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# STUDY ON FRACTIONAL FEJÉR-HADAMARD TYPE INEQUALITIES ASSOCIATED WITH GENERALIZED EXPONENTIALLY CONVEXITY

by Ghulam Farid[1], Liliana Guran[2], Xiaoli Qiang[3] and Yu-Ming Chu[4]

*In this paper fractional integral inequalities of Fejér-Hadamard type for a generalized notion of convexity are established. A new generalization of convexity named exponentially $(\alpha, h-m)$-convex function unifies exponentially $(h-m)$-convex, exponentially $(\alpha-m)$-convex and exponentially $(s, m)$-convex functions. By using the generalized fractional integral operators involving Mittag-Leffler function via a monotonically increasing function we have obtained some fractional versions of Fejér-Hadamard inequality for the generalized convexity. The obtained results lead to many inequalities of Fejér-Hadamard and Hadamard type for well-known fractional integral operators and different kinds of convexities.*

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## 1. Introduction and Preliminaries

The theory of inequalities give an important tool for leading symmetrical phenomena in circumstances of real life. In the same time, the theory of convex functions have important implications in various fields of pure and applied sciences. A close connection exists between theory of convex functions, theory of inequalities and fractional calculus, the last one being one of the most studied field of mathematics due his application in the real world.

Many inequalities are proved for convex functions but, the most known from the related literature, is Hermite-Hadamard inequality.

A function $f : I \to \mathbb{R}$ on an interval of real line is said to be convex, if for all $u_1, u_2 \in I$ and $\omega \in [0, 1]$, the following inequality holds:

$$f(\omega u_1 + (1-\omega)u_2) \leq \omega f(u_1) + (1-\omega)f(u_2). \tag{1.1}$$

In 1883 and 1893 Hermite and Hadamard introduced, independently, the following inequality:

**Theorem 1.1.** *Let $f : [u_1, u_2] \to \mathbb{R}$ be a convex function such that $u_1 < u_2$. Then following inequality holds:*

$$f\left(\frac{u_1+u_2}{2}\right) \leq \frac{1}{u_2-u_1}\int_{u_1}^{u_2} f(\omega)d\omega \leq \frac{f(u_1)+f(u_2)}{2}. \tag{1.2}$$

[1]Department of Mathematics, COMSATS University Islamabad, Attock Campus, Pakistan, e-mail: ghlmfarid@ciit-attock.edu.pk

[2]Department of Pharmaceutical Sciences, "Vasile Goldiş" Western University, Arad, L. Rebreanu Street, no. 86, 310048, Romania, e-mail: guran.liliana@uvvg.ro

[3]Institute of Computing Science and Technology, Guangzhou University, 510006, Guangzhou, China, e-mail: qiangxl@gzhu.edu.cn

[4]Department of Mathematics, Huzhou University, Huzhou 313000, China, [5]Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha University of Science & Technology, Changsha 410114, China, e-mail: chuyuming2005@126.com

The inequality (**1**.2) is well-known in related literature as the Hadamard inequality. In 1906 Fejér gave the following generalization of Hadamard inequality:

**Theorem 1.2.** [14] *Let* $f : [u_1, u_2] \to \mathbb{R}$ *be a convex function such that* $u_1 < u_2$*. Also let* $g : [u_1, u_2] \to \mathbb{R}$ *be a positive, integrable and symmetric to* $\frac{u_1+u_2}{2}$*. Then the following inequality holds:*

$$f\left(\frac{u_1+u_2}{2}\right)\int_{u_1}^{u_2} g(\omega)d\omega \leq \int_{u_1}^{u_2} f(\omega)g(\omega)d\omega \leq \frac{f(u_1)+f(u_2)}{2}\int_{u_1}^{u_2} g(\omega)d\omega. \quad (\mathbf{1}.3)$$

The inequality (**1**.3) is well-known as the Fejér-Hadamard inequality in literature.

In the last years, the classical concept of convex functions was extended in different ways using innovative ideas. The Hadamard and the Fejér-Hadamard inequalities are generalized in various ways (see, [6, 7, 8, 9, 10, 13, 16, 17, 18, 22, 23, 24, 33, 35]).

Convex function play an important role in mathematical inequalities. It's extensions and generalizations have been defined in different ways and are used to extend the associated subjects. In this context the Hadamard inequality is analyzed extensively.

In this paper we present the Fejér-Hadamard inequalities for known generalized fractional integral operators by using a new generalized convexity recently defined in [19]. Let us recall this given notion as follows:

**Definition 1.1.** [19] *Let* $J \subseteq \mathbb{R}$ *be an interval containing* $(0,1)$ *and let* $h : J \to \mathbb{R}$ *be a non-negative function. Then a function* $f : I \to \mathbb{R}$ *on an interval of real line is said to be exponentially* $(\alpha, h-m)$*-convex, if for all* $u_1, u_2 \in I$*,* $\omega \in (0,1)$*,* $\alpha, m \in [0,1]$ *and* $\sigma \in \mathbb{R}$*, the following inequality holds:*

$$f(\omega u_1 + m(1-\omega)u_2) \leq h(\omega^\alpha)\frac{f(u_1)}{e^{\sigma u_1}} + mh(1-\omega^\alpha)\frac{f(u_2)}{e^{\sigma u_2}}. \quad (\mathbf{1}.4)$$

The above definition represents different classes of functions. For $\sigma = 0$, one gets the class of $(\alpha, h-m)$-convex functions and for other values of $\sigma$ it may not represents $(\alpha, h-m)$-convex functions; for this we give the following example corresponding to $\sigma = -1$.

**Example 1.1.** *The function* $f(x) = x\exp(-x)$ *is exponentially* $(1, I_d - 1)$*-convex function but not* $(1, I_d - 1)$*-convex function. More precisely the function* $f$ *is exponentially convex function on* $[0,\infty)$ *but not a convex function on this domain.*

By taking $\alpha = 1$ in (**1**.4), we get the following definition of exponentially $(h-m)$-convex functions.

