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FUZZY SOFT Γ-HYPERMODULES

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Soft set theory, introduced by Molodtsov, has been considered as an effective mathematical tool for modeling uncertainties. In this paper, we apply fuzzy soft sets to Γ*-hypermodules. The concept of* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft* Γ*-subhypermodules of* Γ*-hypermodules is first introduced. Some new characterizations are investigated. In particular, a kind of new* Γ*-hypermodules by congruence relations is obtained.*

Keywords: Fuzzy soft set; Γ-subhypermodule; $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft Γ-subhypermodule; congruence relation; Γ-hypermodule

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1. Introduction

Uncertainties are pervasive in many complicated problems in engineering, economics, environment, medical science and social science, due to information incompleteness, randomness, limitations of measuring instruments, etc. Traditional mathematical tools for modeling uncertainties, such as probability theory, fuzzy set theory [38], vague set theory, rough set theory [34] and interval mathematics have been proven to be useful mathematical tools for dealing with uncertainties. However, all these theories have their inherent difficulties, as pointed out by Molodtsov in [33]. At present, works on the soft set theory are progressing rapidly. Maji et al. [28, 29] described the application of soft set theory to a decision making problem. Ali et al. [2] proposed some new operations on soft sets. Chen et al. [8] presented a new definition of soft set parametrization reduction, and compared this definition to the related concept of attribute reduction in rough set theory. In particular, fuzzy soft set theory has been investigated by some researchers, for examples, see [27, 30]. Recently, the algebraic structures of soft sets have been studied increasingly, see [1, 20]. In particular, Zhan and Jun [44] characterized the (implicative,

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positive implicative and fantastic) filteristic soft BL-algebras based on $\in$-soft sets and q-soft sets.

On the other hand, the theory of algebraic hyperstructures (or hypersystems) is a well established branch of classical algebraic theory. In the literature, the theory of hyperstructures was first initiated by Marty in 1934 [31] when he defined the hypergroups and began to investigate their properties with applications to groups, rational fractions and algebraic functions. Later on, many people have observed that the theory of hyperstructures also have many applications in both pure and applied sciences, for example, semi-hypergroups are the simplest algebraic hyperstructures which possess the properties of closure and associativity. Some review of the theory of hyperstructures can be found in [9, 10, 13, 35]. A well known type of a hyperring is the *Krasner hyperring* [22]. Krasner hyperrings are essentially rings, with approximately modified axioms in which addition is a hyperoperation (i.e., $a+b$ is a set). The concept of hypermodules has been studied by Massouros [32]. He concentrated on the kind of hypermodules over Krasner hyperrings. In particular, the relationships between the fuzzy sets and algebraic hyperstructures have been considered by Ameri, Cristea, Corsini, Davvaz, Leoreanu, Vougiouklis, Zhan and many other researchers. The reader is referred to [5, 11, 12, 15, 23, 37], [40]-[43].

The concept of Γ-rings was introduced by Barnes [6]. After that, this concept was discussed further by some researchers. The notion of fuzzy ideals in a Γ-ring was introduced by Jun and Lee in [21]. They studied some preliminary properties of fuzzy ideals of Γ-rings. Jun [19] defined fuzzy prime ideals of a Γ-ring and obtained a number of characterizations for a fuzzy ideal to be a fuzzy prime ideal. In particular, Dutta and Chanda [17] studied the structures of the set of fuzzy ideals of a Γ-ring. Ma et al. [24, 25] considered the characterizations of Γ-hemirings and Γ-rings, respectively. The notion of a Γ-module was introduced by Ameri et al. in [4]. They studied some preliminary properties of Γ-modules. Recently, some Γ-hyperstuctures have been studied by some researchers. Ameri et al. [3] considered the concept of fuzzy hyperideals of Γ-hyperrings. By a different way of [3], Yin et al. [36] investigated some new results on Γ-hyperrings. Ma et al. [26] considered the (fuzzy) isomorphism theorems of Γ-hyperrings. In the same time, Davvaz et al. [14, 16] considered the properties of Γ-hypernear-rings and Γ-H_v-rings, respectively.

After the introduction of fuzzy sets by Zadeh [38], there have been a number of generalizations of this fundamental concept. A new type of fuzzy subgroup, that is, the $(\in, \in\vee \mathrm{q})$-fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das [7] by using the combined notions of “belongingness” and “quasicoincidence” of fuzzy points and fuzzy sets. In fact, the $(\in, \in\vee \mathrm{q})$-fuzzy subgroup is an important generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems with other algebraic structures, see [12, 15, 42].

In [39], the first author of this paper introduced the concept of Γ-hyperrmodules and investigated some related properties. As a continuation of this paper, we introduce the concept of $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy soft Γ-subhypermodules of Γ-hypermodules in the present paper. Some characterizations are investigated. In particular, we obtain a kind of new Γ-hypermodules by congruence relations.

2. Preliminaries

A *quasicanonical hypergroup* (not necessarily commutative) is an algebraic structure $(\mathcal{H}, +)$ satisfying the following conditions:

(i) for every $x, y, z \in \mathcal{H}, x + (y + z) = (x + y) + z$;

(ii) there exists a $0 \in \mathcal{H}$ such that $0 + x = x$, for all $x \in \mathcal{H}$;

(iii) for every $x \in \mathcal{H}$, there exists a unique element $x' \in \mathcal{H}$ such that $0 \in (x + x') \cap (x' + x)$. (we call the element $-x$ the opposite of x);

(iv) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.

Quasicanonical hypergroups are also called *polygroups.*

We note that if $x \in \mathcal{H}$ and A, B are non-empty subsets in $\mathcal{H}$, then by $A + B$, $A + x$ and $x + B$ we mean that $A + B = \bigcup\limits_{a \in A, b \in B} a + b, A + x = A + \{x\}$ and $x + B = \{x\} + B$, respectively. Also, for all $x, y \in \mathcal{H}$, we have $-(-x) = x, -0 = 0$, where 0 is unique and $-(x + y) = -y - x$.

A sub-hypergroup $A \subset \mathcal{H}$ is said to be *normal* if $x + A - x \subseteq A$ for all $x \in \mathcal{H}$. A normal sub-hypergroup A of $\mathcal{H}$ is called *left (right) hyperideal* of $\mathcal{H}$ if $xA \subseteq A$ $(Ax \subseteq A$ respectively) for all $x \in \mathcal{H}$. Moreover A is said to be a *hyperideal* of $\mathcal{H}$ if it is both a left and a right hyperideal of $\mathcal{H}$. A *canonical hypergroup* is a commutative quasicanonical hypergroup.

