COLOURED CHAOS IN THE ROL-USD EXCHANGE RATE VIA TIME-FREQUENCY ANALYSIS

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Lucrarea se referă la analiza existenţei elementelor caracteristice haosului determinist în seria temporală a cursului de schimb leu-dolar, urmărit de-a lungul unei perioade de aproape şaisprezece ani. Studiul este pus în corespondenţă cu istoricul evoluţiei de la economia centralizată spre o economie de piaţă, după prabuşirea regimului totalitar, şi relevă existenţa unor sub-intervale caracteristice. Coeficienţii Liapunov sunt pozitivi pe toate sub-intervalele. De asemenea, am estimat dimensiunea atractorilor sistemului dinamic care dă naştere unei astfel de serii temporale. Analiza spectral a datelor, prelucrate prin filtrare, probează existenţa "haosului colorat" în spectrul din spaţiul frecvenţelor.

This work is focused on the analysis of the existence of deterministic chaos in the evolution of the Romanian national currency (ROL) exchange rate with respect to the United States Dollar (USD). The study is related to the economic evolution toward an open system after the collapse of the totalitarian regime. We test the daily variation of the time series for almost sixteen years. Positive Lyapunov exponents were detected along the entire period. We also estimate the dimension of the attractors of the underlying dynamic system producing this time series. The frequency spectrum reveals evidence of coloured chaos in the detrended data.

Keywords: nonlinear dynamics, deterministic chaos, largest Lyapunov exponent, space reconstruction, embedding dimension, correlation dimension, coloured chaos, data detrending, underlying economic system.

Introduction

This paper is illustrating how to use the physical laws of nonlinear dynamics in the economic field. The purpose of econo-physics[1] theories is either to develop models that can explain observed regularities or to forecast long term tendencies. The autonomous physico-mathematical models applied to the financial markets are now currently used to explain particular aspects of the complex non-linear dynamics of stock markets, interest rates, money supplies, and price levels, as well as the exchange rates[2,3,4] dynamics. Time analysis of the

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financial series was introduced in order to predict the traditional unexpected phenomena that cause crisis or shocks in hot financial systems[5].

There are two polar models in linear dynamics: white noise and harmonic cycles. Obviously, real data fall between these two extremes. Closer to the first category are the stochastic processes (probabilistic predictable), while towards the last one are the systems with a completely deterministic dynamics (definitely predictable); the deterministic chaos (short run predictable[6,7]) belongs to the deterministic category which, in addition, has the property of sensitivity to initial conditions.

Existence of chaotic behaviour in economics has strong implications for the predictive ability of the time series, especially in as long as possible short run (notwithstanding!) predictions. As mentioned above, one feature of chaotic systems is their sensitive dependence on initial conditions. Hence, one way of revealing the existence of deterministic chaos in a time series is to measure the degree of divergence of nearby orbits in phase-space. Such divergence can be measured by the Lyapunov exponents, and the presence of at least one positive exponent is taken as an indication of chaos[8,9]. Thus, the largest Lyapunov exponent allows regular and chaotic behaviours to be distinguished.

It has been proved the possibility to extract information about a complex dynamical system, which generates several observed time series, while using only the time series of one single (available) characteristic parameter. In other words, if there is a dynamical system hidden in a “black box”, it is possible to reconstruct its “geometrical shape” using only the available time series.

Thus, under several assumptions, the exchange rate ROL-USD series does offer information about the evolution of the Romanian monetary system. The above mentioned assumptions are regarding the applicability of the subsequent theoretical model.

