$

$\widetilde{\mathcal{F}} = \{\tilde{f}_i\}_{i\in I}$ is called the *canonical dual* of $\mathcal{F}$. We say that a Bessel sequence $\{g_i\}_{i\in I}$ is an *alternate dual* or a *dual* for a Bessel sequence $\{f_i\}_{i\in I}$, if for each $f \in H$, we have $f = \sum_{i\in I}\langle f, f_i\rangle g_i$ or equivalently $f = \sum_{i\in I}\langle f, g_i\rangle f_i$. For more results about frames in Hilbert spaces, see [8].

Fusion frames [7] and g-frames [23] are two important generalizations of frames. For each $i \in I$, let H_i be a Hilbert space. In this paper $L(H, H_i)$ is the set of all bounded

[1]Assistant Professor, Department of Mathematics, University of Qom, Qom, Iran, E-mail: morteza_ma62@yahoo.com, m.mirzaee@qom.ac.ir

operators from H into H_i and $L(H,H)$ is denoted by $L(H)$. We call $\Lambda = \{\Lambda_i \in L(H,H_i) : i \in I\}$ a *g-frame* for H with respect to $\{H_i : i \in I\}$ if there exist two positive constants A and B such that

$$A\|f\|^2 \le \sum_{i\in I} \|\Lambda_i f\|^2 \le B\|f\|^2,$$

for each $f \in H$. If only the second inequality is required, we call it a *g-Bessel sequence* with upper bound B.
Note that $\oplus_{i\in I} H_i = \left\{ \{f_i\}_{i\in I} | f_i \in H_i, \|\{f_i\}_{i\in I}\|_2{}^2 = \sum_{i\in I}\|f_i\|^2 < \infty \right\}$ with pointwise operations and the inner product defined by $\langle \{f_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle = \sum_{i\in I}\langle f_i, g_i\rangle$ is a Hilbert space. If $H_i = H$ for each $i \in I$, we denote $\oplus_{i\in I} H_i$ by $\ell^2(I,H)$.
For a g-Bessel sequence $\Lambda = \{\Lambda_i \in L(H,H_i) : i \in I\}$ the *synthesis operator* is $T_\Lambda : \oplus_{i\in I} H_i \longrightarrow H$, $T_\Lambda(\{f_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^* f_i$ and its adjoint operator which is $T_\Lambda^*(f) = \{\Lambda_i f\}_{i\in I}$ is called the *analysis operator* of Λ. The operator S_Λ is defined by $S_\Lambda = T_\Lambda T_\Lambda^*$. If Λ is a g-frame, then S_Λ is invertible. The *canonical g-dual* for Λ is defined by $\widetilde{\Lambda} = \{\tilde{\Lambda_i}\}_{i\in I}$ where $\tilde{\Lambda_i} = \Lambda_i S_\Lambda^{-1}$ which is a g-frame and for each $f \in H$, we have

$$f = \sum_{i\in I} \Lambda_i^* \tilde{\Lambda}_i f = \sum_{i\in I} \tilde{\Lambda_i}^* \Lambda_i f.$$

Also a g-Bessel sequence $\Gamma = \{\Gamma_i \in L(H,H_i) : i \in I\}$ is called an *alternate g-dual* or a *g-dual* for a g-Bessel sequence Λ if

$$f = \sum_{i\in I} \Gamma_i^* \Lambda_i f = \sum_{i\in I} \Lambda_i^* \Gamma_i f,$$

for each $f \in H$.

Let $\{W_i\}_{i\in I}$ be a family of closed subspaces of a Hilbert space H. Let $\{\omega_i\}_{i\in I}$ be a family of weights, i.e., $\omega_i > 0$ for each $i \in I$. Then $\mathcal{W} = \{(W_i,\omega_i)\}_{i\in I}$ is a *fusion frame*, if there exist two positive numbers A and B such that for each $f \in H$,

$$A\|f\|^2 \le \sum_{i\in I} \omega_i^2 \|\pi_{W_i}(f)\|^2 \le B\|f\|^2,$$

where π_{W_i} is the orthogonal projection onto the subspace W_i. If only the right-hand inequality is required, then $\mathcal{W}$ is called a *Bessel fusion sequence*. Parseval and tight g-frames and fusion frames are defined similar to frames.
Note that $\mathcal{W} = \{(W_i,\omega_i)\}_{i\in I}$ is a Bessel fusion sequence (resp. a fusion frame) if and only if $\Lambda_{\mathcal{W}} = \{\omega_i \pi_{W_i}\}_{i\in I}$ is a g-Bessel sequence (resp. a g-frame). Hence every Bessel fusion sequence generates a g-Bessel sequence.
Frames usually provide non-unique representations of vectors and this property is desirable in applications especially in signal processing. As we see in the definition of duals, if a dual of a frame is obtained, then every signal can be easily reconstructed. For a finite-dimensional Hilbert space, the inverse of the frame operator can be obtained using linear algebra methods. Hence the canonical dual of a frame is simply calculated. But in the infinite-dimensional case, the canonical dual and also alternate duals are often difficult to be found. In this situation approximate duals can be useful. If $\mathcal{G}$ is an approximate dual of $\mathcal{F}$, then the composition of the synthesis and analysis operators of $\mathcal{G}$ and $\mathcal{F}$ is invertible and we use this invertible operator for the reconstruction of signals instead of the frame operator. For more applications of approximate duals, see [6, 24, 15, 9].
Approximate duality of frames was recently investigated by Christensen and Laugesen in [9]. Now we recall the definition:

Definition 1.1. *Let $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{G} = \{g_i\}_{i\in I}$ be two Bessel sequences for H. Suppose that $S_{\mathcal{G}\mathcal{F}} = T_{\mathcal{G}}T_{\mathcal{F}}^*$. We say that $\mathcal{F}$ and $\mathcal{G}$ are approximately dual frames if $\|Id_H - S_{\mathcal{G}\mathcal{F}}\| < 1$ or $\|Id_H - S_{\mathcal{F}\mathcal{G}}\| < 1$. In this case we call $\mathcal{G}$ (resp. $\mathcal{F}$) an* approximate dual *of $\mathcal{F}$ (resp. $\mathcal{G}$).*

Recently the present author and A. Khosravi introduced approximate duality for g-frames in [19] and some applications of approximate duals such as the stability under small perturbations and facilitating the reconstruction of signals were obtained (see also [21]). Trivially duals and approximate duals can be defined for a fusion frame as some kind of g-frame. We obtained some results for approximate duals of fusion frames in Corollaries 2.4, 3.3, 3.9 and Proposition 2.14 in [19] (see also [2, 3]). In this paper we introduce Q-approximate duality for g-frames and fusion frames and generalize some of the results obtained for duals and approximate duals of frames and g-frames. We also introduce θ and $(\theta, \|\theta\|)-$approximate g-duals, where θ is a bounded operator on a separable Hilbert space.