Definition 2.1. [22] A *hyperring* is an algebraic structure $(R, +, \cdot)$, which satisfies the following axioms:

(1) $(R, +)$ is a canonical hypergroup;

(2) Related to the multiplication, $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, that is, $0 \cdot x = x \cdot 0 = 0$ for all $x \in R$;

(3) The multiplication is distributive with respect to the hyperoperation "+" that is, $z \cdot (x + y) = z \cdot x + z \cdot y$ and $(x + y) \cdot z = x \cdot z + y \cdot z$ for all $x, y, z \in R$.

Definition 2.2. [3, 36] Let $(R, \oplus)$ and $(\Gamma, \oplus)$ be two canonical hypergroups. Then R is called a Γ-*hyperring*, if the following conditions are satisfied for all $x, y, z \in R$ and for all $\alpha, \beta \in \Gamma$,

(1) $x\alpha y \in R$;

(2) $(x \oplus y)\alpha z = x\alpha z \oplus y\alpha z, x\alpha(y \oplus z) = x\alpha y \oplus x\alpha z$;

(3) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Definition 2.3. [26] A fuzzy set μ of a Γ-hyperring R is called a *fuzzy Γ-hyperideal* of R if the following conditions hold:

(1) $\min\{\mu(x), \mu(y)\} \leq \inf\limits_{z \in x+y} \mu(z)$, for all $x, y \in R$;

(2) $\mu(x) \leq \mu(-x)$, for all $x \in R$;
(3) $\max\{\mu(x), \mu(y)\} \leq \mu(x\alpha y)$, for all $x, y \in R$ and for all $\alpha \in \Gamma$.
(4) $\mu(x) \leq \inf_{z \in -y+x+y} \mu(z)$, for all $x, y \in R$.

In [39], we introduced the concept of Γ-hypermodules as follows.

Definition 2.4. [39] Let $(R, \oplus, \Gamma)$ be a Γ-hyperring and $(M, \oplus)$ be a canonical hypergroup. M is called a Γ-*hypermodule* over R if there exists $f : R \times \Gamma \times M \to M$ (the image of (r, α, m) being denoted by $r\alpha m$) such that for all $a, b \in R, m_1, m_2 \in M, \alpha, \beta \in \Gamma$, we have
(1) $a\alpha(m_1 \oplus m_2) = a\alpha m_1 \oplus a\alpha m_2$;
(2) $(a \oplus b)\alpha m_1 = a\alpha m_1 \oplus b\alpha m_1$;
(3) $a(\alpha \oplus \beta)m_1 = a\alpha m_1 \oplus a\beta m_1$;
(4) $(a\alpha b)\beta m_1 = a\alpha(b\beta m_1)$.

Throughout this paper, R and M is a Γ-hyperring and Γ-hypermodule, respectively, unless otherwise specified.

A subset A in M is said to be a Γ-*subhypermodule* of M if it satisfies the following conditions: (1) $(A, \oplus)$ is a subhypergroup of $(M, \oplus)$; (2) $r\alpha x \in A$, for all $x \in R$, $\alpha \in \Gamma$ and $x \in A$.

A Γ-subhypermodule A of M is called *normal* if $x + A - x \subseteq A$, for all $x \in M$.

Definition 2.5. [39] If M and M' are Γ-hypermodules, then a mapping $f : M \longrightarrow M'$ such that

$$f(x \oplus y) = f(x) \oplus f(y) \text{ and } f(r\alpha x) = r\alpha f(x),$$

for all $r \in R$, $\alpha \in \Gamma$ and $x \in M$, is called a Γ-*hypermodule homomorphism.*

Clearly, a Γ-hypermodule homomorphism f is an isomorphism if f is injective and surjective. We write $M \cong M'$ if M is isomorphic to M'.

If A is a normal Γ-subhypermodule of M, then we define the relation A^* by

$$x \equiv y(modA) \Longleftrightarrow (x - y) \bigcap A \neq \emptyset.$$

This relation is denoted by xA^*y.

If A is a normal Γ-subhypermodule of M, then $[M : A^*] = \{A^*[x] | x \in M\}$.

Define a hyperoperation $\boxplus$ and an operation $\odot_\alpha$ on $[M : A^*]$ by
$A^*[x] \boxplus A^*[y] = \{A^*[z] | z \in A^*[x] \oplus A^*[y]\}$,
$A^*[x] \odot_\alpha A^*[y] = A^*[x\alpha y]$,
for all $r \in R$, $\alpha \in \Gamma$ and $x \in M$.
Then $([M : A^*], \boxplus, \odot_\alpha)$ is a Γ-hypermodule, see [39].

3. Fuzzy soft sets

A fuzzy set μ of M of the form

$$\mu(y) = \begin{cases} t(\neq 0) & \text{if} \quad y = x, \\ 0 & \text{if} \quad y \neq x, \end{cases}$$

is said to be a *fuzzy point with support* x *and value* t and is denoted by x_t. A fuzzy point x_t is said to"*belong to*" (to be" $quasi-coincident\ with$") a fuzzy set μ, written as $x_t \in \mu$ ($x_t q\mu$ respectively) if $\mu(x) \geq t$ ($\mu(x)+t > 1$ respectively).
If $x_t \in \mu$ or $x_t\ q\ \mu$, then we write $x_t \in\vee\,\mathrm{q}\mu$. If $\mu(x) < t$ ($\mu(x)+t \leq 1$), then we say that $x_t\ \overline{\in}\mu$ ($x_t\ \overline{q}\ \mu$ respectively).

We note here that the symbol $\overline{\in\vee\,\mathrm{q}}$ means that $\in\vee\,\mathrm{q}$ does not hold.

Let $\gamma, \delta \in [0,1]$ be such that $\gamma < \delta$. For a fuzzy point x_t and μ of X, we say

(1) $x_t \in_\gamma \mu$ if $\mu(x) \geq t > \gamma$.
(2) $x_t q_\delta \mu$ if $\mu(x)+t > 2\delta$.
(3) $x_t \in_\gamma \vee q_\delta \mu$ if $x_t \in_\gamma \mu$ or $x_t q_\delta \mu$.
(4) $x_t \overline{\in}_\gamma \vee \overline{q}_\delta \mu$ if $x_t \overline{\in}_\gamma \mu$ or $x_t \overline{q}_\delta \mu$.

Molodtsov [33] defined a soft set in the following way: let U be an initial universe set, E be a set of parameters and $A \subseteq E$.

A pair (F, A) is called a *soft set* over U, where F is a mapping given by $F : A \to \mathcal{P}(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A, F(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (F, A).

Definition 3.1. [27] Let U be an initial universe set, E a set of parameters and $A \subseteq E$. Then $(\tilde{F}, A)$ is called a *fuzzy soft set* over U, where $\tilde{F}$ is a mapping given by $\tilde{F} : A \to \mathcal{F}(U)$.