**Definition 1.2.** [30] *Let* $J \subseteq \mathbb{R}$ *be an interval containing* $(0,1)$ *and let* $h : J \to \mathbb{R}$ *be a non-negative function. Then a function* $f : I \to \mathbb{R}$ *on an interval of real line is said to be exponentially* $(h-m)$*-convex, if for all* $u_1, u_2 \in I$*,* $\omega \in (0,1)$*,* $m \in [0,1]$ *and* $\sigma \in \mathbb{R}$*, the following inequality holds:*

$$f(\omega u_1 + m(1-\omega)u_2) \leq h(\omega)\frac{f(u_1)}{e^{\sigma u_1}} + mh(1-\omega)\frac{f(u_2)}{e^{\sigma u_2}}. \quad (\mathbf{1}.5)$$

By taking $h(\omega) = \omega$ in (**1**.4), then we get the following definition of exponentially $(\alpha - m)$-convex functions:

**Definition 1.3.** [30] *A function* $f : I \to \mathbb{R}$ *on an interval of real line is said to be exponentially* $(\alpha - m)$*-convex, if for all* $u_1, u_2 \in I$*,* $\omega \in (0,1)$*,* $\alpha, m \in [0,1]$ *and* $\sigma \in \mathbb{R}$*, the following inequality holds:*

$$f(\omega u_1 + m(1-\omega)u_2) \leq \omega^\alpha\frac{f(u_1)}{e^{\sigma u_1}} + m(1-\omega^\alpha)\frac{f(u_2)}{e^{\sigma u_2}}. \quad (\mathbf{1}.6)$$

**Remark 1.1.** *i) Taking $\alpha = 1$ and $h(\omega) = \omega^s$ in (**1**.4) we obtain exponentially $(s-m)$-convex function defined by Qiang et al. in* [29].
*ii) Taking $\alpha = m = 1$ and $h(\omega) = \omega^s$ in (**1**.4) we obtain exponentially $s$-convex function defined by Mehreen and Anwar in* [25].
*iii) Taking $\alpha = m = 1$ and $h(\omega) = \omega$ in (**1**.4) we obtain exponentially convex function defined by Awan et al. in* [2].
*iv) Taking $\sigma = 0$ in (**1**.4) we obtain $(\alpha, h-m)$-convex function defined by Farid et al. in* [11].
*v) Taking $\sigma = \alpha = 0$ and $\alpha = 1$ in (**1**.4) we obtain $(h-m)$-convex function defined by Ozdemir et al. in* [27].
*vi) Taking $\sigma = 0$ and $h(\omega) = \omega$ in (**1**.4) we obtain $(\alpha - m)$-convex function defined by Mihesan in* [26].
*vii) Taking $\sigma = 0$, $\alpha = 1$ and $h(\omega) = \omega^s$ in (**1**.4) we obtain $(s-m)$-convex function defined by Efthekhari in* [3].
*viii) Taking $\sigma = 0$, $\alpha = m = 1$ and $h(\omega) = \omega^s$ in (**1**.4) we obtain $s$-convex function defined by Hudzik and Maligranda in* [15].
*ix) Taking $\sigma = 0$, $\alpha = 1$ and $h(\omega) = \omega$ in (**1**.4) we obtain $m$-convex function defined by Toader in* [36].
*x) Taking $\sigma = 0$ and $\alpha = m = 1$ in (**1**.4) we obtain $h$-convex function defined by Varosanec in* [38].
*xi) Taking $\sigma = 0$, $\alpha = m = 1$ and $h(\omega) = \omega$ in (**1**.4) we obtain convex function.*

Fractional integral operators also play important role in the field of mathematical analysis. A large number of integral inequalities exist in literature due to fractional integral operators. For details see [4, 12, 20, 21, 37] and references therein. Recently, in [1], Andrić et al. defined the generalized fractional integral operators containing an extended Mittag-Leffler function as follows.

**Definition 1.4.** *Let $\mu, \nu, \kappa, l, \eta, c \in \mathbb{C}$, $\Re(\nu), \Re(\kappa), \Re(l) > 0$, $\Re(c) > \Re(\eta) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\nu)$. Let $f \in L_1[u_1, u_2]$ and $x \in [u_1, u_2]$. Then the generalized fractional integral operators $\zeta^{\eta,r,q,c}_{\nu,\kappa,l,\mu,u_1^+} f$ and $\zeta^{\eta,r,q,c}_{\nu,\kappa,l,\mu,u_2^-} f$ are defined by:*

$$\left(\zeta^{\eta,r,q,c}_{\nu,\kappa,l,\mu,u_1^+} f\right)(x;p) = \int_{u_1}^{x} (x-\omega)^{\kappa-1} E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu(x-\omega)^{\nu};p) f(\omega) d\omega, \tag{1.7}$$

$$\left(\zeta^{\eta,r,q,c}_{\nu,\kappa,l,\mu,u_2^-} f\right)(x;p) = \int_{x}^{u_2} (\omega-x)^{\kappa-1} E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu(\omega-x)^{\nu};p) f(\omega) d\omega, \tag{1.8}$$

*where $E^{\eta,r,q,c}_{\nu,\kappa,l}(\omega;p)$ is the generalized Mittag-Leffler function defined as follows:*

$$E^{\eta,r,q,c}_{\nu,\kappa,l}(\omega;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\sigma+nq, c-\sigma)}{\beta(\sigma, c-\sigma)} \frac{(c)_{nq}}{\Gamma(\nu n+\kappa)} \frac{\omega^n}{(l)_{nr}}.$$

In [5] Farid defined the following unified integral operators:

**Definition 1.5.** *Let $f, g : [u_1, u_2] \to \mathbb{R}$, $0 < u_1 < u_2$ be the functions such that $f$ be a positive and integrable and $g$ be a differentiable and strictly increasing. Also, let $\frac{\gamma}{x}$ be an increasing function on $[u_1, \infty)$ and $\mu, \kappa, l, \eta, c \in \mathbb{C}$, $\Re(\kappa), \Re(l) > 0$, $\Re(c) > \Re(\eta) > 0$ with $p \geq 0$, $\nu, r > 0$ and $0 < q \leq r + \nu$. Then for $x \in [u_1, u_2]$ the integral operators ${}_g\zeta^{\gamma,\eta,r,q,c}_{\nu,\kappa,l,u_1^+} f$ and ${}_g\zeta^{\gamma,\eta,r,q,c}_{\nu,\kappa,l,u_2^-} f$ are defined by:*

$$\left({}_g\zeta^{\gamma,\eta,r,q,c}_{\nu,\kappa,l,u_1^+} f\right)(x;p) = \int_{u_1}^{x} \frac{\gamma(g(x)-g(\omega))}{g(x)-g(\omega)} E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu(g(x)-g(\omega))^{\nu};p) f(\omega) d(g(\omega)), \tag{1.9}$$

$$\left({}_g\zeta^{\gamma,\eta,r,q,c}_{\nu,\kappa,l,u_2^-} f\right)(x;p) = \int_{x}^{u_2} \frac{\gamma(g(\omega)-g(x))}{g(\omega)-g(x)} E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu(g(\omega)-g(x))^{\nu};p) f(\omega) d(g(\omega)). \tag{1.10}$$

Replacing $\gamma(x) = x^\kappa$ in (**1**.9) and (**1**.10), we get the following generalized fractional integral operators containing Mittag-Leffler function:

**Definition 1.6.** *Let $f, g : [u_1, u_2] \to \mathbb{R}$, $0 < u_1 < u_2$ be the functions such that $f$ be a positive and integrable and $g$ be a differentiable and strictly increasing. Also let $\mu, \kappa, l, \eta, c \in \mathbb{C}$, $\Re(\kappa), \Re(l) > 0$, $\Re(c) > \Re(\eta) > 0$ with $p \geq 0$, $\nu, r > 0$ and $0 < q \leq r + \nu$. Then for $x \in [u_1, u_2]$ the integral operators ${}_g\zeta^{\eta,r,q,c}_{\nu,\kappa,l,\mu,u_1^+} f$ and ${}_g\zeta^{\eta,r,q,c}_{\nu,\kappa,l,\mu,u_2^-} f$ are defined by:*

$$\left({}_g\zeta^{\eta,r,q,c}_{\nu,\kappa,l,\mu,u_1^+} f\right)(x;p) = \int_{u_1}^{x} (g(x)-g(\omega))^{\kappa-1} E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu(g(x)-g(\omega))^\nu; p) f(\omega) d(g(\omega)), \quad (\mathbf{1}.11)$$

$$\left({}_g\zeta^{\eta,r,q,c}_{\nu,\kappa,l,\mu,u_2^-} f\right)(x;p) = \int_{x}^{u_2} (g(\omega)-g(x))^{\kappa-1} E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu(g(\omega)-g(x))^\nu; p) f(\omega) d(g(\omega)). \quad (\mathbf{1}.12)$$

**Remark 1.2.** *Integral operators (**1**.11) and (**1**.12) are the generalizations of some well-known fractional integral operators:*
*(i) Taking $g(x) = x$ we obtain the fractional integral operators (**1**.7) and (**1**.8).*
*(ii) Taking $g(x) = x$ and $p = 0$ we obtain the fractional integral operators defined by Salim and Faraj in* [32].
*(iii) Taking $g(x) = x$ and $l = r = 1$ we obtain the fractional integral operators defined by Rahman et al. in* [31].
*(iv) Taking $g(x) = x$, $p = 0$ and $l = r = 1$ we obtain the fractional integral operators defined by Srivastava-Tomovski in* [34].
*(v) Taking $g(x) = x$, $p = 0$ and $l = r = q = 1$ we obtain the fractional integral operators defined by Prabhakar in* [28].
*(vi) Taking $g(x) = x$ and $\mu = p = 0$ we obtain the Riemann-Liouville fractional integral operators.*

The aim of this paper is to establish the fractional integral inequalities of Fejér-Hadamard type for exponentially $(\alpha, h-m)$-convex functions, exponentially $(h-m)$-convex functions and exponentially $(\alpha - m)$-convex functions. We gave these inequalities utilizing the generalized fractional integral operators (**1**.11) and (**1**.12) containing Mittag-Leffler function via a monotone increasing function. These inequalities lead to produce the Fejér-Hadamard type inequalities for various kinds of convexities and well-known fractional integral operators given in Remark **1**.1 and Remark **1**.2 respectively.

In Section 2, we prove two versions of the Fejér-Hadamard type inequalities for generalized fractional integral operators (**1**.11) and (**1**.12) via exponentially $(\alpha, h-m)$-convex functions. Also their consequences are given in remarks. In Section 3 we give the Fejér-Hadamard type inequalities for exponentially $(h-m)$-convex functions. In Section 4, we give these inequalities for exponentially $(\alpha - m)$-convex functions.

## 2. Fejér-Hadamard type inequalities for exponentially $(\alpha, h-m)$-convex functions

First we give the following Fejér-Hadamard inequality for exponentially $(\alpha, h-m)$-convex functions via generalized fractional integral operators.

**Theorem 2.1.** *Let $f, g : [u_1, mu_2] \subset [0, \infty) \to \mathbb{R}$, $0 < u_1 < mu_2$ be the functions satisfying the assumptions:*
*i) $f$ be positive, integrable and exponentially $(\alpha, h-m)$-convex function,*
*ii) $g$ be differentiable, strictly increasing and $g(x) = g(u_1) + mg(u_2) - mg(x)$.*