2. **Approximate duals for g-frames**

In this section we get some new results for approximate duals of frames and g-frames. First we recall the definition of approximate duality for g-frames from [19]:

Definition 2.1. *Let Λ and Γ be two g-Bessel sequences and $S_{\Gamma\Lambda} = T_\Gamma {T_\Lambda}^*$. Then Λ and Γ are approximately dual g-frames if $\|Id_H - S_{\Gamma\Lambda}\| < 1$ or $\|Id_H - S_{\Lambda\Gamma}\| < 1$. In this case, we say that Γ (resp. Λ) is an approximate dual g-frame or an* approximate g-dual *of Λ (resp. Γ).*

The conditions in the above definition are equivalent because $(Id_H - S_{\Gamma\Lambda})^* = Id_H - S_{\Lambda\Gamma}$. Since $\|Id_H - S_{\Lambda\Gamma}\| < 1$, we obtain that $S_{\Lambda\Gamma}$ is invertible with ${S_{\Lambda\Gamma}}^{-1} = \sum_{n=0}^{\infty} (Id_H - S_{\Lambda\Gamma})^n$. Now for each $f \in H$, we have the following reconstruction formulas:

$$f = \sum_{n=0}^{\infty} S_{\Lambda\Gamma}(Id_H - S_{\Lambda\Gamma})^n f, \qquad f = \sum_{n=0}^{\infty} (Id_H - S_{\Lambda\Gamma})^n S_{\Lambda\Gamma} f.$$

It is also obtained from Theorem 2.3 in [19] that if Λ and Γ are approximately dual g-frames, then Λ and Γ are g-frames.
Throughout this section $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in I\}$ and $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in I\}$ are g-Bessel sequences with upper bounds B and D, respectively.

Theorem 2.1. *Let $\mathcal{F}_i = \{f_{ij}\}_{j\in J_i}$ and $\mathcal{G}_i = \{g_{ij}\}_{j\in J_i}$ be B'_i and D'_i-Bessel sequences for H_i, respectively with $\sup_{i\in I}\{B'_i\} < \infty$ and $\sup_{i\in I}\{D'_i\} < \infty$.*

- (i) *If Λ is a g-dual of Γ with $BD < 1$ and $\mathcal{F}_i$ is an approximate dual of $\mathcal{G}_i$, for each $i \in I$, then $\{\Lambda_i^*(f_{ij})\}_{i\in I, j\in J_i}$ is an approximate dual of $\{\Gamma_i^*(g_{ij})\}_{i\in I, j\in J_i}$.*
- (ii) *Let $\mathcal{F}_i$ be a dual of $\mathcal{G}_i$, for each $i \in I$. Then Λ is an approximate g-dual (resp. a g-dual) of Γ if and only if $\{\Lambda_i^*(f_{ij})\}_{i\in I, j\in J_i}$ is an approximate dual (resp. a dual) of $\{\Gamma_i^*(g_{ij})\}_{i\in I, j\in J_i}$.*

Proof. (i) It is easy to see that $\mathcal{F} = \{\Lambda_i^*(f_{ij})\}_{i\in I, j\in J_i}$ and $\mathcal{G} = \{\Gamma_i^*(g_{ij})\}_{i\in I, j\in J_i}$ are $B'B$ and $D'D$-Bessel sequences, respectively where $B' = \sup_{i\in I}\{B'_i\}$ and $D' = \sup_{i\in I}\{D'_i\}$. Since $\|S_{\mathcal{G}_i\mathcal{F}_i}\| \leq \sqrt{B'_i D'_i} \leq \sqrt{B'D'}$, we get $\sum_{i\in I} \|S_{\mathcal{G}_i\mathcal{F}_i}\Lambda_i f\|^2 \leq B'D'B\|f\|^2$, for each $f \in H$.

Hence $\Phi = \{S_{\mathcal{G}_i\mathcal{F}_i}\Lambda_i\}_{i\in I}$ is a g-Bessel sequence. Now we have

$$\begin{aligned}\|S_{\mathcal{G}\mathcal{F}}f - f\| &= \Big\|\sum_{i\in I}\Gamma_i^*\Big(\sum_{j\in J_i}\langle\Lambda_i f, f_{ij}\rangle g_{ij}\Big) - f\Big\| \\ &= \Big\|\sum_{i\in I}\Gamma_i^*(S_{\mathcal{G}_i\mathcal{F}_i}\Lambda_i f) - f\Big\| = \|T_\Gamma T_\Phi^* f - T_\Gamma T_\Lambda^* f\| \\ &\leq \sqrt{D}\Big(\sum_{i\in I}\|S_{\mathcal{G}_i\mathcal{F}_i} - Id_{H_i}\|^2\|\Lambda_i f\|^2\Big)^{\frac{1}{2}} \leq \sqrt{BD}\|f\|.\end{aligned}$$

This means that $\|S_{\mathcal{G}\mathcal{F}} - Id_H\| \leq \sqrt{BD} < 1$, so $\mathcal{F}$ is an approximate dual of $\mathcal{G}$.
(ii) Let $f \in H$. Then

$$\begin{aligned}S_{\mathcal{G}\mathcal{F}}f = \sum_{i\in I}\sum_{j\in J_i}\langle f, \Lambda_i^* f_{ij}\rangle\Gamma_i^*(g_{ij}) &= \sum_{i\in I}\Gamma_i^*\Big(\sum_{j\in J_i}\langle\Lambda_i f, f_{ij}\rangle g_{ij}\Big) \\ &= \sum_{i\in I}\Gamma_i^*\Lambda_i f = S_{\Gamma\Lambda}f.\end{aligned}$$

The above equality implies that $\mathcal{F}$ is an approximate dual (resp. a dual) of $\mathcal{G}$ if and only if Λ is an approximate g-dual (resp. a g-dual) of Γ. □

Corollary 2.1. (i) *Suppose that $\{f_{ij}\}_{j\in J_i}$ is an A_i-tight frame such that there exist positive numbers B_1 and B_2 with $B_1 \leq A_i \leq B_2$, for each $i \in I$. Then Λ is an approximate g-dual (resp. a g-dual) of Γ if and only if $\{\Lambda_i^*(f_{ij})\}_{i\in I, j\in J_i}$ is an approximate dual (resp. a dual) of $\{\frac{1}{A_i}\Gamma_i^*(f_{ij})\}_{i\in I, j\in J_i}$.*
(ii) *Let $\{f_{ij}\}_{j\in J_i}$ be a Parseval frame, for each $i \in I$. Then Λ is an approximate g-dual (resp. a g-dual) of Γ if and only if $\{\Lambda_i^*(f_{ij})\}_{i\in I, j\in J_i}$ is an approximate dual (resp. a dual) of $\{\Gamma_i^*(f_{ij})\}_{i\in I, j\in J_i}$.*

Proof. (i) It is easy to see that $\mathcal{F}_i = \{f_{ij}\}_{j\in J_i}$ is a dual of $\mathcal{G}_i = \{\frac{1}{A_i}f_{ij}\}_{j\in J_i}$, for each $i \in I$. Now the result follows from part (ii) of Theorem 2.1.
(ii) We get the result from part (i) by considering $A_i = 1$, for each $i \in I$. □

Since an orthonormal basis is a Parseval frame, part (i) of Theorem 2.5 in [19] is a special case of the above corollary.