In general, for every $x \in A$, $\tilde{F}(x)$ is a fuzzy set in U and it is called *fuzzy value set* of parameter x. If for every $x \in A$, $\tilde{F}(x)$ is a crisp subset of U, then $(\tilde{F}, A)$ is degenerated to the standard soft set. Thus, from the above definition, it is clear that fuzzy soft sets are a generalization of standard soft sets.

Let $A \subseteq E$ and $F \in \mathcal{F}(U)$. We define a fuzzy soft set $(\tilde{F}, A)$ by $\tilde{F}[\alpha] = F$ for all $\alpha \in A$. For any fuzzy point x_t in U, we define a fuzzy soft set $(\tilde{x}_t, A)$ by $\tilde{x}_t[\alpha] = x_t$ for all $\alpha \in A$.

The notions of AND, OR and bi-intersection operations of fuzzy soft sets can be found in [2, 28].

Now, we introduce an ordered relation " $\subseteq\vee q_{(\gamma,\delta)}$" on the set of all fuzzy soft sets over U.

For two fuzzy soft sets $(\tilde{F}, A)$ and $(\tilde{G}, B)$ over U, by $(\tilde{F}, A) \subseteq\vee q_{(\gamma,\delta)} (\tilde{G}, B)$ we mean that $A \subseteq B$ and $x_t \in_\gamma \tilde{F}[\alpha] \Rightarrow x_t \in_\gamma \vee\,\mathrm{q}_\delta\ \tilde{G}[\alpha]$ for all $\alpha \in A, x \in U$ and $t \in (\gamma, 1]$.

Definition 3.2. For two fuzzy soft sets $(\tilde{F}, A)$ and $(\tilde{G}, B)$ over U, we say that $(\tilde{F}, A)$ is an $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-*fuzzy soft subset* of $(\tilde{G}, B)$, if $(\tilde{F}, A) \subseteq\vee q_{(\gamma,\delta)} (\tilde{G}, B)$.

$(\tilde{F}, A)$ and $(\tilde{G}, B)$ are said to be $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-*equal* if $(\tilde{F}, A) \subseteq\vee q_{(\gamma,\delta)} (\tilde{G}, B)$ and $(\tilde{G}, B) \subseteq\vee q_{(\gamma,\delta)} (\tilde{F}, A)$. This is denoted by $(\tilde{F}, A) =_{(\gamma,\delta)} (\tilde{G}, B)$.

The following two lemmas are straightforward.

Lemma 3.3. Let $(\tilde{F}, A)$ and $(\tilde{G}, B)$ be two fuzzy soft sets over U. Then $(\tilde{F}, A) \subseteq \vee q_{(\gamma,\delta)} (\tilde{G}, B)$ if and only if $\max\{\tilde{G}[\alpha](x), \gamma\} \geq \min\{\tilde{F}[\alpha](x), \delta\}$ for all $\alpha \in A$ and $x \in U$.

Lemma 3.4. Let $(\tilde{F}, A), (\tilde{G}, B)$ and $(\tilde{H}, C)$ be fuzzy soft sets over U such that $(\tilde{F}, A) \subseteq \vee q_{(\gamma,\delta)} (\tilde{G}, B)$ and $(\tilde{G}, B) \subseteq \vee q_{(\gamma,\delta)} (\tilde{H}, C)$. Then $(\tilde{F}, A) \subseteq \vee q_{(\gamma,\delta)} (\tilde{H}, C)$.

It follows from Lemmas 3.3 and 3.4 that " $=_{(\gamma,\delta)}$ " is an equivalence relation on the set of all fuzzy soft sets over U.

Now, let us define some operations of fuzzy subsets of M.

Definition 3.5. Let $\mu, \nu \in \mathcal{F}(M)$ and $\alpha \in \Gamma$. We define the fuzzy subsets $-\mu, \mu \boxplus \nu$ and $\mu @ \nu$ *by*

$$(-\mu)(x) = \mu(-x),$$
$$(\mu \boxplus \nu)(x) = \sup_{x \in y \oplus z} \min\{\mu(y), \nu(z)\}$$

and

$$(\mu @ \nu)(x) = \begin{cases} \sup\limits_{x = y\alpha z} \min\{\mu(y), \nu(z)\} & \textit{if} \, \exists y, z \in M \textit{ such that} \, x = y\alpha z, \\ 0 & \textit{otherwise}, \end{cases}$$

respectively, for all $x \in M$ and $\alpha \in \Gamma$.

Definition 3.6. Let $(\tilde{F}, A)$ and $(\tilde{G}, B)$ be two fuzzy soft sets over M. The sum of $(\tilde{F}, A)$ and $(\tilde{G}, B)$, denoted by $(\tilde{F}, A) \boxplus (\tilde{G}, B)$, is defined to be the fuzzy soft set $(\tilde{F}, A) \boxplus (\tilde{G}, B) = (\tilde{F} \boxplus \tilde{G}, C)$ over U, where $C = A \cup B$ and

$$(\tilde{F} \boxplus \tilde{G})[\alpha](x) = \begin{cases} \tilde{F}[\alpha](x) & \textit{if} \, \alpha \in A - B, \\ \tilde{G}[\alpha](x) & \textit{if} \, \alpha \in B - A, \\ (\tilde{F}[\alpha] \boxplus \tilde{G}[\alpha])(x) & \textit{if} \, \alpha \in A \cap B, \end{cases}$$

for all $\alpha \in C$ and $x \in M$.

Definition 3.7. Let $(\tilde{F}, A)$ and $(\tilde{G}, B)$ be two fuzzy soft sets over M. The α-product of $(\tilde{F}, A)$ and $(\tilde{G}, B)$, denoted by $(\tilde{F}, A) @ (\tilde{G}, B)$, is defined to be fuzzy soft set $(\tilde{F}, A) @ (\tilde{G}, B) = (\tilde{F} @ \tilde{G}, C)$ over U, where $C = A \cup B$ and

$$(\tilde{F} @ \tilde{G})[\alpha](x) = \begin{cases} \tilde{F}[\alpha](x) & \textit{if} \, \alpha \in A - B, \\ \tilde{G}[\alpha](x) & \textit{if} \, \alpha \in B - A, \\ (\tilde{F}[\alpha] @ \tilde{G}[\alpha])(x) & \textit{if} \, \alpha \in A \cap B, \end{cases}$$

for all $\alpha \in C$ and $x \in M$.

4. $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy soft Γ-subhypermodules

In this section, we concentrate our study on the $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy soft Γ-subhypermodules of Γ-hypermodules.