1. Theoretical model

1.1 Nonlinear causality

The main hypotheses concerning the chaotic dynamics of the underlying system are: a/ causality, or deterministic evolution, and b/ stationarity, i.e. the system remains unchanged in time (otherwise, a filter could extract the trend and simulate the steady state conditions). Regarding the first point, it’s worth noting that the stochastic processes are excluded. For the second, it’s clear that there is no absolute stationarity because the historical conditions are not reproducible. Thus, we limit ourselves to a certain quasi-stationarity. For example, the economic growth is essentially a non-stationary parameter, but we can remove the influence of long time tendency by using an adequate detrending procedure. Moreover, the real signals are containing both deterministic and stochastic components. The most common stochastic component is the white Gaussian noise i.e. its amplitude follows the Gaussian distribution with zero mean while the frequency spectrum is absolutely flat. According to the Wiener-Hincine theorem, the last property implies that the correlation function is a Dirac spike and any sample does not depend on any of the previous ones (the samples are statistically independent). A Gaussian coloured noise is the same, except the spectrum and, consequently, the correlation function which is showing a decay rate due to the "memory" of the successive samples.

In order to deal with chaos, one has to remove the whole stochastic part, as well as the linear dependence from the deterministic component. The difficulty arises especially when trying to separate the deterministic nonlinear component from the "with-memory" stochastic one. One method is to use the Auto Regressive Conditional Heteroskedastic[10] (ARCH) methods followed by the Brock-Deckert-Scheinkman[11] (BDS) test on the residuals of the series. The first is assumed to remove the one-step conditional probabilistic dependence, while the second is a robust test against the null hypothesis of independent, identically distributed noise. If the null hypothesis is rejected, then there are arguments for deterministic dynamics.

For the sake of didactic purposes, we can state that the constant and the linear term of a signal expansion are indicating the long run evolution, while the nonlinear terms characterize the short run behaviour. Since we are interested in the short run nonlinear features, the linear dependence is also not desirable, so we drop it out. Finally, we get a series of samples that is a quite fairly expression of the underlying nonlinear dynamics.

If the insulating methods of the nonlinear determinism are not very convenient, a more direct method is to use the so-called "surrogate data". The basic assumption is of the existence of nonlinear determinism in the genuine series, and a posteriori to test this assumption on several sets of randomized samples. This method works only if one obtains significant differences from the original set.

Now, the idea is not to analyze the given dynamic system, which remains mostly unknown, but an image-system with the same topology that preserves the main characteristics of the genuine one. As it is stated in literature[12], there are many attempts to simulate a minimal model, almost all of them being based on the Ruelle-Takens' embedding theorem[13].

The simulation has to follow two steps: the reconstruction of the phase space where the image-system is evolving, and the evaluation of the largest Lyapunov exponent. In turn, the reconstruction involves at least two aspects: the proper choice of the dimension for the reconstructed phase space, i.e. the embedding dimension, and the evaluation of the correlation dimension referring to the degree of complexity expressed by the minimum number of variables that is needed to replicate the dynamic system. The last one is the same with the dimension of the (strange) attractor characterizing which one of the topological-equivalent systems[14].

1.2 Takens' theorem

So far we focused on the applicability of the theoretical model. Suppose now that the dynamics of the system is indeed deterministic and nonlinear. A simple form of its temporal evolution might be written as

$$\boldsymbol{x}_{i+1}=T(\boldsymbol{x}_i), \tag{1}$$

where T is a deterministic rule, and $\boldsymbol{x}_i$ is an n-dimensional state vector $\boldsymbol{x}_i \in S$, S being the phase space $S \subseteq \mathrm{R}^n$. Given the initial point $\boldsymbol{x}_0$ and a sampling time (daily, in our case), we get an orbit $\boldsymbol{X}$ as the sequence

$$\boldsymbol{X}=(\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, ...) = (\boldsymbol{x}_0, T(\boldsymbol{x}_0), T(T(\boldsymbol{x}_0)), ...). \tag{2}$$

More clearly, for N points we have

$$\boldsymbol{X} = \left(\begin{bmatrix} x_{0\,0} \\ x_{0\,1} \\ . \\ . \\ x_{0\,n-1} \end{bmatrix}, \begin{bmatrix} x_{1\,0} \\ x_{1\,1} \\ . \\ . \\ x_{1\,n-1} \end{bmatrix}, ..., \begin{bmatrix} x_{N\,0} \\ x_{N\,1} \\ . \\ . \\ x_{N\,n-1} \end{bmatrix} \right). \tag{2'}$$