Proposition 2.1. *Γ is an approximate g-dual of Λ if and only if there exists an operator T on H with $\|T - Id_H\| < 1$ such that $\{\Gamma_i T^{-1}\}_{i\in I}$ is a g-dual of Λ.*

Proof. Since Γ is an approximate g-dual of Λ, we have $\|S_{\Lambda\Gamma} - Id_H\| < 1$. By Neumann algorithm $T = S_{\Lambda\Gamma}$ is invertible and $\sum_{i\in I}\Lambda_i^*\Gamma_i S_{\Lambda\Gamma}^{-1}f = f$, for each $f \in H$. Hence $\{\Gamma_i T^{-1}\}_{i\in I}$ is a g-dual of Λ.
For the converse, suppose that there exists an operator T on H with $\|T - Id_H\| < 1$ such that $\Phi = \{\Gamma_i T^{-1}\}_{i\in I}$ is a g-dual of Λ. Now we have

$$\|S_{\Gamma\Lambda} - Id_H\| = \|T^* S_{\Phi\Lambda} - Id_H\| = \|(T - Id_H)^*\| < 1.$$

This means that Γ is an approximate g-dual of Λ. □

We say that $\{f_i\}_{i\in I}$ is a *Riesz basis* for H, if it is complete in H and there exist two constants $0 < A \leq B < \infty$, such that

$$A\sum_{i\in F}|c_i|^2 \leq \Big\|\sum_{i\in F}c_i f_i\Big\|^2 \leq B\sum_{i\in F}|c_i|^2,$$

for each sequence of scalars $\{c_i\}_{i\in F}$, where F is a finite subset of I.
$\Lambda = \{\Lambda_i \in L(H, H_i) : i \in I\}$ is called *g-complete* if $\{f : \Lambda_i f = 0, \forall i \in I\} = \{0\}$. We call Λ a *g-orthonormal basis* for H, if

$$\langle \Lambda_{i_1}^* f_{i_1}, \Lambda_{i_2}^* f_{i_2} \rangle = \delta_{i_1,i_2} \langle f_{i_1}, f_{i_2} \rangle, \quad i_1, i_2 \in I, f_{i_1} \in H_{i_1}, f_{i_2} \in H_{i_2},$$

and $\sum_{i\in I} \|\Lambda_i f\|^2 = \|f\|^2$, for each $f \in H$. $\{\Lambda_i\}_{i\in I}$ is a *g-Riesz basis* for H, if it is g-complete and there exist two constants $0 < A \leq B < \infty$, such that for each finite subset $F \subseteq I$ and $f_i \in H_i$, $i \in F$,

$$A \sum_{i\in F} \|f_i\|^2 \leq \left\| \sum_{i\in F} \Lambda_i^* f_i \right\|^2 \leq B \sum_{i\in F} \|f_i\|^2.$$

Recall that if P is an invertible operator on H and $\Lambda_i = \Gamma_i P$, for each $i \in I$, then we say that Λ and Γ are *P-equivalent*. Also if $\{f_i\}_{i\in I}, \{g_i\}_{i\in I} \subseteq H$ and $f_i = P g_i$, for each $i \in I$, then $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are *P-equivalent* (see [5]). Note that if $\{f_i\}_{i\in I}$ is a Riesz basis, then Theorem 3.2.2 in [8] implies that $\{\widetilde{f_i}\}_{i\in I}$ is the unique dual of $\{f_i\}_{i\in I}$ and it is also a Riesz basis. A similar result can be obtained for g-Riesz bases using Theorem 3.1 in [23]. But a Riesz basis can have many approximate duals. For example if $\{e_i\}_{i\in I}$ is an orthonormal basis for H and $0 < a < 2$, then $\{a e_i\}_{i\in I}$ is an approximate dual for $\{e_i\}_{i\in I}$. In the following proposition and corollary we show that every approximate g-dual (resp. approximate dual) of a g-Riesz basis (resp. Riesz basis) is also a g-Riesz basis (resp. Riesz basis).

Proposition 2.2. *Let Λ be a g-Riesz basis. Then*

(i) *Γ is an approximate g-dual of Λ if and only if there exists an operator T on H with $\|T - Id_H\| < 1$ such that $\Gamma_i = \widetilde{\Lambda_i} T$, for each $i \in I$.*
(ii) *If Γ is an approximate g-dual of Λ, then Γ and Λ are P- equivalent for some invertible operator P on H and Γ is a g-Riesz basis.*

Proof. (i) Since Λ is a g-Riesz basis, by Theorem 3.1 in [23] and Theorem 5.5.4 in [8], $\widetilde{\Lambda}$ is the unique g-dual of Λ. Hence by Proposition 2.1, Γ is an approximate g-dual of Λ if and only if there exists an operator T on H such that $\|T - Id_H\| < 1$ with $\Gamma_i T^{-1} = \widetilde{\Lambda_i}$ consequently $\Gamma_i = \widetilde{\Lambda_i} T$, for each $i \in I$.
(ii) It follows from part (i) that there exists an invertible operator T on H with $\Gamma_i = \widetilde{\Lambda_i} T = \Lambda_i S_\Lambda^{-1} T$. Since $P = S_\Lambda^{-1} T$ is invertible, Γ and Λ are P-equivalent. Because Λ is a g-Riesz basis, by Corollary 3.4 in [23], there exists a g-orthonormal basis $\{Q_i\}_{i\in I}$ and an invertible operator U on H such that $\Lambda_i = Q_i U$, so $\Gamma_i = Q_i U P$ and since UP is invertible, again by Corollary 3.4 in [23], we obtain that Γ is a g-Riesz basis. □

Now using Propositions 2.1, 2.2 and the equivalent conditions for a frame to be a Riesz basis stated in Definition 3.3.1 and Theorem 3.3.7 in [8], we get the following result for frames:

Corollary 2.2. (i) *$\{g_i\}_{i\in I}$ is an approximate dual of $\{f_i\}_{i\in I}$ if and only if there exists an operator T on H with $\|T - Id_H\| < 1$ such that $\{T^{-1} g_i\}_{i\in I}$ is a dual of $\{f_i\}_{i\in I}$.*
(ii) *Let $\{f_i\}_{i\in I}$ be a Riesz basis. Then $\{g_i\}_{i\in I}$ is an approximate dual of $\{f_i\}_{i\in I}$ if and only if there exists an operator T on H with $\|T - Id_H\| < 1$ such that $g_i = T\widetilde{f_i}$. In this case $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are P-equivalent, for some invertible operator P on H and $\{g_i\}_{i\in I}$ is also a Riesz basis.*

3. $Q-$approximate duality for g-frames and fusion frames

In this section, we introduce a new kind of approximate duality for g-frames and fusion frames and we study their properties. In this section $\mathcal{W}$ and $\mathcal{V}$ are $\{(W_i, \omega_i)\}_{i\in I}$ and $\{(V_i, \upsilon_i)\}_{i\in I}$, respectively. Also $\Lambda_{\mathcal{W}} = \{\omega_i \pi_{W_i}\}_{i\in I}$ and $\Lambda_{\mathcal{V}} = \{\upsilon_i \pi_{V_i}\}_{i\in I}$.

First we recall parts (i) and (ii) of the following definition from [16] and [14], respectively. Part (iii) uses the fact that every Bessel fusion sequence generates a g-Bessel sequence.

Definition 3.1. *Let $\mathcal{W}$ and $\mathcal{V}$ be Bessel fusion sequences for H.*

(i) *If there exists an operator $Q \in L(\ell^2(I,H))$ such that $T_{\Lambda_\mathcal{W}} Q T^*_{\Lambda_\mathcal{V}} = Id_H$, then $\mathcal{W}$ is called a* Q-dual *of $\mathcal{V}$.*

(ii) *Let $\mathcal{V}$ be a fusion frame. Then we say that $\mathcal{W}$ is an* alternate dual *or a* dual *of $\mathcal{V}$ if $\sum_{i\in I} \upsilon_i\omega_i \pi_{W_i} S^{-1}_{\Lambda_\mathcal{V}} \pi_{V_i} f = f$, for each $f \in H$.*

(iii) *We say that $\mathcal{W}$ is a g-dual of $\mathcal{V}$ if $\Lambda_\mathcal{W}$ is a g-dual of $\Lambda_\mathcal{V}$.*

Now we introduce Q-duals and Q-approximate duals for g-Bessel sequences:

Definition 3.2. *Let Λ and Γ be g-Bessel sequences for H.*

(i) *If there exists an operator $Q \in L(\oplus_{i\in I} H_i)$ such that $T_\Lambda Q T^*_\Gamma = Id_H$, then Λ is called a* Q-dual *of Γ.*

(ii) *If there exists an operator $Q \in L(\oplus_{i\in I} H_i)$ such that $\|T_\Lambda Q T^*_\Gamma - Id_H\| < 1$, then Λ is called a* Q-approximate dual *of Γ.*

Note that if Λ is an approximate g-dual (resp. g-dual) of Γ, then Λ is a Q-approximate dual (resp. Q-dual) of Γ with $Q = Id_{(\oplus_{i\in I} H_i)}$.