Definition 4.1. A fuzzy soft set $(\tilde{F}, A)$ over M is called an $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-*fuzzy soft* Γ-*subhypermodule* of M if for all $\alpha \in A, t, s \in (\gamma, 1]$, $r \in R$, $\lambda \in \Gamma$ and $x \in M$:

(F1a) $x_t \in_\gamma \tilde{F}[\alpha]$ and $y_s \in_\gamma \tilde{F}[\alpha]$ imply that $z_{\min\{t,s\}} \in_\gamma \vee \mathrm{q}_\delta \tilde{F}[\alpha]$ for all $z \in x \oplus y$;

(F2a) $x_t \in_\gamma \tilde{F}[\alpha]$ implies that $(-x)_t \in_\gamma \vee \mathrm{q}_\delta \tilde{F}[\alpha]$;

(F3a) $x_t \in_\gamma \tilde{F}[\alpha]$ implies that $(r\lambda x)_t \in_\gamma \vee \mathrm{q}_\delta \tilde{F}[\alpha]$.

Clearly, if for each $\alpha \in A$, $\tilde{F}[\alpha]$ is a constant map, that is $\exists r \in [0, 1]$: $\tilde{F}[\alpha](u) = r$, for all $u \in M$, then $(\tilde{F}, A)$ is a fuzzy soft set over M.

Examples 4.2. Let us consider the ring $R = Z_2$, $\Gamma = \{\cdot\}$ and let A be a nonempty set. Moreover, let L be a modular lattice with 0. If we define the next hyperoperation on L :

$$x \oplus y = \{z \mid z \vee y = z \vee x = x \vee y\},$$

then $(L, \oplus)$ is a canonical hypergroup, in which the scalar identity is 0 and the opposite of any element x of L is just x. (see [10], page 129).

For all $\alpha \in A$, we impose that the fuzzy set $\tilde{F}[\alpha]$ satisfies the following conditions:

$$\forall x, y \in L,\ \tilde{F}[\alpha](x \vee y) = \tilde{F}[\alpha](x) \wedge \tilde{F}[\alpha](y) \text{ and } \tilde{F}[\alpha](0) = 1.$$

Define $\hat{1} \cdot x = x$ and $\hat{0} \cdot x = 0 \in L$. The conditions (F1a), (F2a) and (F3a) of the above definition hold.

Indeed, if $z \in x \oplus y$, then

$$\tilde{F}[\alpha](z) \geq \tilde{F}[\alpha](x) \wedge \tilde{F}[\alpha](y) = \tilde{F}[\alpha](x) \wedge \tilde{F}[\alpha](z) = \tilde{F}[\alpha](y) \wedge \tilde{F}[\alpha](z) \geq min\{t, s\},$$

whence we obtain (F1a).

(F2a) is clearly satisfied since $\forall x \in L$, $-x = x$. For the third condition, we use that $\tilde{F}[\alpha](0) = 1$.

Therefore $(\tilde{F}, A)$ is a $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy soft Γ-subhypermodule of L.

Lemma 4.3. Let $(\tilde{F}, A)$ be a fuzzy soft set over M. Then (F1a) holds if and only if one of the following conditions holds: for all $\alpha \in A$ and $x, y \in M$,

(F1b) $\max\left\{\inf\limits_{z \in x \oplus y} \tilde{F}[\alpha](z), \gamma\right\} \geq \min\{\tilde{F}[\alpha](x), \tilde{F}[\alpha](y), \delta\}$;

(F1c) $(\tilde{F}, A) \boxplus (\tilde{F}, A) \subseteq \vee q_{(\gamma,\delta)} (\tilde{F}, A)$.

Proof. (F1a)⇒(F1b) Let $\alpha \in A$ and $x, y \in M$. Suppose that the following situation is possible: $z \in M$ is such that $z \in x \oplus y$ and $\max\left\{\tilde{F}[\alpha](z), \gamma\right\} < t = \min\{\tilde{F}[\alpha](x), \tilde{F}[\alpha](y), \delta\}$. Then $\tilde{F}[\alpha](x) \geq t, \tilde{F}[\alpha](y) \geq t, \tilde{F}[\alpha](z) < t \leq \delta$, hence $x_t, y_t \in_\gamma \tilde{F}[\alpha]$ and $z_t \overline{\in_\gamma \vee \mathrm{q}_\delta}\, \tilde{F}[\alpha]$, a contradiction. Hence (F1b) is valid.

(F1b)⇒(F1c) Let $\alpha \in A, t \in (\gamma, 1]$ and $x \in M$ be such that $x_t \in_\gamma (\tilde{F} \boxplus \tilde{F})[\alpha]$. Suppose that $x_t \overline{\in_\gamma \vee \mathrm{q}_\delta}\, \tilde{F}[\alpha]$. Then $x_t \overline{\in_\gamma} \tilde{F}[\alpha]$ and $x_t \overline{q_\delta} \tilde{F}[\alpha]$, i.e.,

$\tilde{F}[\alpha](x) < t$ and $\tilde{F}[\alpha](x) + t \leq 2\delta$ which implies that $\tilde{F}[\alpha](x) < \delta$. If $x \in y \oplus z$ for some $y, z \in M$, by (F1b), then we have

$$\max\left\{\tilde{F}[\alpha](x), \gamma\right\} \geq \min\{\tilde{F}[\alpha](y), \tilde{F}[\alpha](z), \delta\}.$$

From $\tilde{F}[\alpha](x) < \delta$ it follows that

$$\max\left\{\tilde{F}[\alpha](x), \gamma\right\} \geq \min\{\tilde{F}[\alpha](y), \tilde{F}[\alpha](z)\}.$$

Hence we have

$$\begin{aligned} r \leq (\tilde{F} \boxplus \tilde{F})[\alpha](x) &= \sup_{x \in a \oplus b} \min\{\tilde{F}[\alpha](a), \tilde{F}[\alpha](b)\} \\ &\leq \sup_{x \in a \oplus b} \max\left\{\tilde{F}[\alpha](x), \gamma\right\} = \max\left\{\tilde{F}[\alpha](x), \gamma\right\}, \end{aligned}$$

a contradiction. Hence (F1c) is satisfied.

(F1c)⇒(F1a) Let $\alpha \in A, t, s \in (\gamma, 1]$ and $x, y \in M$ be such that $x_t, y_s \in_\gamma \tilde{F}[\alpha]$. Then for any $z \in x \oplus y$, we have

$$\begin{aligned} (\tilde{F} \boxplus \tilde{F})[\alpha](z) &= \sup_{z \in a \oplus b} \min\{\tilde{F}[\alpha](a), \tilde{F}[\alpha](b)\} \\ &\geq \min\{\tilde{F}[\alpha](x), \tilde{F}[\alpha](y)\} \geq \min\{t, s\} > \gamma. \end{aligned}$$

Hence $z_{\min\{t,s\}} \in_\gamma (\tilde{F} \boxplus \tilde{F})[\alpha]$ and so $z_{\min\{t,s\}} \in \vee\, \mathrm{q}\tilde{F}[\alpha]$ by (F1c). Thus (F1a) holds.