In real life, we cannot know the state $\boldsymbol{x}_k \in S \subseteq \mathrm{R}^n$ of the system, but the system states completely determine a measured sequence via a read-out function $f{:}S \subseteq \mathrm{R}^n \to \mathrm{R}$, so that for each orbit $\boldsymbol{X}$ there is a corresponding time series (here the exchange rate):

$$Y = (y_0, y_1, ...) = (f(\boldsymbol{x}_0), f(\boldsymbol{x}_1), ...). \tag{3}$$

Since the single measurement cannot describe the entire internal state of the complex system, the problem is: can we reconstruct it starting from incomplete, truncate information? Under certain assumptions, the answer is partially affirmative, and this is the famous Takens-Ruelle theorem. We briefly sketch here the method of the space reconstruction and the calculation of the largest Lyapunov exponent.

1.3 Phase space reconstruction

Provided that the time development admits an attractor, and according to Takens, it is possible to reconstruct the dynamics for the system in Eq. (1) using only the scalar time series from Eq. (3). Specifically, the data points in an observed scalar time series contain information about unobserved state variables that can be used to define a state at the present time. Therefore, let us consider m-tuples of real numbers $(y_i, y_{i+r}, \ldots, y_{i+(m-1)r})$ and denote the m-points projection of the real orbit via the real function f

$$Y_{m,i} = (f(\boldsymbol{x}_{i-1}), f(T(\boldsymbol{x}_{i-1})), \ldots, f(T^{m-1}(\boldsymbol{x}_{i-1}))), \tag{4}$$

where the i is the starting point, and r the time lag. For the sake of simplicity, hereafter r=1. Now it's the basic statement of the theorem: the scalars in Eq. (4) are no more considered as the projections of m different state vectors $\boldsymbol{x}_{i-1}, \ldots \boldsymbol{x}_{i+m-1}$, belonging to the real trajectory $\boldsymbol{X}$, but the co-ordinations of a single point in a m-dimensional "embedding space":

$$\boldsymbol{x}_i^{\mathrm{REC}} = \begin{bmatrix} y_i \\ y_{i+1} \\ . \\ . \\ y_{i+m-1} \end{bmatrix}. \tag{5}$$

If N is the number of points of the genuine temporal series, then there will be M points of the "reconstructed" trajectory $\boldsymbol{X}^{\mathrm{REC}}$ in the embedding space

$$\boldsymbol{X}^{\mathrm{REC}} = \left(\boldsymbol{x}_0^{\mathrm{REC}}, \boldsymbol{x}_1^{\mathrm{REC}}, \ldots, \boldsymbol{x}_{M\text{-}1}^{\mathrm{REC}}\right) = \left(\begin{bmatrix} y_0 \\ y_1 \\ . \\ . \\ y_{m-1} \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ y_m \end{bmatrix}, \ldots, \begin{bmatrix} y_{N-m+1} \\ y_{N-m+2} \\ . \\ . \\ y_N \end{bmatrix}\right), \tag{6}$$

where the constants N, m and M are related as

$$M = N - m + 1. \tag{6'}$$

In some sense, this conversion is similar to Fourier transform with constant kern. The reconstruction is effective provided that the condition

$$m > 2n \tag{7}$$

is fulfilled. One can observe that, again, this condition is similar to the well known Shannon's sampling condition.

It is important that the embedding dimension and the reconstruction delay are correctly chosen so that the original system and its reconstruction are qualitatively equivalent[15,16]. Takens proved that there is a map that performs a one-to-one coordinate transformation between the original n-dimensional state $\boldsymbol{x}_t$ and the m-dimensional reconstructed state $\boldsymbol{x}_{\mathrm{i}}^{\mathrm{REC}}$ (Eq. (5)). This map preserves topological information about the unknown dynamic system under the mapping, e.g. the Lyapunov exponents. In particular, the map induces a functional $\hat{T}:S^{\mathrm{REC}}\subseteq \mathrm{R}^m \rightarrow \mathrm{R}^m$ on the reconstructed trajectory

$$\boldsymbol{x}_{\mathrm{i+1}}^{\mathrm{REC}} = \hat{T}\left(\boldsymbol{x}_{\mathrm{i}}^{\mathrm{REC}}\right). \tag{8}$$

Thus, $\hat{T}$ in Eq. (8) is a reconstructed dynamic system, e.g., a ''reconstructed'' economy, which has the same Lyapunov exponents as the unknown real system.