Theorem 3.1. *Let Λ and Γ be g-Bessel sequences for H. If Λ is a Q-approximate dual of Γ, then*

(i) $\|T_\Gamma Q^* T^*_\Lambda - Id_H\| < 1$.

(ii) *T^*_Γ is injective and $T_\Lambda Q$ is surjective.*

(iii) *T^*_Λ is injective and $T_\Gamma Q^*$ is surjective.*

(iv) *Λ and Γ are g-frames.*

Proof. (i) We have

$$\|T_\Gamma Q^* T^*_\Lambda - Id_H\| = \|(T_\Lambda Q T^*_\Gamma - Id_H)^*\| = \|T_\Lambda Q T^*_\Gamma - Id_H\| < 1.$$

(ii) Since $\|T_\Lambda Q T^*_\Gamma - Id_H\| < 1$, by Newmann algorithm $T_\Lambda Q T^*_\Gamma$ is invertible. Hence T^*_Γ is injective and $T_\Lambda Q$ is surjective.

(iii) We can obtain the result similar to (ii) by using part (i).

(iv) Let $S_{\Lambda Q \Gamma} = T_\Lambda Q T^*_\Gamma$ and D be an upper bound for Γ. Then $S^*_{\Lambda Q\Gamma} = S_{\Gamma Q^*\Lambda}$ and since $\|S_{\Lambda Q\Gamma} - Id_H\| < 1$, $S_{\Lambda Q\Gamma}$ and $S_{\Gamma Q^*\Lambda}$ are invertible. Now for each $f \in H$, we have

$$\begin{aligned}
\|f\| = \|S^{-1}_{\Gamma Q^*\Lambda} S_{\Gamma Q^*\Lambda} f\| &\leq \|S^{-1}_{\Gamma Q^*\Lambda}\| \|S_{\Gamma Q^*\Lambda} f\| \\
&= \|S^{-1}_{\Gamma Q^*\Lambda}\| \Big(\sup_{\|g\|=1} |\langle S_{\Gamma Q^*\Lambda} f, g\rangle| \Big) \\
&= \|S^{-1}_{\Gamma Q^*\Lambda}\| \Big(\sup_{\|g\|=1} |\langle Q^*(\{\Lambda_i f\}_{i\in I}), T^*_\Gamma g\rangle| \Big) \\
&\leq \|S^{-1}_{\Gamma Q^*\Lambda}\| \|Q^*\| \|\{\Lambda_i f\}_{i\in I}\| \|T^*_\Gamma\| \\
&\leq \sqrt{D} \|S^{-1}_{\Gamma Q^*\Lambda}\| \|Q^*\| \Big(\sum_{i\in I} \|\Lambda_i f\|^2 \Big)^{\frac{1}{2}}.
\end{aligned}$$

Therefore $\frac{1}{D\|S^{-1}_{\Gamma Q^*\Lambda}\|^2 \|Q^*\|^2}$ is a lower bound for Λ. Similarly we can see that Γ is a g-frame. □

Now we introduce Q-approximate duality for Bessel fusion sequences:

Definition 3.3. *Let $\mathcal{W}$ and $\mathcal{V}$ be Bessel fusion sequences for H. If there exists an operator $Q \in L(\ell^2(I,H))$ such that $\|T_{\Lambda_\mathcal{W}} Q T^*_{\Lambda_\mathcal{V}} - Id_H\| < 1$, then $\mathcal{W}$ is called a* Q-approximate dual *of $\mathcal{V}$.*

As a consequence of Theorem 3.1 we get the following result which is a generalization of Lemma 3.2 in [16] to the approximate duality of fusion frames.

Theorem 3.2. *Let $\mathcal{W}$ and $\mathcal{V}$ be Bessel fusion sequences for H. If $\mathcal{W}$ is a Q-approximate dual of $\mathcal{V}$, then*

(i) $\|T_{\Lambda_{\mathcal{V}}} Q^* T^*_{\Lambda_{\mathcal{W}}} - Id_H\| < 1$.
(ii) $T^*_{\Lambda_{\mathcal{V}}}$ *is injective and* $T_{\Lambda_{\mathcal{W}}} Q$ *is surjective.*
(iii) $T^*_{\Lambda_{\mathcal{W}}}$ *is injective and* $T_{\Lambda_{\mathcal{V}}} Q^*$ *is surjective.*
(iv) $\mathcal{W}$ *and* $\mathcal{V}$ *are fusion frames.*

If $\mathcal{W}$ and $\mathcal{V}$ are Bessel fusion sequence and fusion frame, respectively, then by Lemma 3.9 in [7], the series $\sum_{i\in I} \upsilon_i \omega_i \pi_{W_i} S^{-1}_{\Lambda_{\mathcal{V}}} \pi_{V_i} f$ converges for each $f \in H$. Hence the operator $S_{\mathcal{V}_{\mathcal{W}}}$ defined on H by $S_{\mathcal{V}_{\mathcal{W}}} f = \sum_{i\in I} \upsilon_i \omega_i \pi_{W_i} S^{-1}_{\Lambda_{\mathcal{V}}} \pi_{V_i} f$ is bounded. Now we have two kinds of approximate duals for fusion frames which are special cases of Q-approximate duals (see also [1, 2, 3]):

Definition 3.4. (i) *Let $\mathcal{W}$ and $\mathcal{V}$ be Bessel fusion sequences and $S_{\mathcal{W}\mathcal{V}} = T_{\Lambda_{\mathcal{W}}} T^*_{\Lambda_{\mathcal{V}}}$. Then we say that $\mathcal{W}$ is an approximate g-dual of $\mathcal{V}$ if $\Lambda_{\mathcal{W}}$ is an approximate g-dual of $\Lambda_{\mathcal{V}}$, equivalently $\|S_{\mathcal{W}\mathcal{V}} - Id_H\| < 1$.*
(ii) *Let $\mathcal{W}$ and $\mathcal{V}$ be Bessel fusion sequence and fusion frame, respectively. Then we say that $\mathcal{W}$ is an approximate dual of $\mathcal{V}$ if $\|S_{\mathcal{V}_{\mathcal{W}}} - Id_H\| < 1$.*

If $\mathcal{W}$ is an approximate g-dual of $\mathcal{V}$, then $\mathcal{W}$ is a Q-approximate dual of $\mathcal{V}$ with $Q = Id_{\ell^2(I,H)}$. Also if $\mathcal{W}$ is an approximate dual of $\mathcal{V}$, then $\mathcal{W}$ is a Q-approximate dual of $\mathcal{V}$ with $Q(\{f_i\}_{i\in I}) = \{S^{-1}_{\Lambda_{\mathcal{V}}} f_i\}_{i\in I}$. Hence using Theorem 3.2, we get the following result which is a generalization of Theorem 2.3 in [19] and Proposition 2.8 in [14] to the approximate duality of fusion frames.