For any fuzzy soft set $(\tilde{F}, A)$ over M, denote by $(\tilde{F}^{-1}, A)$ the fuzzy soft set defined by $\tilde{F}^{-1}[\alpha](x) = \tilde{F}[\alpha](-x)$ for all $\alpha \in A$ and $x \in M$.

Lemma 4.4. Let $(\tilde{F}, A)$ be a fuzzy soft set over M. Then (F2a) holds if and only if one of the following conditions hold: for all $\alpha \in A$ and $x \in R$,

(F2b) $\max\left\{\tilde{F}[\alpha](-x), \gamma\right\} \geq \min\{\tilde{F}[\alpha](x), \delta\}$;

(F2c) $(\tilde{F}, A) \subseteq \vee q_{(\gamma,\delta)}(\tilde{F}^{-1}, A)$.

Proof. It is similar to the proof of Lemma 4.3.

Lemma 4.5. Let $(\tilde{F}, A)$ be a fuzzy soft set over M. Then (F3a) holds if and only if one of the following conditions hold: for all $\alpha \in A$, $r \in R$, $\lambda \in \Gamma$ and $x \in M$,

(F3b) $\max\left\{\tilde{F}[\alpha](r\lambda x), \gamma\right\} \geq \min\{\tilde{F}[\alpha](x), \delta\}$;

(F3c) $(\tilde{\chi_M}, A)@(\tilde{F}, A) \subseteq \vee q_{(\gamma,\delta)}(\tilde{F}, A)$.

Proof. It is similar to the proof of Lemma 4.3.

From the above discussion, we can immediately obtain the following two theorems:

Theorem 4.6. A fuzzy soft set $(\tilde{F}, A)$ over M is an $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy soft Γ-subhypermodule of M if and only if it satisfies (F1b), (F2b) and (F3b).

Theorem 4.7. A fuzzy soft set $(\tilde{F}, A)$ over M is an $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy soft Γ-subhypermodule of M if and only if it satisfies (F1c), (F2c) and (F34c).

For any fuzzy soft set $(\tilde{F}, A)$ over M, $\alpha \in A$ and $t \in [0,1]$, we define $\tilde{F}[\alpha]_t = \{x \in M | x_t \in_\gamma \tilde{F}[\alpha]\}$, $\tilde{F}[\alpha]_t^\delta = \{x \in M | x_t q_\delta \tilde{F}[\alpha]\}$ and $[\tilde{F}[\alpha]]_t^\delta = \{x \in M | x_t \in \vee\, \mathrm{q}\, \tilde{F}[\alpha]\}$. It is clear that $[\tilde{F}[\alpha]]_t^\delta = \tilde{F}[\alpha]_t \cup \tilde{F}[\alpha]_t^\delta$.

The next theorem provides a relationships between $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy soft Γ-subhypermodules of M and crisp Γ-subhypermodules of M .

Theorem 4.8. Let $(\tilde{F}, A)$ be a fuzzy soft set over M. Then

(1) $(\tilde{F}, A)$ is an $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy Γ-subhypermodule of M if and only if the non-empty set $\tilde{F}[\alpha]_t$ is a Γ-subhypermodule of M for all $\alpha \in A$ and $t \in (\gamma, \delta]$.

(2) If $2\delta = 1+\gamma$, then $(\tilde{F}, A)$ is an $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy soft Γ-subhypermodule of M if and only if the non-empty set $\tilde{F}[\alpha]_t^\delta$ is a Γ-subhypermodule of M for all $\alpha \in A$ and $t \in (\delta, 1]$.

(3) If $2\delta = 1+\gamma$, then $(\tilde{F}, A)$ is an $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy soft Γ-subhypermodule of M if and only if the non-empty set $[\tilde{F}[\alpha]]_t^\delta$ is a Γ-subhypermodule of M for all $\alpha \in A$ and $t \in (\gamma, 1]$.

Proof. We show only (3). The proofs of (1) and (2) are similar. Let $(\tilde{F}, A)$ be an $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy soft Γ-subhypermodule of M and $x, y \in [\tilde{F}[\alpha]]_t^\delta$ for some $\alpha \in A$ and $t \in (\gamma, 1]$. Then $x_t \in_\gamma \vee \mathrm{q}_\delta \tilde{F}[\alpha]$ and $y_t \in_\gamma \vee \mathrm{q}_\delta \tilde{F}[\alpha]$, i.e., $\tilde{F}[\alpha](x) \geq t$ or $\tilde{F}[\alpha](x) > 2\delta - t > 2\delta - 1 = \gamma$, and $\tilde{F}[\alpha](y) \geq t$ or $\tilde{F}[\alpha](y) > 2\delta - t > 2\delta - 1 = \gamma$. Since $(\tilde{F}, A)$ is an $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy soft Γ-subhypermodule of M and $\min\{\tilde{F}[\alpha](x), \tilde{F}[\alpha](y), \delta\} > \gamma$, we have $\tilde{F}[\alpha](z) \geq \min\{\tilde{F}[\alpha](x), \tilde{F}[\alpha](y), \delta\}$ for all $z \in x \oplus y$. We consider the following cases:

Case 1: $t \in (\gamma, \delta]$. Since $t \in (\gamma, \delta]$, we have $2\delta - t \geq \delta \geq t$.

Case 1a: $\tilde{F}[\alpha](x) \geq t$ or $\tilde{F}[\alpha](y) \geq t$. Then
$\tilde{F}[\alpha](z) \geq \min\{\tilde{F}[\alpha](x), \tilde{F}[\alpha](y), \delta\} \geq t$. Hence $z_t \in_\gamma \tilde{F}[\alpha]$.

Case 1b: $\tilde{F}[\alpha](x) > 2\delta - t$ and $\tilde{F}[\alpha](y) > 2\delta - t$. Then
$\tilde{F}[\alpha](z) \geq \min\{\tilde{F}[\alpha](x), \tilde{F}[\alpha](y), \delta\} = \delta \geq t$. Hence $z_t \in_\gamma \tilde{F}[\alpha]$.

Case 2: $t \in (\delta, 1]$. Since $t \in (\delta, 1]$, we have $2\delta - t < \delta < t$.

Case 2a: $\tilde{F}[\alpha](x) \geq t$ and $\tilde{F}[\alpha](y) \geq t$. Then
$\tilde{F}[\alpha](z) \geq \min\{\tilde{F}[\alpha](x), \tilde{F}[\alpha](y), \delta\} = \delta$, and so $f(z) + t \geq t + \delta > 2\delta$. Hence $z_t\, q_\delta\, \tilde{F}[\alpha]$.