*Also, let $\gamma : [u_1, mu_2] \to \mathbb{R}$ be a non-negative and integrable function. Then for generalized fractional integral operators, the following inequalities hold:*

$$f\left(\frac{g(u_1)+mg(u_2)}{2}\right)G(\sigma)\left({}_g\zeta^{\eta,r,q,c}_{\nu,\kappa,l,\bar{\mu}m^{\nu},u_2^-}\gamma\circ g\right)\left(g^{-1}\left(\frac{g(u_1)}{m}\right);p\right)$$
$$\leq\left(h\left(\frac{1}{2^{\alpha}}\right)+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\right)\left({}_g\zeta^{\eta,r,q,c}_{\nu,\kappa,l,\bar{\mu}m^{\nu},u_2^-}(f\circ g)(\gamma\circ g)\right)\left(g^{-1}\left(\frac{g(u_1)}{m}\right);p\right) \quad (\mathbf{2}.1)$$
$$\leq\frac{(mg(u_2)-g(u_1))^{\kappa}}{m^{\kappa}}\left[\left(h\left(\frac{1}{2^{\alpha}}\right)\frac{f(g(u_1))}{e^{\sigma g(u_1)}}+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f(g(u_2))}{e^{\sigma g(u_2)}}\right)\right.$$
$$\times\int_0^1\omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)\gamma\left((1-\omega)\frac{g(u_1)}{m}+\omega g(u_2)\right)h(\omega^{\alpha})d\omega$$
$$+m\left(h\left(\frac{1}{2^{\alpha}}\right)\frac{f(g(u_2))}{e^{\sigma g(u_2)}}+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f\left(\frac{g(u_1)}{m^2}\right)}{e^{\sigma\frac{g(u_1)}{m^2}}}\right)$$
$$\left.\times\int_0^1\omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)\gamma\left((1-\omega)\frac{g(u_1)}{m}+\omega g(u_2)\right)h(1-\omega^{\alpha})d\omega\right],$$

*where* $G(\sigma)=e^{\sigma g(u_2)}$ *for* $\sigma<0$, $G(\sigma)=e^{\sigma g(u_1)}$ *for* $\sigma\geq 0$ *and* $\bar{\mu}=\frac{\mu}{(mg(u_2)-g(u_1))^{\nu}}$.

*Proof.* Since $f$ is exponentially $(\alpha,h{-}m)$-convex function, we have the following inequalities:

$$f\left(\frac{g(u_1)+mg(u_2)}{2}\right)\leq h\left(\frac{1}{2^{\alpha}}\right)\frac{f(\omega g(u_1)+m(1-\omega)g(u_2))}{e^{\sigma(\omega g(u_1)+m(1-\omega)g(u_2))}} \quad (\mathbf{2}.2)$$
$$+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f\left((1-\omega)\frac{g(u_1)}{m}+\omega g(u_2)\right)}{e^{\sigma\left((1-\omega)\frac{g(u_1)}{m}+\omega g(u_2)\right)}}.$$

Then, we have

$$h\left(\frac{1}{2^{\alpha}}\right)f(\omega g(u_1)+m(1-\omega)g(u_2))+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)f\left((1-\omega)\frac{g(u_1)}{m}+\omega g(u_2)\right) \quad (\mathbf{2}.3)$$
$$\leq\left(h\left(\frac{1}{2^{\alpha}}\right)\frac{f(g(u_1))}{e^{\sigma g(u_1)}}+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f(g(u_2))}{e^{\sigma g(u_2)}}\right)h(\omega^{\alpha})$$
$$+m\left(h\left(\frac{1}{2^{\alpha}}\right)\frac{f(g(u_2))}{e^{\sigma g(u_2)}}+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f\left(\frac{g(u_1)}{m^2}\right)}{e^{\sigma\frac{g(u_1)}{m^2}}}\right)h(1-\omega^{\alpha}).$$

Multiplying ($\mathbf{2}$.2) with $\omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)\gamma\left((1-\omega)\frac{g(u_1)}{m}+\omega g(u_2)\right)$ on both sides and integrating over $[0,1]$, we get the following inequality:

$$f\left(\frac{g(u_1)+mg(u_2)}{2}\right)\int_0^1\omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)\gamma\left((1-\omega)\frac{g(u_1)}{m}+\omega g(u_2)\right)d\omega \quad (\mathbf{2}.4)$$
$$\leq h\left(\frac{1}{2^{\alpha}}\right)\int_0^1\omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)\frac{f(\omega g(u_1)+m(1-\omega)g(u_2))}{e^{\sigma(\omega g(u_1)+m(1-\omega)g(u_2))}}$$
$$\times\gamma\left((1-\omega)\frac{g(u_1)}{m}+\omega g(u_2)\right)d\omega+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_0^1\omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)$$
$$\times\frac{f\left((1-\omega)\frac{g(u_1)}{m}+\omega g(u_2)\right)}{e^{\sigma\left((1-\omega)\frac{g(u_1)}{m}+\omega g(u_2)\right)}}\gamma\left((1-\omega)\frac{g(u_1)}{m}+\omega g(u_2)\right)d\omega.$$

For $g(x) = (1-\omega)\frac{g(u_1)}{m} + \omega g(u_2)$ in (**2**.4) and using (**1**.12) and (ii) ($g(x) = g(u_1) + mg(u_2) - mg(x)$) then we get the first inequality of (**2**.1).

Multiplying (**2**.3) with $\omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)\gamma\left((1-\omega)\frac{g(u_1)}{m} + \omega g(u_2)\right)$ on both sides and integrating over $[0,1]$, we have

$$\begin{aligned}
&h\left(\frac{1}{2^{\alpha}}\right)\int_0^1 \omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)f(\omega g(u_1) + m(1-\omega)g(u_2)) \qquad (\mathbf{2}.5)\\
&\times\gamma\left((1-\omega)\frac{g(u_1)}{m} + \omega g(u_2)\right)d\omega + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_0^1 \omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)\\
&\times f\left((1-\omega)\frac{g(u_1)}{m} + \omega g(u_2)\right)\gamma\left((1-\omega)\frac{g(u_1)}{m} + \omega g(u_2)\right)d\omega\\
&\le \left(h\left(\frac{1}{2^{\alpha}}\right)\frac{f(g(u_1))}{e^{\sigma g(u_1)}} + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f(g(u_2))}{e^{\sigma g(u_2)}}\right)\\
&\times\int_0^1 \omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)\gamma\left((1-\omega)\frac{g(u_1)}{m} + \omega g(u_2)\right)h(\omega^{\alpha})d\omega\\
&+ m\left(h\left(\frac{1}{2^{\alpha}}\right)\frac{f(g(u_2))}{e^{\sigma g(u_2)}} + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f\left(\frac{g(u_1)}{m^2}\right)}{e^{\sigma\frac{g(u_1)}{m^2}}}\right)\\
&\times\int_0^1 \omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)\gamma\left((1-\omega)\frac{g(u_1)}{m} + \omega g(u_2)\right)h(1-\omega^{\alpha})d\omega.
\end{aligned}$$