Proposition 3.1. (i) *If $\mathcal{W}$ is an approximate g-dual of $\mathcal{V}$, then $\mathcal{W}$ and $\mathcal{V}$ are fusion frames.*
(ii) *If $\mathcal{W}$ is an approximate dual of $\mathcal{V}$, then $\mathcal{W}$ is a fusion frame.*

Note that if $\mathcal{W}$ is a g-dual (resp. a Q-dual, an alternate dual) of $\mathcal{V}$, then $\mathcal{W}$ is an approximate g-dual (resp. a Q-approximate dual, an approximate dual) of $\mathcal{V}$ because $S_{\mathcal{W}\mathcal{V}} = Id_H$ (resp. $T_{\Lambda_{\mathcal{W}}} Q T^*_{\Lambda_{\mathcal{V}}} = Id_H$, $S_{\mathcal{V}_{\mathcal{W}}} = Id_H$). If $\mathcal{W}$ is an approximate g-dual (resp. a Q-approximate dual) of $\mathcal{V}$, then $\mathcal{V}$ is also an approximate g-dual (resp. a Q^*-approximate dual) of $\mathcal{W}$ since $\|S_{\mathcal{V}\mathcal{W}} - Id_H\| = \|(S_{\mathcal{W}\mathcal{V}} - Id_H)^*\| = \|(S_{\mathcal{W}\mathcal{V}} - Id_H)\| < 1$ (resp. $\|T_{\Lambda_{\mathcal{V}}} Q^* T^*_{\Lambda_{\mathcal{W}}} - Id_H\| < 1$).

Example 3.1. (i) *Let H be a Hilbert space, $\mathcal{W} = \{(H, \frac{1}{2})\}$ and $\mathcal{V} = \{(H, 2)\}$. Then $S_{\mathcal{V}_{\mathcal{W}}} = \frac{1}{4}.Id_H$, so $\|S_{\mathcal{V}_{\mathcal{W}}} - Id_H\| = \frac{3}{4} < 1$. Thus $\mathcal{W}$ is an approximate dual of $\mathcal{V}$. We also have $S_{\mathcal{W}_{\mathcal{V}}} = 4.Id_H$. Hence $\|S_{\mathcal{W}_{\mathcal{V}}} - Id_H\| = 3 > 1$. This shows that $\mathcal{V}$ is not an approximate dual of $\mathcal{W}$.*
(ii) *Let $\mathcal{V}$ be an $A-$tight fusion frame with $A > 2$. Then $S_{\mathcal{V}_{\mathcal{V}}} = Id_H$ and $S_{\mathcal{V}\mathcal{V}} = A.Id_H$. Therefore $\mathcal{V}$ is an approximate dual of itself but it is not an approximate g-dual of itself.*
(iii) *Let $\mathcal{W} = \{(H, 2)\}$ and $\mathcal{V} = \{(H, \frac{1}{2})\}$. Then $S_{\mathcal{V}\mathcal{W}} = Id_H$ and $S_{\mathcal{V}_{\mathcal{W}}} = 4.Id_H$. Hence $\mathcal{W}$ is an approximate g-dual of $\mathcal{V}$ but it is not an approximate dual of $\mathcal{V}$.*

In the following two propositions and corollary $\mathcal{F}_i = \{f_{ij}\}_{j\in J_i}$, ${\mathcal{F}_i}' = \{f'_{ij}\}_{j\in J_i}$ and $\mathcal{G}_i = \{g_{ij}\}_{j\in J_i}$, ${\mathcal{G}_i}' = \{g'_{ij}\}_{j\in J_i}$ are Bessel sequences for W_i and V_i, respectively such that the sequence of their upper bounds are bounded above.

Proposition 3.2. *Assume that ${\mathcal{F}_i}'$ and ${\mathcal{G}_i}'$ are duals of $\mathcal{F}_i$ and $\mathcal{G}_i$, respectively such that $\mathcal{F}'_i$ and $\mathcal{G}'_i$ are biorthogonal for each $i \in I$. Then $\mathcal{W}$ is an approximate g-dual (resp. a g-dual) of $\mathcal{V}$ if and only if $\{\omega_i f_{ij}\}_{i\in I, j\in J_i}$ is an approximate dual (resp. a dual) of $\{\upsilon_i g_{ij}\}_{i\in I, j\in J_i}$.*

Proof. Let B be an upper bound for $\mathcal{W}$ and $C = \sup_{i\in I}\{C_i\}$, where C_i is an upper bound for $\mathcal{F}_i$. Now for each $f \in H$, we have

$$\sum_{i\in I}\sum_{j\in J_i} |\langle f, \omega_i f_{ij}\rangle|^2 = \sum_{i\in I}\omega_i^2 \sum_{j\in J_i} |\langle \pi_{W_i} f, f_{ij}\rangle|^2 \le C\sum_{i\in I}\omega_i^2 \|\pi_{W_i} f\|^2 \le CB\|f\|^2,$$

so $\{\omega_i f_{ij}\}_{i\in I, j\in J_i}$ is a Bessel sequence. Similarly we can see that $\{\upsilon_i g_{ij}\}_{i\in I, j\in J_i}$ is a Bessel sequence for H. Let $f \in H$. Then

$$\begin{aligned} S_{\mathcal{V}\mathcal{W}} f &= \sum_{i\in I} \upsilon_i\omega_i \pi_{V_i}\left(\sum_{j\in J_i}\langle f, f_{ij}\rangle f'_{ij}\right) \\ &= \sum_{i\in I}\sum_{j\in J_i} \upsilon_i\omega_i \langle f, f_{ij}\rangle \pi_{V_i} f'_{ij} = \sum_{i\in I}\sum_{j\in J_i}\sum_{k\in J_i} \upsilon_i\omega_i\langle f, f_{ij}\rangle\langle f'_{ij}, g'_{ik}\rangle g_{ik} \\ &= \sum_{i\in I}\sum_{j\in J_i}\langle f, \omega_i f_{ij}\rangle \upsilon_i g_{ij} = S_{\mathcal{G}_{\mathcal{V}}\mathcal{F}_{\mathcal{W}}} f, \end{aligned}$$

where $\mathcal{F}_{\mathcal{W}} = \{\omega_i f_{ij}\}_{i\in I, j\in J_i}$ and $\mathcal{G}_{\mathcal{V}} = \{\upsilon_i g_{ij}\}_{i\in I, j\in J_i}$. This yields that $\mathcal{W}$ is an approximate g-dual (resp. a g-dual) of $\mathcal{V}$ if and only if $\mathcal{F}_{\mathcal{W}}$ is an approximate dual (resp. a dual) of $\mathcal{G}_{\mathcal{V}}$. □

Corollary 3.1. *Suppose that $\{f_{ij}\}_{j\in J_i}$ is a Riesz basis for W_i with upper bound B_i and $\sup_{i\in I}\{B_i\} < \infty$. Then $\mathcal{W}$ is an approximate g-dual (resp. a g-dual) of itself if and only if $\{\omega_i f_{ij}\}_{i\in I, j\in J_i}$ is an approximate dual (resp. a dual) of $\{\omega_i \widetilde{f_{ij}}\}_{i\in I, j\in J_i}$.*

Proof. Let $\mathcal{F}_i = \{f_{ij}\}_{j\in J_i} = \mathcal{G}_i{}'$ and $\mathcal{G}_i = \{\widetilde{f_{ij}}\}_{j\in J_i} = \mathcal{F}_i{}'$. Now we can get the result from the above proposition and Theorem 5.5.4 in [8]. □

The following proposition is a generalization of Theorem 3.12 in [16] to the approximate duality of fusion frames.