Case 2b: $\tilde{F}[\alpha](x) > 2\delta - t$ or $\tilde{F}[\alpha](y) > 2\delta - t$. Then
$\tilde{F}[\alpha](z) \geq \min\{\tilde{F}[\alpha](x), \tilde{F}[\alpha](y), \delta\} > 2\delta - t$. Hence $z_t\, q_\delta\, \tilde{F}[\alpha]$.

Thus in any case, $z_t \in_\gamma \vee \mathrm{q}_\delta \tilde{F}[\alpha]$, i.e., $z \in [\tilde{F}[\alpha]]_t^\delta$ for all $z \in x \oplus y$. Similarly we can show that the other conditions hold. Therefore, $[\tilde{F}[\alpha]]_t^\delta$ is a Γ-subhypermodule of M .

Conversely, assume that the given condition holds. Let $\alpha \in A$ and $x, y \in M$. If there exists $z \in M$ such that $z \in x \oplus y$ and $\max\left\{\tilde{F}[\alpha](z), \gamma\right\} < t = \min\{\tilde{F}[\alpha](x), \tilde{F}[\alpha](y), \delta\}$, then $\tilde{F}[\alpha](x) \geq t, \tilde{F}[\alpha](y) \geq t, \tilde{F}[\alpha](z) < t \leq \delta$, i.e., $x_t, y_t \in \tilde{F}[\alpha]$ but $z_t \overline{\in_\gamma \vee \mathrm{q}_\delta} \tilde{F}[\alpha]$. Hence $x, y \in [\tilde{F}[\alpha]]_t^\delta$ but $z \overline{\in} [\tilde{F}[\alpha]]_t^\delta$, a contradiction. Therefore, $\max\left\{\tilde{F}[\alpha](z),\gamma\right\}\geq\min\{\tilde{F}[\alpha](x),\tilde{F}[\alpha](y),\delta\}$ for all $z\in x\oplus y$. This proves that (F1b) holds. Similarly we can show that (F2b) and (F3b) hold. Therefore, $(\tilde{F},A)$ is an $(\in_\gamma,\in_\gamma\vee\mathrm{q}_\delta)$-fuzzy soft Γ-subhypermodule of M by Theorem 4.6.

Remark 4.9. For any $(\in_\gamma,\in_\gamma\vee\mathrm{q}_\delta)$-fuzzy soft Γ-subhypermodule $(\tilde{F},A)$ of M, we can conclude that

(1) $\tilde{F}[\alpha]$ is a fuzzy Γ-subhypermodule of M, for all $\alpha\in A$ when $\gamma=0$ and $\delta=1$;

(2) $(\tilde{F},A)$ is an $(\in,\in\vee\mathrm{q})$-fuzzy soft Γ-subhypermodule of M when $\gamma=0,\ \delta=0.5$.

Theorem 4.10. Let $(\tilde{F},A)$ and $(\tilde{G},B)$ be two $(\in_\gamma,\in_\gamma\vee\mathrm{q}_\delta)$-fuzzy soft Γ-subhypermodules of M. Then $(\tilde{F},A)\boxplus(\tilde{G},B)$ is an $(\in_\gamma,\in\vee\mathrm{q})$-fuzzy soft Γ-subhypermodule of M if and only if $(\tilde{F},A)\boxplus(\tilde{G},B)=_{(\gamma,\delta)}(\tilde{G},B)\boxplus(\tilde{F},A)$.

Proof. If $(\tilde{F},A)\boxplus(\tilde{G},B)=_{(\gamma,\delta)}(\tilde{G},B)\boxplus(\tilde{F},A)$, then

$$\begin{aligned}((\tilde{F},A)\boxplus(\tilde{G},B))\boxplus((\tilde{F},A)\boxplus(\tilde{G},B))&=(\tilde{F},A)\boxplus((\tilde{G},B)\boxplus(\tilde{F},A))\boxplus(\tilde{G},B)\\&=_{(\gamma,\delta)}(\tilde{F},A)\boxplus((\tilde{F},A)\boxplus(\tilde{G},B))\boxplus(\tilde{G},B)\\&=((\tilde{F},A)\boxplus(\tilde{F},A))\boxplus((\tilde{G},B)\boxplus(\tilde{G},B))\\&\subseteq\vee q_{(\gamma,\delta)}(\tilde{F},A)\boxplus(\tilde{G},B).\end{aligned}$$

Hence $((\tilde{F},A)\boxplus(\tilde{G},B))\boxplus((\tilde{F},A)\boxplus(\tilde{G},B))\subseteq\vee q_{(\gamma,\delta)}(\tilde{F},A)\boxplus(\tilde{G},B)$. This proves that (F1c) holds.

Let $\alpha\in A\cup B$ and $x\in R$. We have the following situations:

Case 1: $\alpha\in A-B$. Then

$$\begin{aligned}\max\{(\tilde{F}\boxplus\tilde{G})[\alpha](-x),\gamma\}&=\max\{\tilde{F}[\alpha](-x),\gamma\}\\\geq\min\{\tilde{F}[\alpha](x),\delta\}&=\min\{(\tilde{F}\boxplus\tilde{G})[\alpha](x),\delta\}\end{aligned}$$

Case 2: $\alpha\in B-A$. Analogous to case 1, we have

$$\max\{(\tilde{F}\boxplus\tilde{G})[\alpha](-x),\gamma\}\geq\min\{(\tilde{F}\boxplus\tilde{G})[\alpha](x),\delta\}.$$

Case 3: $\alpha \in A \cap B$.

$$\begin{aligned}
\max\{(\tilde{F} \boxplus \tilde{G})[\alpha](-x), \gamma\} &= \max\left\{\sup_{-x \in a \oplus b} \min\{\tilde{F}[\alpha](a), \tilde{G}[\alpha](b)\}, \gamma\right\} \\
&= \sup_{x \in (-a) \oplus (-b)} \min\left\{\max\left\{\tilde{F}[\alpha](a), \gamma\right\}, \max\left\{\tilde{G}[\alpha](b), \gamma\right\}\right\} \\
&\geq \sup_{x \in (-a) \oplus (-b)} \min\left\{\min\left\{\tilde{F}[\alpha](a), \delta\right\}, \min\left\{\tilde{G}[\alpha](b), \delta\right\}\right\} \\
&= \min\left\{\sup_{x \in (-a) \oplus (-b)} \min\left\{\tilde{F}[\alpha](a), \tilde{G}[\alpha](b)\right\}, \delta\right\} \\
&= \min\left\{(\tilde{F} \boxplus \tilde{G})[\alpha](x), \delta\right\}.
\end{aligned}$$

Thus, in any case, $\max\{(\tilde{F} \boxplus \tilde{G})[\alpha](-x), \gamma\} \geq \min\{(\tilde{F} \boxplus \tilde{G})[\alpha](x), \delta\}$. This proves that (F2b) holds. Similarly, we can prove that (F3b) and (F4b) hold. Therefore, $(\tilde{F}, A) \boxplus (\tilde{G}, B)$ is an $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy soft Γ-subhypermodule of M.