1.4 Estimation of the correlation dimension

As regards the complexity of a chaotic (irregular) attractor, the notion of fractal dimension is often used as a measure of the degree of its complexity. In the present context, the main motivation for estimating the dimension of a reconstructed attractor is the perception that fractal dimension is a geometrical and dynamical characteristic, that remains unaltered by the process of phase space reconstruction. Since the fractal dimension of a reconstructed attractor cannot easily be computed in practice, the notion of correlation dimension[17] is often used as an alternative measure. For a vectorial, discrete quantity the correlation dimension is

$$C(N,m,\varepsilon) = \frac{1}{(N-m+1)(N-m)} \sum_{k=1}^{N-m+1} \sum_{j=1}^{N-m+1} \eta\left(\varepsilon - \left\| \boldsymbol{x}_k^{\mathrm{REC}} - \boldsymbol{x}_j^{\mathrm{REC}} \right\|\right), \tag{9}$$

where η is the step unity function (the Heaviside function), i.e. it is 1 for positive arguments and 0 otherwise.

Due to the property of deterministic systems that following states are uniquely determined by previous ones, one approach to estimate the correct correlation dimension is to check the size of the distances between the images of close points on the reconstructed trajectory as m increases from zero to N. It's worth noting that all distances decrease more slowly with a further increase of the embedding dimension, provided that the embedding dimension is equal to or larger than the correct one. As a matter of fact, if there is a plateau in the (m, D_{corr}) diagram, then the slope of the linear part of the plot ($\ln\varepsilon$, $\ln C(N,m,\varepsilon)$), for small ε, is the correlation dimension D_{corr}. Thus, the saturation is a strong indicator of the determinism.

1.5 Estimation of the largest Lyapunov exponent

Supposing the reconstruction of the dynamics given by Eq. (8), we have to search for the nearest neighbour of each state on the trajectory by minimizing the distance to the particular reference state[16]

$$\delta_k = \min_{i \neq k} \left\| \boldsymbol{x}_k^{\mathrm{REC}} - \boldsymbol{x}_i^{\mathrm{REC}} \right\|, \text{ for } k=0, 1,\ldots, M. \tag{10}$$

where δ_k, $\boldsymbol{x}_k^{\mathrm{REC}}$ and $\boldsymbol{x}_i^{\mathrm{REC}}$ are the minimum distance, the k-th state (reference) and its current neighbour, respectively. One can consider each pair of neighbours as initial conditions for virtual trajectories only if the temporal separation between them should be greater than the mean period of the time series (which can be defined as the reciprocal of the mean frequency found in the power spectrum). Taking into account the condition, a new indexed collection of ordered pairs is obtained $\left(\left(\boldsymbol{x}_0^{\mathrm{REC}}, \boldsymbol{x}_{i(0)}^{\mathrm{REC}}\right), \left(\boldsymbol{x}_1^{\mathrm{REC}}, \boldsymbol{x}_{i(1)}^{\mathrm{REC}}\right), \ldots, \left(\boldsymbol{x}_q^{\mathrm{REC}}, \boldsymbol{x}_{i(q)}^{\mathrm{REC}}\right)\right)$ that are corresponding one to one to the collection of the minimum distances ($\delta_0(0)$, $\delta_1(0)$,…, $\delta_q(0)$) , where q is of the order of the halved number of points of the reconstructed orbit $q{\sim}M/2$. The j-th pair of nearest neighbours then diverges at a rate approximated by the largest Lyapunov exponent $\lambda_{\mathrm{MAX}(k)}$:

$$\delta_k(j) \cong \delta_k(0) \cdot \mathrm{e}^{\lambda_{\mathrm{MAX}(k)} \cdot j}, \text{ for } k=0,\ldots q, \tag{11}$$

where j is the number of separation steps. Taking the logarithm on both sides of Eq. (10) gives

$$\ln \delta_k(j) \cong \ln \delta_k(0) + \lambda_{\mathrm{MAX}(k)} \cdot j, \text{ for } k=0,\ldots q, \tag{11'}$$

which represents a family of q approximately parallel lines with a slope approximately proportional to $\lambda_{\mathrm{MAX}(k)}$. The Lyapunov exponents are then estimated using a least-square fit with a constant to the average line defined by plotting $\left(j, \ln \delta_k\right)$ for every k. As stated before, recall that the solution paths to the unknown dynamic system remain within a bounded set, so the solution paths to the reconstructed dynamic system also remain within a bounded set. Consequently, the approximations given in Eqs. (11) and (11') are more reliable for a limited number of separation steps. To be more specific, the proper number of separation steps is achieved when the plot (11') reaches a quasi-constant value. Finally, the largest exponent will be found by averaging the q values of the exponents:

$$\lambda_{\mathrm{MAX}} = \left\langle \lambda_{\mathrm{MAX}} \right\rangle_k . \tag{12}$$

1.6 Coloured chaos

The spectral analysis is an additional task that might reveal interesting properties of the samples. Besides the already mentioned Gaussian white noise, there are several more well known types of pink noise, like $1/f$ and $1/f^2$, i.e. the signal power distribution versus frequency follows a $1/f^{\alpha}$ law, α=1,2. For α=0 one obtains the "white" noise. If $\alpha \notin \{0;1;2\}$, then it's called "coloured" noise. By analogy with the coloured noise, coloured chaos is the chaos that characterizes a time series with the spectrum following a $f^{-\alpha}$ power law.

2. Data analysis

Our time series of ROL-USD exchange rate covers the interval between January 1990 and 31 Oct. 2005 i.e. 4080 daily samples. We use here an averaged interbanking exchange rate[17]. The absolute values of ROL are expressed in present denominated values. Only the legal working days were considered.

2.1 Romanian environment

By simple visual inspection, one can a priori identify at least two intervals in the Romanian financial and economic environment (see Fig.1): the first, between January 1990 and December 2001, the structural changes period, is characterized by the elaboration and implementation of a new infrastructure (laws, regulations, institutions, etc.) that match the requirements of a functional market economy, and the second, between January 2002 and 31 Oct. 2005 (the final day of our study), is the beginning of the stationary regime, when the Central Bank reached enough power to influence the monetary system by open market operations.

The first period shows a quasi-parabolic shaped dependence with several angle points and jumps. The very small increase in 1990 and even in 1991 is because of the inertia of the total deterministic evolution of the centralized economy that has been legally operated until the end of 1989; however, the effect prolonged, as one can see in Fig.1, for almost two more years. As the economy diminished to work as a national holding and the structural changes begin to manifest their effects, as well as the emergence of the competing economic agents, the system is searching for new macro-economic equilibriums and, consistent with the irreversibility and the increasing entropy law[18], the exchange rate is moving up very fast.