Proposition 3.3. *Suppose that $Q \in L(\ell^2(I, H))$ which is defined by $Q(\{h_i\}_{i\in I}) = \{\sum_{j\in J_i}\langle h_i, f_{ij}\rangle g_{ij}\}_{i\in I}$. Then the following conditions are equivalent:*

(i) *$\{\upsilon_i g_{ij}\}_{i\in I, j\in J_i}$ is an approximate dual of $\{\omega_i f_{ij}\}_{i\in I, j\in J_i}$.*
(ii) *$\mathcal{V} = \{(V_i, \upsilon_i)\}_{i\in I}$ is a Q-approximate dual of $\mathcal{W} = \{(W_i, \omega_i)\}_{i\in I}$.*

Proof. Similar to the proof of Theorem 3.12 in [16], we can obtain that Q is well-defined and bounded, also $T_{\Lambda_{\mathcal{V}}} Q T^*_{\Lambda_{\mathcal{W}}}(f) = \sum_{i\in I}\sum_{j\in J_i}\langle f, \omega_i f_{ij}\rangle \upsilon_i g_{ij} = S_{\mathcal{G}\mathcal{F}} f$, where $\mathcal{F} = \{\omega_i f_{ij}\}_{i\in I, j\in J_i}$ and $\mathcal{G} = \{\upsilon_i g_{ij}\}_{i\in I, j\in J_i}$. Hence $\|T_{\Lambda_{\mathcal{V}}} Q T^*_{\Lambda_{\mathcal{W}}} - Id_H\| < 1$ if and only if $\|S_{\mathcal{G}\mathcal{F}} - Id_H\| < 1$. □

4. Approximate duals for operators

Recently g-frames for operators and local g-atoms have been introduced in [4] as generalizations of frames for operators and local atoms for subspaces, for more results see [12, 13, 20].
In this section, we introduce $\theta-$approximate g-duals and $(\theta, \|\theta\|)-$approximate g-duals, where θ is a bounded operator on a separable Hilbert space. First we recall the following definition from [4].

Definition 4.1. *Let $\theta \in L(H)$. Then $\{\Lambda_i \in L(H, H_i) : i \in I\}$ is called a $\theta-$*g-frame *in H if the following holds:*

(i) *The series $\sum_{i\in I}\Lambda_i^* g_i$ converges for all $\{g_i\}_{i\in I} \in \oplus_{i\in I} H_i$.*
(ii) *There exists $B > 0$ such that for each $f \in H$ there exists $\{g_i\}_{i\in I} \in \oplus_{i\in I} H_i$ such that $\theta f = \sum_{i\in I}\Lambda_i^* g_i$ and $\sum_{i\in I}\|g_i\|^2 \le B\|f\|^2$.*

It was proved in Theorem 2.5 in [4] that $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in I\}$ is a $\theta-$g-frame if and only if $\{\Lambda_i\}_{i\in I}$ is a g-Bessel sequence and there exists a g-Bessel sequence $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in I\}$ such that $\theta f = \sum_{i\in I} \Lambda_i^* \Gamma_i f = S_{\Lambda\Gamma} f$, for each $f \in H$. In this case Γ is called a *$\theta-$g-dual* of Λ. Because in this case $S_{\Lambda\Gamma} = \theta$, we have $S_{\Gamma\Lambda} = \theta^*$. Thus if Γ is a $\theta-$g-dual of Λ, then Λ is a θ^*-g-dual of Γ.

Definition 4.2. *Let Λ and Γ be g-Bessel sequences and $\theta \in L(H)$. Then*

(i) Γ *is called a $\theta-$*approximate g-dual *of Λ if $\|\theta - S_{\Lambda\Gamma}\| < 1$.*
(ii) *Let $\|\theta\| < 1$. Then Γ is called a $(\theta, \|\theta\|)-$*approximate g-dual *of Λ if $\|\theta - S_{\Lambda\Gamma}\| \le \|\theta\|$.*

Since $\|\theta^* - S_{\Gamma\Lambda}\| = \|(\theta - S_{\Lambda\Gamma})^*\| = \|\theta - S_{\Lambda\Gamma}\|$ and $\|\theta\| = \|\theta^*\|$, if Γ is a $\theta-$approximate g-dual (resp. $(\theta, \|\theta\|)-$approximate g-dual) of Λ, then Λ is a θ^*-approximate g-dual (resp. $(\theta^*, \|\theta^*\|)-$approximate g-dual) of Γ.

Proposition 4.1. *Let θ be a self-adjoint operator on H. Then*

(i) *If there exist two g-Bessel sequences Λ and Γ such that Γ is a $\theta-$approximate g-dual (resp. $(\theta, \|\theta\|)-$approximate g-dual) of Λ and $\{\frac{\Gamma_i}{A}\}$ is a g-dual of Λ, for some $A \ge 1$ (resp. $A \ge \|\theta\|$), then θ is a positive operator.*
(ii) *If Λ is an $A-$tight g-frame, for some $A \ge 1$ (resp. $A \ge \|\theta\|$) such that Λ is a $\theta-$approximate g-dual (resp. $(\theta, \|\theta\|)-$approximate g-dual) of itself, then θ is a positive operator.*
(iii) *If there exists a Parseval g-frame which is a $\theta-$approximate g-dual or $(\theta, \|\theta\|)$-approximate g-dual of itself, then θ is a positive operator.*

Proof. (i) Since $\{\frac{\Gamma_i}{A}\}_{i\in I}$ is a g-dual of Λ, $S_{\Lambda\Gamma} = A.Id_H$. Therefore if Γ is a $\theta-$approximate g-dual of Λ, then $\|\theta - A.Id_H\| = \|\theta - S_{\Lambda\Gamma}\| < 1 \le A$ and if Γ is a $(\theta, \|\theta\|)-$approximate g-dual of Λ with $\|\theta\| \le A$, then $\|\theta - A.Id_H\| = \|\theta - S_{\Lambda\Gamma}\| \le \|\theta\| \le A$. Now Lemma 2.2.2 in [22] implies that θ is a positive operator.
(ii) Since Λ is an $A-$tight g-frame, $\{\frac{\Lambda_i}{A}\}_{i\in I}$ is a g-dual of Λ. Now the result follows from part (i).
(iii) We get the result by considering $A = 1$ in part (ii). □

Proposition 4.2. *Let θ be a positive operator on H. Then*

(i) *If $\|\theta\| < 1$, then every $\|\theta\|-$tight g-frame is a $(\theta, \|\theta\|)-$approximate g-dual of itself.*
(ii) *Every $A-$tight g-frame with $\|\theta\| \le A < 1$ is a $\theta-$approximate g-dual of itself.*

Proof. (i) Let Λ be a $\|\theta\|-$tight g-frame. Since θ is positive, Lemma 2.2.2 in [22] implies that $\|\theta - S_{\Lambda\Lambda}\| = \|\theta - \|\theta\|.Id_H\| \le \|\theta\|$, so Λ is a $(\theta, \|\theta\|)-$approximate g-dual of itself.
(ii) Let Λ be an $A-$tight g-frame with $\|\theta\| \le A < 1$. Since $\|\theta\| \le A$, by Lemma 2.2.2 in [22], $\|\theta - S_{\Lambda\Lambda}\| = \|\theta - A.Id_H\| \le A < 1$ and we get the result. □

Let $\Phi_j = \{\Lambda_{ij} \in L(H_j, H_{ij}) : i \in I\}$ be a g-Bessel sequence for H_j, with upper bound B_j such that $B = sup\{B_j : j \in J\} < \infty$. Then $\{\Phi_j\}_{j\in J}$ is called a *B-bounded family of g-Bessel sequences* or shortly *B-BFGBS*. In this case $\oplus_{j\in J}\Phi_j = \{\oplus_{j\in J}\Lambda_{ij} \in L(\oplus_{j\in J}H_j, \oplus_{j\in J}H_{ij}) : i \in I\}$ is a g-Bessel sequence with upper bound B (see Theorem 2.1 in [18]).