Replacing $g(x) = (1-\omega)\frac{g(u_1)}{m} + \omega g(u_2)$ in (**2**.5) and using (**1**.12), by (ii), we get the second inequality of (**2**.1). □

**Remark 2.1.** *i) Taking $\sigma = p = 0$, $\alpha = m = 1$, $g(x) = x$, $\gamma(g(x)) = 1$ and $h(\omega) = \omega$ in (**2**.1) we get* [6, Theorem 2.1].
*ii) Taking $\sigma = p = 0$, $\alpha = 1$, $g(x) = x$, $\gamma(g(x)) = 1$ and $h(\omega) = \omega$ in (**2**.1) we get* [7, Theorem 3].
*iii) Taking $\sigma = p = \mu = 0$, $\alpha = 1$, $g(x) = x$, $\gamma(g(x)) = 1$ and $h(\omega) = \omega$ in (**2**.1) we get* [8, Theorem 2.1].
*iv) Taking $\sigma = 0$, $\alpha = 1$, $g(x) = x$ and $\gamma(g(x)) = 1$ in (**2**.1) we get* [17, Theorem 2.1].
*v) Taking $\sigma = 0$, $\alpha = m = 1$, $g(x) = x$, $\gamma(g(x)) = 1$ and $h(\omega) = \omega$ in (**2**.1) we get* [18, Theorem 2.1].
*vi) Taking $\sigma = 0$, $\alpha = 1$, $g(x) = x$ and $h(\omega) = \omega$ in (**2**.1) we get* [18, Theorem 3.2].
*vii) Taking $\sigma = p = \mu = 0$, $\alpha = m = 1$, $g(x) = x$, $\gamma(g(x)) = 1$ and $h(\omega) = \omega$ in (**2**.1) we get* [33, Theorem 2].

Further let us we give another version of the Fejér-Hadamard inequality.

**Theorem 2.2.** *Under the assumptions of Theorem* **2**.*1, the following inequalities hold:*

$$f\left(\frac{g(u_1)+mg(u_2)}{2}\right)G(\sigma)\left({}_{g}\zeta^{\eta,r,q,c}_{\nu,\kappa,l,\bar{\mu}(2m)^{\nu},\left(g^{-1}\left(\frac{g(u_1)+mg(u_2)}{2m}\right)\right)^{-}}\gamma\circ g\right)\left(g^{-1}\left(\frac{g(u_1)}{m}\right);p\right) \quad (\mathbf{2}.6)$$

$$\leq\left(h\left(\frac{1}{2^{\alpha}}\right)+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\right)$$

$$\times\left({}_{g}\zeta^{\eta,r,q,c}_{\nu,\kappa,l,\bar{\mu}(2m)^{\nu},\left(g^{-1}\left(\frac{g(u_1)+mg(u_2)}{2m}\right)\right)^{-}}(f\circ g)(\gamma\circ g)\right)\left(g^{-1}\left(\frac{g(u_1)}{m}\right);p\right)$$

$$\leq\frac{(mg(u_2)-g(u_1))^{\kappa}}{(2m)^{\kappa}}\left[\left(h\left(\frac{1}{2^{\alpha}}\right)\frac{f(g(u_1))}{e^{\sigma g(u_1)}}+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f(g(u_2))}{e^{\sigma g(u_2)}}\right)\right.$$

$$\times\int_0^1\omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)\gamma\left(\frac{(2-\omega)}{2}\frac{g(u_1)}{m}+\frac{\omega}{2}g(u_2)\right)h\left(\frac{\omega^{\alpha}}{2^{\alpha}}\right)d\omega$$

$$+m\left(h\left(\frac{1}{2^{\alpha}}\right)\frac{f(g(u_2))}{e^{\sigma g(u_2)}}+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f\left(\frac{g(u_1)}{m^2}\right)}{e^{\sigma\frac{g(u_1)}{m^2}}}\right)$$

$$\left.\times\int_0^1\omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)\gamma\left(\frac{(2-\omega)}{2}\frac{g(u_1)}{m}+\frac{\omega}{2}g(u_2)\right)h\left(\frac{(2-\omega)^{\alpha}}{2^{\alpha}}\right)d\omega\right],$$

*where* $G(\sigma)=e^{\sigma g(u_2)}$ *for* $\sigma<0$, $G(\sigma)=e^{\sigma g(u_1)}$ *for* $\sigma\geq 0$ *and* $\bar{\mu}=\frac{\mu}{(mg(u_2)-g(u_1))^{\nu}}$.

*Proof.* From exponentially $(\alpha,h-m)$-convexity of $f$, one can obtain the following inequalities:

$$f\left(\frac{g(u_1)+mg(u_2)}{2}\right)\leq h\left(\frac{1}{2^{\alpha}}\right)\frac{f\left(\frac{\omega}{2}g(u_1)+m\frac{(2-\omega)}{2}g(u_2)\right)}{e^{\sigma\left(\frac{\omega}{2}g(u_1)+m\frac{(2-\omega)}{2}g(u_2)\right)}} \quad (\mathbf{2}.7)$$

$$+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f\left(\frac{(2-\omega)}{2}\frac{g(u_1)}{m}+\frac{\omega}{2}g(u_2)\right)}{e^{\sigma\left(\frac{(2-\omega)}{2}\frac{g(u_1)}{m}+\frac{\omega}{2}g(u_2)\right)}}.$$

Then we obtain

$$h\left(\frac{1}{2^{\alpha}}\right)f\left(\frac{\omega}{2}g(u_1)+m\frac{(2-\omega)}{2}g(u_2)\right)+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)f\left(\frac{(2-\omega)}{2}\frac{g(u_1)}{m}+\frac{\omega}{2}g(u_2)\right) \quad (\mathbf{2}.8)$$

$$\leq\left(h\left(\frac{1}{2^{\alpha}}\right)\frac{f(g(u_1))}{e^{\sigma g(u_1)}}+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f(g(u_2))}{e^{\sigma g(u_2)}}\right)h\left(\frac{\omega^{\alpha}}{2^{\alpha}}\right)$$

$$+m\left(h\left(\frac{1}{2^{\alpha}}\right)\frac{f(g(u_2))}{e^{\sigma g(u_2)}}+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f\left(\frac{g(u_1)}{m^2}\right)}{e^{\sigma\frac{g(u_1)}{m^2}}}\right)h\left(\frac{(2-\omega)^{\alpha}}{2^{\alpha}}\right).$$