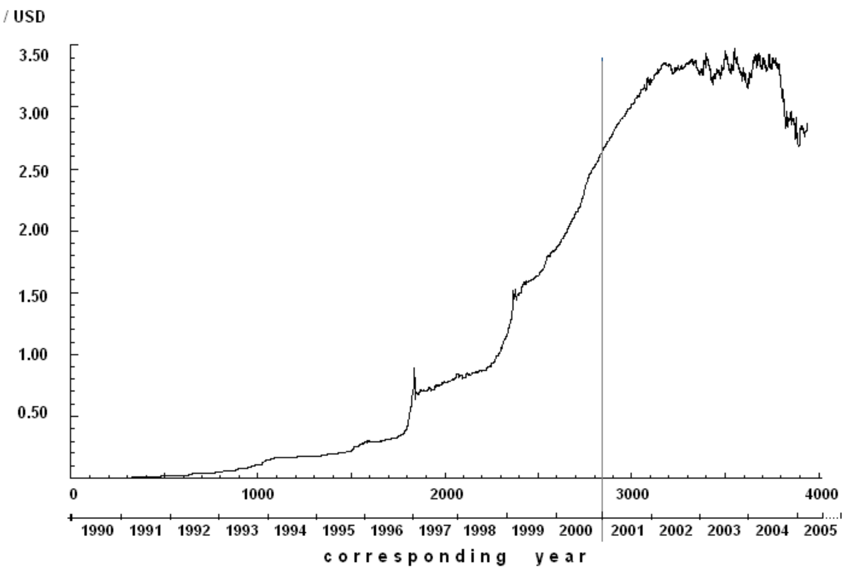


Fig.1 ROL-USD exchange rates (absolute, denominated values); the vertical line indicates the border between transition and steady-state intervals

The second period might be considered as the beginning of the quasi-stationary regime; it is apparent a major down step when, in March 2005, the Central Bank announced an almost nil interventionist policy concerning the exchange rate. Despite several future jumps and oscillations of little amplitude, and neglecting the exogenous sources of economic crashes and shocks, it’s very likely to observe a quite steady future monetary regime.

2.2 Data detrending

As the issue of choosing an appropriate time-sampling rate is often out of our choice (because it depends on the availability of the numerical data), the selection of a reference trend or a proper transformation to simplify the empirical pattern of the measured time series remains an open problem. Consequently, finding a proper transformation is called the problem of trend-cycle decomposition, or, simply, detrending. A distinctive problem in economic analysis is how to deal with growing trends in an aggregate economic time series. Unlike laboratory experiments in natural sciences, there is no way to maintain steady flows in economic growth and describe business cycles by invariant attractors. Many controversial issues in macroeconomic studies, such as noise versus chaos in business cycles, are closely related to competing detrending methods[19,20].

It is the theoretical perspective that dictates the choice of a detrending approach. The econometric practice of pre-whitening data is justified by equilibrium theory, and is convenient for regression analysis. For pattern recognition, a typical technique in science and engineering is to project the data onto some well-constructed deterministic space to recover possible patterns from empirical time series. Notable examples are the Fourier analysis and wavelets. The essence of trend-cycle decomposition is finding an appropriate time window, or equivalently, a proper frequency window, for observing time-dependent movements.

There are several approaches in econometric analysis[21]. In principle, a choice of observation reference is associated with a theory of economic dynamics and consequently with a certain detrending method. Often the detrending procedure also solves the problem of linear filtering and is apparent to fulfill the stationary hypothesis.

In our case it's obvious a power-law trend for the transition period (see Fig.1), so the trend is fitted with a thirteen-degree polynomial expression; the resulting samples are given by

$$y_{\mathrm{POL}}(t) = y(t) - \sum_{i=0}^{13} a_i t^i \tag{13}$$

In Fig.2 the detrended evolution of the exchange rate between Jan. 1990 and Oct. 2005 is represented.

We assume that the series $y_{\mathrm{POL}}(t)$ fulfills the conditions a/ and b/ from the chapter 1.1. Thus, we can perform the analysis in order to characterize the underlying system from the point of view of the existence of the deterministic chaos.