The following result is analogous to Proposition 3.2 in [18] and Proposition 2.8 in [19]. In the following proposition $\Psi_j = \{\Gamma_{ij} \in L(H_j, H_{ij}) : i \in I\}$.

Proposition 4.3. *Let $\{\Phi_j\}_{j\in J}$ and $\{\Psi_j\}_{j\in J}$ be BFGBS and $\theta_j \in L(H_j)$. Then*

(i) *Ψ_j is a θ_j-g-dual of Φ_j, for each $j \in J$ if and only if $\oplus_{j\in J}\Psi_j$ is a $\oplus_{j\in J}\theta_j-$g-dual of $\oplus_{j\in J}\Phi_j$.*

(ii) *Let J be a finite set. If Ψ_j is a θ_j-approximate g-dual (resp. $(\theta_j, \|\theta_j\|)-$approximate g-dual) of Φ_j, for each $j \in J$, then $\oplus_{j\in J}\Psi_j$ is a $\oplus_{j\in J}\theta_j-$approximate g-dual (resp. $\Big((\oplus_{j\in J}\theta_j), \| \oplus_{j\in J} \theta_j\|\Big)-$approximate g-dual) of $\oplus_{j\in J}\Phi_j$. The converse holds for θ_j-approximate g-duals.*

Proof. (i) Let B and D be upper bounds for Φ_j's and Ψ_j's, respectively. Since Ψ_j is a θ_j-g-dual of Φ_j, we have $\theta_j = S_{\Phi_j\Psi_j}$, so $\|\theta_j\| = \|S_{\Phi_j\Psi_j}\| \leq \sqrt{BD}$. Hence $\oplus_{j\in J}\theta_j$ is a bounded operator on $\oplus_{j\in J}H_j$. Let $\{f_j\}_{j\in J}$,$\{g_j\}_{j\in J} \in \oplus_{j\in J}H_j$. Similar to the proof of Proposition 3.2 in [18], we can see that $\sum_{j\in J}\sum_{i\in I}\langle\Gamma_{ij}f_j, \Lambda_{ij}g_j\rangle = \sum_{i\in I}\sum_{j\in J}\langle\Gamma_{ij}f_j, \Lambda_{ij}g_j\rangle$ and now it is easy to see that

$$\langle(\oplus_{j\in J}\theta_j)(\{f_j\}_{j\in J}), \{g_j\}_{j\in J}\rangle = \langle S_{(\oplus_{j\in J}\Phi_j)(\oplus_{j\in J}\Psi_j)}(\{f_j\}_{j\in J}), \{g_j\}_{j\in J}\rangle.$$

Hence $\oplus_{j\in J}\theta_j = S_{(\oplus_{j\in J}\Phi_j)(\oplus_{j\in J}\Psi_j)}$, so $\oplus_{j\in J}\Psi_j$ is a $(\oplus_{j\in J}\theta_j)-$g-dual of $\oplus_{j\in J}\Phi_j$. The converse is clear.
(ii) The result follows from the equalities

$$\|(\oplus_{j\in J}\theta_j) - S_{(\oplus_{j\in J}\Phi_j)(\oplus_{j\in J}\Psi_j)}\| = max\{\|\theta_j - S_{\Phi_j\Psi_j}\| : j \in J\},$$

and $\| \oplus_{j\in J} \theta_j\| = max\{\|\theta_j\| : j \in J\}$. □

The converse of part (ii) is not necessarily true for $(\theta_j, \|\theta_j\|)-$approximate g-duals. For example if $\theta_1 = -\frac{1}{8}.Id_H$, $\theta_2 = \frac{1}{2}.Id_H$, $\Phi_1 = \Psi_1 = \{\frac{1}{\sqrt{8}}.Id_H\}$ and $\Phi_2 = \Psi_2 = \{0\}$, then $\Psi_1 \oplus \Psi_2$ is a $\Big((\theta_1 \oplus \theta_2), \|\theta_1 \oplus \theta_2\|\Big)-$approximate g-dual of $\Phi_1 \oplus \Phi_2$ but Ψ_1 is not a $(\theta_1, \|\theta_1\|)-$approximate g-dual of Φ_1.

Let H and H' be Hilbert spaces. Then the tensor product $H \otimes H'$ is a Hilbert space, the inner product for simple tensors is defined by $\langle x \otimes x', y \otimes y'\rangle = \langle x, y\rangle\langle x', y'\rangle$, where $x, y \in H$ and $x', y' \in H'$. If U and U' are bounded operators on H and H', respectively, then $U \otimes U'$ is a bounded operator on $H \otimes H'$ which is defined on simple tensors by $U \otimes U'(x \otimes x') = (Ux) \otimes (U'x')$ and we have $(U \otimes U')^* = U^* \otimes U'^*$ and $\|U \otimes U'\| = \|U\|\|U'\|$. For more results, see [22].
The following result is analogous to Proposition 2.10 in [19].
In the following proposition Λ' and Γ' denote $\{\Lambda'_j \in L(H', H'_j) : j \in J\}$ and $\{\Gamma'_j \in L(H', H'_j) : j \in J\}$, respectively. Also $\Gamma \otimes \Gamma' = \{\Gamma_i \otimes \Gamma'_j\}_{i\in I, j\in J}$, $\Lambda \otimes \Lambda' = \{\Lambda_i \otimes \Lambda'_j\}_{i\in I, j\in J}$ and $\theta' \in L(H')$.

Proposition 4.4. *Let Γ and Γ' be $(\theta, \|\theta\|)-$approximate g-dual (resp. $\theta-$approximate g-dual) and $\theta'-$g-dual of Λ and Λ', respectively with $\|\theta'\| \leq 1$. Then $\Gamma \otimes \Gamma'$ is a $\Big((\theta \otimes \theta'), \|\theta \otimes \theta'\|\Big)-$approximate g-dual (resp. $(\theta \otimes \theta')-$approximate g-dual) of $\Lambda \otimes \Lambda'$.*

Proof. Similar to the proof of Proposition 2.10 in [19], we can see that $\Gamma \otimes \Gamma'$ and $\Lambda \otimes \Lambda'$ are g-Bessel sequences and $S_{(\Lambda\otimes\Lambda')(\Gamma\otimes\Gamma')} = S_{\Lambda\Gamma} \otimes S_{\Lambda'\Gamma'} = S_{\Lambda\Gamma} \otimes \theta'$. Now the result can be obtained using the equalities $\|(\theta \otimes \theta') - S_{(\Lambda\otimes\Lambda')(\Gamma\otimes\Gamma')}\| = \|(\theta - S_{\Lambda\Gamma}) \otimes \theta'\| = \|\theta - S_{\Lambda\Gamma}\|\|\theta'\|$ and $\|\theta \otimes \theta'\| = \|\theta\|\|\theta'\|$. □

Note that it is obtained from the proof of the above proposition that if Γ and Γ' are θ and $\theta'-$g-duals of Λ and Λ', respectively, then $\Gamma \otimes \Gamma'$ is a $(\theta \otimes \theta')-$g-dual of $\Lambda \otimes \Lambda'$.
We recall the following definition from [4].