Multiplying (**2**.7) with $\omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)\gamma\left(\frac{(2-\omega)}{2}\frac{g(u_1)}{m}+\frac{\omega}{2}g(u_2)\right)$ on both sides and integrating over $[0,1]$, we have

$$f\left(\frac{g(u_1)+mg(u_2)}{2}\right)\int_0^1\omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)\gamma\left(\frac{(2-\omega)}{2}\frac{g(u_1)}{m}+\frac{\omega}{2}g(u_2)\right)d\omega \tag{2.9}$$
$$\leq h\left(\frac{1}{2^{\alpha}}\right)\int_0^1\omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)\frac{f\left(\frac{\omega}{2}g(u_1)+m\frac{(2-\omega)}{2}g(u_2)\right)}{e^{\sigma\left(\frac{\omega}{2}g(u_1)+m\frac{(2-\omega)}{2}g(u_2)\right)}}$$
$$\times\gamma\left(\frac{(2-\omega)}{2}\frac{g(u_1)}{m}+\frac{\omega}{2}g(u_2)\right)d\omega+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_0^1\omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)$$
$$\times\frac{f\left(\frac{(2-\omega)}{2}\frac{g(u_1)}{m}+\frac{\omega}{2}g(u_2)\right)}{e^{\sigma\left(\frac{(2-\omega)}{2}\frac{g(u_1)}{m}+\frac{\omega}{2}g(u_2)\right)}}\gamma\left(\frac{(2-\omega)}{2}\frac{g(u_1)}{m}+\frac{\omega}{2}g(u_2)\right)d\omega.$$

Taking $g(x)=\frac{(2-\omega)}{2}\frac{g(u_1)}{m}+\frac{\omega}{2}g(u_2)$ in (**2**.9) and multiplying (**2**.8) with $\omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)\gamma\left(\frac{(2-\omega)}{2}\frac{g(u_1)}{m}+\frac{\omega}{2}g\right.$ on both sides and integrating over $[0,1]$ we have

$$h\left(\frac{1}{2^{\alpha}}\right)\int_0^1\omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)f\left(\frac{\omega}{2}g(u_1)+m\frac{(2-\omega)}{2}g(u_2)\right) \tag{2.10}$$
$$\times\gamma\left(\frac{(2-\omega)}{2}\frac{g(u_1)}{m}+\frac{\omega}{2}g(u_2)\right)d\omega+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_0^1\omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)$$
$$\times f\left(\frac{\omega}{2}g(u_2)+\frac{(2-\omega)}{2}\frac{g(u_1)}{m}\right)\gamma\left(\frac{(2-\omega)}{2}\frac{g(u_1)}{m}+\frac{\omega}{2}g(u_2)\right)d\omega$$
$$\leq\left(h\left(\frac{1}{2^{\alpha}}\right)\frac{f(g(u_1))}{e^{\sigma g(u_1)}}+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f(g(u_2))}{e^{\sigma g(u_2)}}\right)$$
$$\times\int_0^1\omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)\gamma\left(\frac{(2-\omega)}{2}\frac{g(u_1)}{m}+\frac{\omega}{2}g(u_2)\right)h\left(\frac{\omega^{\alpha}}{2^{\alpha}}\right)d\omega$$
$$+m\left(h\left(\frac{1}{2^{\alpha}}\right)\frac{f(g(u_2))}{e^{\sigma g(u_2)}}+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f\left(\frac{g(u_1)}{m^2}\right)}{e^{\sigma\frac{g(u_1)}{m^2}}}\right)$$
$$\times\int_0^1\omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)\gamma\left(\frac{(2-\omega)}{2}\frac{g(u_1)}{m}+\frac{\omega}{2}g(u_2)\right)h\left(\frac{(2-\omega)^{\alpha}}{2^{\alpha}}\right)d\omega.$$

Taking $g(x)=\frac{\omega}{2}g(u_2)+\frac{(2-\omega)}{2}\frac{g(u_1)}{m}$ in (**2**.10), and using (**1**.12), $g(x)=g(u_1)+mg(u_2)-mg(x)$, the second inequality of (**2**.6) is obtained. □

Using the previously results we establish a connections between our results and the results already obtained in related literature. Then, let us give the following remark.

**Remark 2.2.** *i) Taking* $\sigma=p=0$, $\alpha=1$, $g(x)=x$ *and* $\gamma(g(x))=1$ *in (**2**.6) we get* [8, Theorem 3.10].
*ii) Taking* $\sigma=p=\mu=0$, $\alpha=1$, $g(x)=x$ *and* $\gamma(g(x))=1$ *in (**2**.6) we get* [9, Theorem 2.1].
*iii) Taking* $\sigma=0$, $\alpha=1$, $g(x)=x$ *and* $\gamma(g(x))=1$ *in (**2**.6) we get* [13, Theorem 2.11].
*iv) Taking* $\sigma=p=\mu=0$, $\alpha=m=1$, $g(x)=x$ *and* $\gamma(g(x))=1$ *in (**2**.6) we get* [35, Theorem 4].

## 3. Fejér-Hadamard type inequalities for exponentially $(h-m)$-convex functions

In this section we give two versions of the Fejér-Hadamard type inequality for exponentially $(h-m)$-convex functions via generalized fractional integral operators.