Conversely, assume that $(\tilde{F}, A) \boxplus (\tilde{G}, B)$ is an $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy soft Γ-subhypermodule of M. Let $\alpha \in A \cup B$. We have the following situations:

Case 1: $\alpha \in A - B$. Then $(\tilde{F} \boxplus \tilde{G})[\alpha] = \tilde{F}[\alpha] = (\tilde{G} \boxplus \tilde{F})[\alpha]$.

Case 2: $\alpha \in B - A$. Then $(\tilde{F} \boxplus \tilde{G})[\alpha] = \tilde{G}[\alpha] = (\tilde{G} \boxplus \tilde{F})[\alpha]$.

Case 3: $\alpha \in A \cap B$. Then for any $x \in M$,

$$\begin{aligned}
\max\{(\tilde{F} \boxplus \tilde{G})[\alpha](x), \gamma\} &\geq \max\{\min\{(\tilde{F} \boxplus \tilde{G})[\alpha](-x), \delta\}, \gamma\} \\
&= \min\{\max\{(\tilde{F} \boxplus \tilde{G})[\alpha](-x), \gamma\}, \delta\} \\
&= \min\left\{\max\left\{\sup_{-x \in a \oplus b} \min\{\tilde{F}[\alpha](a), \tilde{G}[\alpha](b)\}, \gamma\right\}, \delta\right\} \\
&= \min\left\{\sup_{x \in -b \oplus -a} \min\left\{\max\left\{\tilde{F}[\alpha](a), \gamma\right\}, \max\left\{\tilde{G}[\alpha](b), \gamma\right\}\right\}, \delta\right\} \\
&\geq \min\left\{\sup_{x \in -b \oplus -a} \min\left\{\min\left\{\tilde{F}[\alpha](-a), \delta\right\}, \min\left\{\tilde{G}[\alpha](-b), \delta\right\}\right\}, \delta\right\} \\
&= \min\left\{\sup_{x \in -b \oplus -a} \min\left\{\tilde{F}[\alpha](-a), \tilde{G}[\alpha](-b)\right\}, \delta\right\} \\
&= \min\left\{(\tilde{G} \boxplus \tilde{F})[\alpha](x), \delta\right\}.
\end{aligned}$$

It follows that $\max\{\min\{(\tilde{F} \boxplus \tilde{G})[\alpha](x), \delta\}, \gamma\} \geq \max\{\min\{(\tilde{G} \boxplus \tilde{F})[\alpha](x), \delta\}, \gamma\}$. In a similar way, we have $\max\{\min\{(\tilde{G} \boxplus \tilde{F})[\alpha](x), \delta\}, \gamma\} \geq \max\{\min\{(\tilde{F} \boxplus \tilde{G})[\alpha](x), \delta\}, \gamma\}$.

Thus, in any case, we have $(\tilde{F}, A) \boxplus (\tilde{G}, B) =_{(\gamma,\delta)} (\tilde{G}, B) \boxplus (\tilde{F}, A)$, as required.

The following proposition is obvious.

Proposition 4.11. If $(\tilde{F}, A)$ is an $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy soft Γ-subhypermodule of M, then

$$(\tilde{F}, A) \boxplus (\tilde{x}_1, A) =_{(\gamma,\delta)} (\tilde{x}_1, A) \boxplus (\tilde{F}, A)$$

for all $x \in M$.

If ρ is an equivalence relation on a Γ-hypermodule M, then we say that $A\overline{\rho}B$ if it satisfies the following conditions:

(1) $\forall x \in A$, $\exists y \in B$, such that $x\rho y$;

(2) $\forall b \in B$, $\exists a \in A$, such that $a\rho b$.

Definition 4.12. An equivalence relation ρ on M is said to be a *congruence relation* on M if for any $r \in R$, $x, y, z \in M$ and $\alpha \in \Gamma$, we have

$$x\rho y \Rightarrow x \oplus z\overline{\rho}y \oplus z, z \oplus x\overline{\rho}z \oplus y, r\alpha x\overline{\rho}r\alpha y.$$

Theorem 4.13. Let $(\tilde{F}, A)$ be an $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy soft Γ-subhypermodule of M, $\alpha \in A$ and $t \in [\gamma, \min\{\tilde{F}[\alpha](0), \delta\})$. Define a relation $\tilde{F}[\alpha]^*_t$ on M by

$$x\tilde{F}[\alpha]^*_t y \Leftrightarrow \min\{(\widetilde{x}_1 \boxplus \tilde{F})[\alpha](y), \delta\} > t,$$

for all $x, y \in M$. Then $\tilde{F}[\alpha]^*_r$ is a congruence relation on M.

Proof. Let $(\tilde{F}, A)$ be an $(\in_\gamma, \in \vee \mathrm{q})$-fuzzy soft Γ-subhypermodule of M, $\alpha \in A$ and $t \in [\gamma, \min\{\tilde{F}[\alpha](0), \delta\})$. Then we have

(1) $\tilde{F}[\alpha]^*_t$ is reflexive on M. In fact, for any $x \in M$, we have

$$\min\{(\widetilde{x}_1 \boxplus \tilde{F}[\alpha])(x), \delta\} = \min\left\{\sup_{x \in x \oplus a} \tilde{F}[\alpha](a), \delta\right\} \geq \min\{\tilde{F}[\alpha](0), \delta\} > t.$$

Hence $x\tilde{F}[\alpha]^*_t x$, and so $\tilde{F}[\alpha]^*_t$ is reflexive.

(2) $\tilde{F}[\alpha]^*_t$ is symmetric on M by Proposition 4.11.