Definition 4.3. *Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in I\}$ be a g-Bessel sequence and H_0 be a closed subspace of H. Then Λ is called a* family of local g-atoms *for H_0 with respect to $\{H_i\}_{i\in I}$, if there exists a g-Bessel sequence $\Gamma = \{\Gamma_i \in L(H_0, H_i) : i \in I\}$ such that $f = \sum_{i\in I}\Lambda_i^*\Gamma_i f$, for each $f \in H_0$.*

Now we introduce a family of approximately local g-atoms:

Definition 4.4. *Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in I\}$ be a g-Bessel sequence and H_0 be a closed subspace of H. Then Λ is called a* family of approximately local g-atoms *for H_0 with respect to $\{H_i\}_{i\in I}$, if there exists a g-Bessel sequence $\Gamma = \{\Gamma_i \in L(H_0, H_i) : i \in I\}$ and $K < 1$ such that $\|f - \sum_{i\in I} \Lambda_i^* \Gamma_i f\| \leq K\|f\|$, for each $f \in H_0$.*

It was proved in Theorem 2.14 in [4] that Λ is a family of local g-atoms for H_0 with respect to $\{H_i\}_{i\in I}$ if and only if Λ has a $P_{H_0}-$g-dual, where P_{H_0} is the orthogonal projection from H onto H_0.
In the following theorem we obtain a similar result for approximately local g-atoms.

Theorem 4.1. *Let Λ be a g-Bessel sequence. Then the following conditions are equivalent:*

(i) *Λ is a family of approximately local g-atoms for H_0 with respect to $\{H_i\}_{i\in I}$.*
(ii) *Λ has a $P_{H_0}-$approximate g-dual.*

Proof. (i) $\Longrightarrow$ (ii) Suppose that $\Gamma = \{\Gamma_i \in L(H_0, H_i) : i \in I\}$ is a g-Bessel sequence and $K < 1$ such that $\|f - \sum_{i\in I} \Lambda_i^* \Gamma_i f\| \leq K\|f\|$, for each $f \in H_0$. Let $\Psi = \{\Gamma_i P_{H_0}\}_{i\in I}$. Then it is easy to see that Ψ is a g-Bessel sequence and $\|P_{H_0} f - S_{\Lambda\Psi} f\| \leq K\|f\|$, for each $f \in H$. Hence $\|P_{H_0} - S_{\Lambda\Psi}\| \leq K < 1$, so Ψ is a $P_{H_0}-$approximate g-dual of Λ.
(ii) $\Longrightarrow$ (i) Suppose that $\Psi = \{\psi_i\}_{i\in I}$ is a $P_{H_0}-$approximate g-dual of Λ, so $\|P_{H_0} - S_{\Lambda\Psi}\| < 1$. Now for $\Gamma_i = \psi_i Id_{H_0}$ and $\Gamma = \{\Gamma_i\}_{i\in I}$, it is easy to see that Γ is a g-Bessel sequence and if $K = \|P_{H_0} - S_{\Lambda\Psi}\|$, then

$$\left\| f - \sum_{i\in I} \Lambda_i^* \Gamma_i f \right\| = \|P_{H_0} f - S_{\Lambda\Psi} f\| \leq K\|f\|,$$

for each $f \in H_0$. This means that Λ is a family of approximately local g-atoms for H_0 with respect to $\{H_i\}_{i\in I}$. □

REFERENCES

[1] A. A. Arefijamaal and F. Arabyani Neyshaburi, Characterization of dual Riesz fusion bases by using approximate duals, <www.arxiv.org>, math. RT/1601.03024.

[2] M. S. Asgari and G. Kavian, Some results about duality of generalized Bessel sequences, *Indian. J. Sci. Technology.* **7** (2014) 734–744.

[3] M. S. Asgari and G. Kavian, Duality of g-Bessel sequences and some results about RIP g-frames, *Int. J. Industrial Mathematics.* **7** (2015) 51–61.

[4] M. S. Asgari and H. Rahimi, Generalized frames for operators in Hilbert spaces, *Infin. Dimens. Anal. Quantum. Probab. Relat. Top.* **17** (2014) 1450013 (20 pages).

[5] R. Balan, Equivalence relations and distances between Hilbert frames, *Proc. Amer. Math. Soc.* **127** (1999) 2353–2366.

[6] P. Balazs, H. G. Feichtinger, M. Hampejs and G. Kracher, Double preconditioning for Gabor frames, *IEEE. Trans. Signal. Process.* **54** (2006) 4597–4610.

[7] P. Casazza and G. Kutyniok, Frames of subspaces, *Contemp. Math. Amer. Math. Soc.* **345** (2004) 87–113.

[8] O. Christensen, *Frames and Bases: An Introductory Course* (Birkhäuser, 2008).

[9] O. Christensen and R. S. Laugesen, Approximate dual frames in Hilbert spaces and applications to Gabor frames, *Sampl. Theory Signal Image Process.* **9** (2011) 77–90.

[10] I. Daubechies, A. Grossmann and Y. Meyer, Painless nonorthogonal expansions, *J. Math. Phys.* **27** (1986) 1271–1283.

[11] R. J. Duffin and A. C. Schaeffer, A class of nonharmonic Fourier series, *Trans. Amer. Math. Soc.* **72** (1952) 341–366.

[12] H. G. Feichtinger and T. Werther, Atomic systems for subspaces, in *Proceedings SampTA 2001*, Orlando, FL, ed. L. Zayed (2001), pp. 163–165.

[13] L. Gavruta, Frames for operators, *Appl. Comput. Harmon. Anal.* **32** (2012) 139–144.

[14] P. Gavruta, On the duality of fusion frames, *J. Math. Anal. Appl.* **333** (2007) 871–879.

[15] J. E. Gilbert, Y. S. Han, J. A. Hogan, J. D. Lakey, D. Weiland and G. Weiss, Smooth molecular decompositions of functions and singular integral operators, *Mem. Amer. Math. Soc.* **742** (2002) 1–74.

[16] S. B. Heineken, P. M. Morillas, A. M. Benavente and M. I. Zakowicz, Dual fusion frames, *Arch. Math.* **103** (2014) 355–365.

[17] A. Khosravi and M. Mirzaee Azandaryani, Fusion frames and g-frames in tensor product and direct sum of Hilbert spaces, *Appl. Anal. Discrete Math.* **6** (2012) 287–303.

[18] A. Khosravi and M. Mirzaee Azandaryani, G-frames and direct sums, *Bull. Malays. Math. Sci. Soc.* **36** (2013) 313–323.

[19] A. Khosravi and M. Mirzaee Azandaryani, Approximate duality of g-frames in Hilbert spaces, *Acta. Math. Sci.* **34** (2014) 639–652.

[20] S. Li and H. Ogawa, Pseudoframes for subspaces with applications, *J. Fourier Anal. Appl.* **10** (2004) 409–431.

[21] M. Mirzaee Azandaryani, Approximate duals and nearly Parseval frames, *Turk. J. Math.* **39** (2015) 515–526.

[22] G. J. Murphy, *C^*-Algebras and Operator Theory* (Academic Press, 1990).

[23] W. Sun, G-frames and g-Riesz bases, *J. Math. Anal. Appl.* **322** (2006) 437–452.

[24] T. Werther, Y. C. Eldar and N. K. Subbanna, Dual Gabor frames: theory and computational aspects, *IEEE. Trans. Signal Process.* **53** (2005) 4147–4158.