**Theorem 3.1.** *Let $f, g : [u_1, mu_2] \subset [0, \infty) \to \mathbb{R}$, $0 < u_1 < mu_2$ be the functions satisfying assumptions:*
*i) $f$ be positive, integrable and exponentially $(h-m)$-convex function:*
*ii) $g$ be differentiable, strictly increasing and $g(x) = g(u_1) + mg(u_2) - mg(x)$.*

*Also, let $\gamma : [u_1, mu_2] \to \mathbb{R}$ be a non-negative and integrable function. Then for generalized fractional integral operators, the following inequalities hold:*

$$\begin{aligned}
&f\left(\frac{g(u_1)+mg(u_2)}{2}\right)G(\sigma)\left({}_g\zeta^{\eta,r,q,c}_{\nu,\kappa,l,\bar{\mu}m^{\nu},u_2^-}\gamma\circ g\right)\left(g^{-1}\left(\frac{g(u_1)}{m}\right);p\right) \qquad (3.1)\\
&\le h\left(\frac{1}{2}\right)(1+m)\left({}_g\zeta^{\eta,r,q,c}_{\nu,\kappa,l,\bar{\mu}m^{\nu},u_2^-}(f\circ g)(\gamma\circ g)\right)\left(g^{-1}\left(\frac{g(u_1)}{m}\right);p\right)\\
&\le \frac{(mg(u_2)-g(u_1))^{\kappa}}{m^{\kappa}}h\left(\frac{1}{2}\right)\Bigg[\left(\frac{f(g(u_1))}{e^{\sigma g(u_1)}}+m\frac{f(g(u_2))}{e^{\sigma g(u_2)}}\right)\\
&\times\int_0^1\omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)\gamma\left((1-\omega)\frac{g(u_1)}{m}+\omega g(u_2)\right)h(\omega)d\omega\\
&+m\left(\frac{f(g(u_2))}{e^{\sigma g(u_2)}}+m\frac{f\left(\frac{g(u_1)}{m^2}\right)}{e^{\sigma\frac{g(u_1)}{m^2}}}\right)\\
&\times\int_0^1\omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)\gamma\left((1-\omega)\frac{g(u_1)}{m}+\omega g(u_2)\right)h(1-\omega)d\omega\Bigg],
\end{aligned}$$

*where $G(\sigma) = e^{\sigma g(u_2)}$ for $\sigma < 0$, $G(\sigma) = e^{\sigma g(u_1)}$ for $\sigma \ge 0$ and $\bar{\mu} = \frac{\mu}{(mg(u_2)-g(u_1))^{\nu}}$.*

*Proof.* Taking $\alpha = 1$ in (**2**.1) we obtain (**3**.1). It also can be proved explicitly on the same lines of proof of Theorem **2**.1. □

The second version of the Fejér-Hadamard inequality is given as follows.

**Theorem 3.2.** *Under the assumptions of Theorem **3**.1 the following inequalities hold:*

$$\begin{aligned}
&f\left(\frac{g(u_1)+mg(u_2)}{2}\right)G(\sigma)\left({}_g\zeta^{\eta,r,q,c}_{\nu,\kappa,l,\bar{\mu}(2m)^{\nu},\left(g^{-1}\left(\frac{g(u_1)+mg(u_2)}{2m}\right)\right)^-}\gamma\circ g\right)\left(g^{-1}\left(\frac{g(u_1)}{m}\right);p\right) \qquad (3.2)\\
&\le h\left(\frac{1}{2}\right)(1+m)\left({}_g\zeta^{\eta,r,q,c}_{\nu,\kappa,l,\bar{\mu}(2m)^{\nu},\left(g^{-1}\left(\frac{g(u_1)+mg(u_2)}{2m}\right)\right)^-}(f\circ g)(\gamma\circ g)\right)\left(g^{-1}\left(\frac{g(u_1)}{m}\right);p\right)\\
&\le \frac{(mg(u_2)-g(u_1))^{\kappa}}{(2m)^{\kappa}}h\left(\frac{1}{2}\right)\Bigg[\left(\frac{f(g(u_1))}{e^{\sigma g(u_1)}}+m\frac{f(g(u_2))}{e^{\sigma g(u_2)}}\right)\int_0^1\omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)\\
&\times\gamma\left(\frac{(2-\omega)}{2}\frac{g(u_1)}{m}+\frac{\omega}{2}g(u_2)\right)h\left(\frac{\omega}{2}\right)d\omega+m\left(\frac{f(g(u_2))}{e^{\sigma g(u_2)}}+m\frac{f\left(\frac{g(u_1)}{m^2}\right)}{e^{\sigma\frac{g(u_1)}{m^2}}}\right)\\
&\times\int_0^1\omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^{\nu};p)\gamma\left(\frac{(2-\omega)}{2}\frac{g(u_1)}{m}+\frac{\omega}{2}g(u_2)\right)h\left(\frac{2-\omega}{2}\right)d\omega\Bigg],
\end{aligned}$$

*where $G(\sigma) = e^{\sigma g(u_2)}$ for $\sigma < 0$, $G(\sigma) = e^{\sigma g(u_1)}$ for $\sigma \ge 0$ and $\bar{\mu} = \frac{\mu}{(mg(u_2)-g(u_1))^{\nu}}$.*

*Proof.* Replacing $\alpha = 1$ in (**2**.6) we obtain (**3**.2). It also can be proved explicitly on the same lines of proof of Theorem **2**.2. □

## 4. Fejér-Hadamard type inequalities for exponentially $(\alpha-m)$-convex functions

In this section we give two versions of the Fejér-Hadamard inequality for exponentially $(\alpha-m)$-convex functions via generalized fractional integral operators.

**Theorem 4.1.** *Let $f, g : [u_1, mu_2] \subset [0,\infty) \to \mathbb{R}$, $0 < u_1 < mu_2$ be the functions satisfying assumptions:*
*i) $f$ be positive, integrable and exponentially $(\alpha-m)$-convex function,*
*ii) $g$ be differentiable, strictly increasing and $g(x) = g(u_1) + mg(u_2) - mg(x)$.*
*Also, let $\gamma : [u_1, mu_2] \to \mathbb{R}$ be a non-negative and integrable function. Then for generalized fractional integral operators, the following inequalities hold:*