(3) $\tilde{F}[\alpha]^*_t$ is transitive on M. In fact, let $x, y, z \in M$ be such that $x\tilde{F}[\alpha]^*_t y$ and $y\tilde{F}[\alpha]^*_r z$. Then $\min\{(\widetilde{x}_1 \boxplus \tilde{F})[\alpha](y), \delta\} > t$ and $\min\{(\widetilde{y}_1 \boxplus \tilde{F})[\alpha](z), \delta\} > t$. From

$$(\tilde{F}, A) \boxplus \tilde{F}, A) \subseteq \vee q_{(\gamma,\delta)}(\tilde{F}, A)$$

, we have

$$(\widetilde{x}_1, A) \boxplus (\tilde{F}, A) \boxplus (\tilde{F}, A) \subseteq \vee q_{(\gamma,\delta)}(\widetilde{x}_1, A) \boxplus (\tilde{F}, A)$$

, and so

$$\begin{aligned}
\min\{(\widetilde{x}_1 \boxplus \tilde{F})[\alpha](z), \delta\} &\geq \min\{(\widetilde{x}_1 \boxplus \tilde{F} \boxplus \tilde{F})[\alpha](z), \delta\} \\
&= \min\left\{\sup_{a \in M} \min\{(\widetilde{x}_1 \boxplus \tilde{F})[\alpha](a), (\widetilde{a}_1 \boxplus \tilde{F})[\alpha](z)\}, \delta\right\} \\
&\geq \min\left\{(\widetilde{x}_1 \boxplus \tilde{F})[\alpha](y), (\widetilde{y}_1 \boxplus \tilde{F})[\alpha](z), \delta\right\} \\
&> t.
\end{aligned}$$

It follows that

$$\min\{(\widetilde{x}_1 \boxplus \tilde{F})[\alpha](z), \delta\} > t$$

since $t \geq \gamma$, i.e., $x\tilde{F}[\alpha]^*_t z$. Hence $\tilde{F}[\alpha]^*_t$ is transitive.

From the above arguments, it follows that $\tilde{F}[\alpha]^*_t$ is an equivalence relation on M.

Next let $x, y, z, a \in M$ be such that $x\tilde{F}[\alpha]^*_r y$ and $a \in x \oplus z$. Then

$$\min\{(\widetilde{y}_1 \boxplus \tilde{F})[\alpha](x), \delta\} > t$$

. From Proposition 4.11, we have

$$\begin{aligned}
\min\{(\widetilde{y \oplus z})_1 \boxplus \tilde{F})[\alpha](a), \delta\} &= \min\{(\widetilde{y}_1 \boxplus \widetilde{z}_1 \boxplus \tilde{F})[\alpha](a), \delta\} = \min\{(\widetilde{y}_1 \boxplus \tilde{F} \boxplus \widetilde{z}_1)[\alpha](a), \delta\} \\
&= \min\left\{\sup_{b\in M} \min\{(\widetilde{y}_1 \boxplus \tilde{F})[\alpha](b), (\widetilde{b}_1 \boxplus \tilde{z}_1)[\alpha](a)\}, \delta\right\} \\
&\geq \min\left\{(\widetilde{y}_1 \boxplus \tilde{F})[\alpha](x), (\widetilde{x}_1 \boxplus \tilde{z}_1)[\alpha](a), \delta\right\} \\
&= \min\{(\widetilde{y}_1 \boxplus \tilde{F})[\alpha](x), \delta\} > t.
\end{aligned}$$

Hence there exists $c \in M$ such that $c \in y \oplus z$ and $\min\{(\widetilde{c}_1 \boxplus \tilde{F})[\alpha](a), \delta\} > t$, i.e., $a\tilde{F}[\alpha]^*_t c$. Similarly, if $d \in y \oplus z$ for some $d \in M$, then there exists $e \in M$ such that $d \in x \oplus z$ and $d\tilde{F}[\alpha]^*_t e$. Hence

$$x \oplus z\overline{\tilde{F}[\alpha]^*_t} y \oplus z$$

. In a similar way, we have

$$z \oplus x\overline{\tilde{F}[\alpha]^*_t} z \oplus y$$

.

Finally, let $r \in R$, $x, y \in M$, $\alpha \in A, \lambda \in \Gamma$ be such that

$$x\tilde{F}[\alpha]^*_t y$$

. Then

$$\min\{(\widetilde{y}_1 \boxplus \tilde{F})[\alpha](x), \delta\} > t$$

. From Proposition 4.11, we have

$$\begin{aligned}
\min\{(\widetilde{x}_1 \boxplus \tilde{F})[\alpha](a), \delta\} &= \min\left\{\sup_{y\in x\oplus a} \min\{\tilde{F}[\alpha](a)\}, \delta\right\} \\
&= \min\left\{\sup_{b\in R} \min\{\tilde{F}[\alpha](b)\}, \delta\right\} \\
&> t,
\end{aligned}$$

and so,

$$\begin{aligned}\min\{(\widetilde{(r\lambda x)}_1 \boxplus \tilde{F})[\alpha](r\lambda y),\delta\} &= \min\left\{\sup_{r\alpha y\in r\alpha x\oplus a}\min\{\tilde{F}[\alpha](a)\},\delta\right\}\\ &= \min\left\{\tilde{F}[\alpha](r\lambda b),\delta\right\}\\ &= \min\left\{\min\{\tilde{F}[\alpha](b)\},\delta\right\}\\ &> t,\end{aligned}$$

that is,

$$r\lambda x\ \overline{\tilde{F}[\alpha]_t^*}\ r\lambda y$$

.

Therefore, $\tilde{F}[\alpha]_t^*$ is a congruence relation on M. This completes the proof.

Let $\tilde{F}[\alpha]_t^*[x]$ be the equivalence class containing the element x. We denote by $M/\tilde{F}[\alpha]_t^*$ the set of all equivalence classes, i.e.,

$$M/\tilde{F}[\alpha]_t^* = \{\tilde{F}[\alpha]_t^*[x] | x \in M\}$$

.

Since $\tilde{F}[\alpha]_t^*$ is a congruence relation on M, we can easily deduce the following theorem.

bf Theorem 4.14. Let $(\tilde{F}, A)$ be an $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy soft Γ-subhypermodule of M, $\alpha \in A$, $\lambda \in \Gamma$ and $t \in [\gamma, \min\{\tilde{F}[\alpha](0), \delta\})$. Then $(M/\tilde{F}[\alpha]_t^*, \uplus, \Gamma)$ is a Γ-hypermodule, where:

$$\tilde{F}[\alpha]_t^*[x] \uplus \tilde{F}[\alpha]_t^*[y] = \{\tilde{F}[\alpha]_t^*[z] | z \in x \oplus y\} \text{and } r\lambda\tilde{F}[\alpha]_t^*[x] = \tilde{F}[\alpha]_t^*[r\lambda x],$$

for all $r \in R$, $x, y \in M$, $\alpha \in A$ and $\lambda \in \Gamma$.

5. Conclusions

In this paper, we consider the $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy soft Γ-subhypermodules of Γ-hypermodules. In particular, we obtain a kind of new Γ-hypermodules by congruence relations.

In a future study of fuzzy structure of Γ-hypermodules, the following topics could be considered:

(1) To consider the fuzzy rough Γ-hypermodules;
(2) To establish three fuzzy isomorphism theorems of Γ-hypermodules.
(3) To describe the rough soft Γ-hypermodules and their applications.

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