$$\begin{aligned} & f\left(\frac{g(u_1)+mg(u_2)}{2}\right)G(\sigma)\left({}_g\zeta^{\eta,r,q,c}_{\nu,\kappa,l,\bar{\mu}m^\nu,u_2^-}\gamma\circ g\right)\left(g^{-1}\left(\frac{g(u_1)}{m}\right);p\right) \qquad (4.1)\\ & \leq \frac{1}{2^\alpha}\left(1+m\left(2^\alpha-1\right)\right)\left({}_g\zeta^{\eta,r,q,c}_{\nu,\kappa,l,\bar{\mu}m^\nu,u_2^-}(f\circ g)(\gamma\circ g)\right)\left(g^{-1}\left(\frac{g(u_1)}{m}\right);p\right)\\ & \leq \frac{(mg(u_2)-g(u_1))^\kappa}{m^\kappa}\frac{1}{2^\alpha}\left[\left(\frac{f(g(u_1))}{e^{\sigma g(u_1)}}+m\left(2^\alpha-1\right)\frac{f(g(u_2))}{e^{\sigma g(u_2)}}\right)\right.\\ & \times\int_0^1\omega^{\kappa+\alpha-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^\nu;p)\,\gamma\left((1-\omega)\frac{g(u_1)}{m}+\omega g(u_2)\right)d\omega\\ & +m\left(\frac{f(g(u_2))}{e^{\sigma g(u_2)}}+m\left(2^\alpha-1\right)\frac{f\left(\frac{g(u_1)}{m^2}\right)}{e^{\sigma\frac{g(u_1)}{m^2}}}\right)\\ & \left.\times\int_0^1\omega^{\kappa-1}(1-\omega^\alpha)E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^\nu;p)\gamma\left((1-\omega)\frac{g(u_1)}{m}+\omega g(u_2)\right)d\omega\right], \end{aligned}$$

*where $G(\sigma) = e^{\sigma g(u_2)}$ for $\sigma < 0$, $G(\sigma) = e^{\sigma g(u_1)}$ for $\sigma \geq 0$ and $\bar{\mu} = \frac{\mu}{(mg(u_2)-g(u_1))^\nu}$.*

*Proof.* Replacing $h(\omega) = \omega$ in (**2**.1) we obtain (**4**.1). It also can be proved explicitly on the same lines of proof of Theorem **2**.1. ☐

**Theorem 4.2.** *Let $f, g : [u_1, mu_2] \subset [0,\infty) \to \mathbb{R}$, $0 < u_1 < mu_2$ be the functions satisfying assumptions:*
*i) $f$ be positive, integrable and exponentially $(\alpha-m)$-convex function,*
*ii) $g$ be differentiable, strictly increasing and $g(x) = g(u_1) + mg(u_2) - mg(x)$.*
*Also, let $\gamma : [u_1, mu_2] \to \mathbb{R}$ be a non-negative and integrable function. Then for generalized fractional integral operators, the following inequalities hold:*

$$\begin{aligned} & f\left(\frac{g(u_1)+mg(u_2)}{2}\right)G(\sigma)\left({}_g\zeta^{\eta,r,q,c}_{\nu,\kappa,l,\bar{\mu}(2m)^\nu,\left(g^{-1}\left(\frac{g(u_1)+mg(u_2)}{2m}\right)\right)^-}\gamma\circ g\right)\left(g^{-1}\left(\frac{g(u_1)}{m}\right);p\right) \qquad (4.2)\\ & \leq\frac{1}{2^\alpha}\left(1+m\left(2^\alpha-1\right)\right)\left({}_g\zeta^{\eta,r,q,c}_{\nu,\kappa,l,\bar{\mu}(2m)^\nu,\left(g^{-1}\left(\frac{g(u_1)+mg(u_2)}{2m}\right)\right)^-}(f\circ g)(\gamma\circ g)\right)\left(g^{-1}\left(\frac{g(u_1)}{m}\right);p\right)\\ & \leq\frac{(mg(u_2)-g(u_1))^\kappa}{(2m)^\kappa}\frac{1}{2^\alpha}\left[\left(\frac{f(g(u_1))}{e^{\sigma g(u_1)}}+m\left(2^\alpha-1\right)\frac{f(g(u_2))}{e^{\sigma g(u_2)}}\right)\int_0^1\omega^{\kappa-1}E^{\eta,r,q,c}_{\nu,\kappa,l}(\mu\omega^\nu;p)\right.\\ & \times\gamma\left(\frac{(2-\omega)}{2}\frac{g(u_1)}{m}+\frac{\omega}{2}g(u_2)\right)\left(\frac{\omega}{2}\right)^\alpha d\omega+m\left(\frac{f(g(u_2))}{e^{\sigma g(u_2)}}+m\left(2^\alpha-1\right)\frac{f\left(\frac{g(u_1)}{m^2}\right)}{e^{\sigma\frac{g(u_1)}{m^2}}}\right) \end{aligned}$$

$$\times \int_0^1 \omega^{\kappa-1} E_{\nu,\kappa,l}^{\eta,r,q,c}(\mu\omega^\nu;p)\gamma\left(\frac{(2-\omega)}{2}\frac{g(u_1)}{m}+\frac{\omega}{2}g(u_2)\right)\left(\frac{(2-\omega)^\alpha}{2^\alpha}\right)d\omega\right],$$

*where* $G(\sigma)=e^{\sigma g(u_2)}$ *for* $\sigma<0$, $G(\sigma)=e^{\sigma g(u_1)}$ *for* $\sigma\geq 0$ *and* $\bar{\mu}=\frac{\mu}{(mg(u_2)-g(u_1))^\nu}$.

*Proof.* Replacing $h(\omega)=\omega$ in (**2**.6) we obtain (**4**.2). It also can be proved explicitly on the same lines of proof of Theorem **2**.2. □

### Conclusions

In this paper we have established the generalized fractional integral inequalities of Fejér-Hadamard type for a generalized exponentially convexity. The results are obtained for exponentially $(\alpha, h-m)$-convex functions, exponentially $(h-m)$-convex functions and exponentially $(\alpha-m)$-convex functions which are further deducible for several kinds of known convex functions written in Remark **1**.1. Also, they hold for well-known fractional integral operators containing Mittag-Leffler functions in their kernels written in Remark **1**.2. The readers can deduce a plenty of fractional integral inequalities of their choice.

### Declarations

**Availability of supporting data**
Not applicable.

**Competing interests**
The authors declare that they have no competing interests.

**Author's contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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