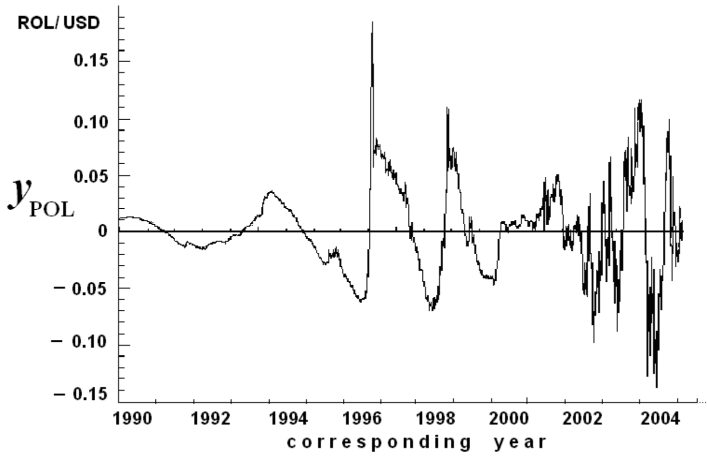


Fig.2 ROL-USD exchange rate: conversion of the absolute values after polynomial detrending

3. Results

The results are synthesized in the Table 1. We assume the validity of Takens' theorem for the detrended data, such that the reconstruction method described before is applicable.

Table 1

Interval	Lyapunov exponent	Correlation dimension	Embedded dimension	Time lag
Structural changes (Jan.1990-Dec.2001)	0.057±0.018	1.93	5	1
Steady state (Jan.2002-Oct. 2005)	0.118±0.022	4.45	9	1

A smaller Lyapunov exponent is consistent with a higher degree of short run predictability in an economic system; for the first period, the exchange rate is predictable also from the long run perspective (positive trend). For the second interval, the predictability is significantly lower and, again, is consistent with the long run behaviour, which in fact has no trend and the curve approaches a random walk appearance (Fig.1). In a steady state running economy, as mentioned by other authors[22,23,24], a positive real part of the largest Lyapunov coefficient seems to be normal.

In order to estimate the dimension of the system producing the exchange rate series, we also calculate the correlation dimension. It saturates for both intervals, so it is well defined (see Fig.3).

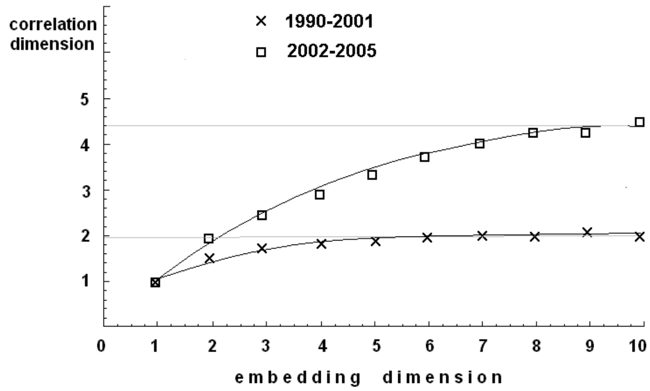
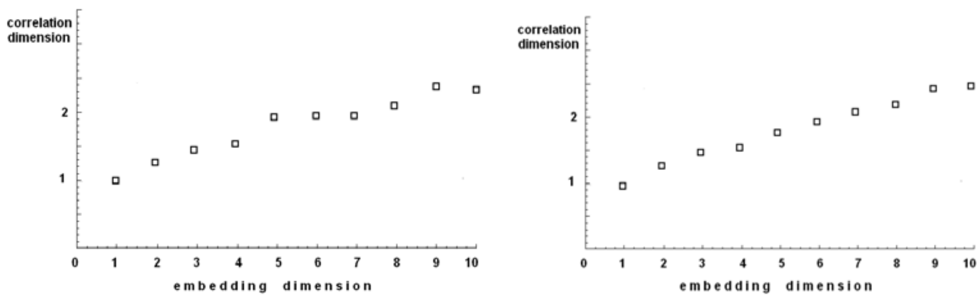


Fig.3 Correlation dimension for ROL/USD exchange rates: for the structural changes interval 1990-2001 and for the beginning of the steady state interval 2002-2005

For the last interval the saturation is not so obvious as for the first one; this is due to the insufficient length of the time interval or, equivalent, to the insufficient number of samples. An analogous behaviour is shown in Fig.4 for the first interval, where we were simulating the lack of data. More exactly, we represented the analysis for the fractions 1990-1993 and 1990-1996.

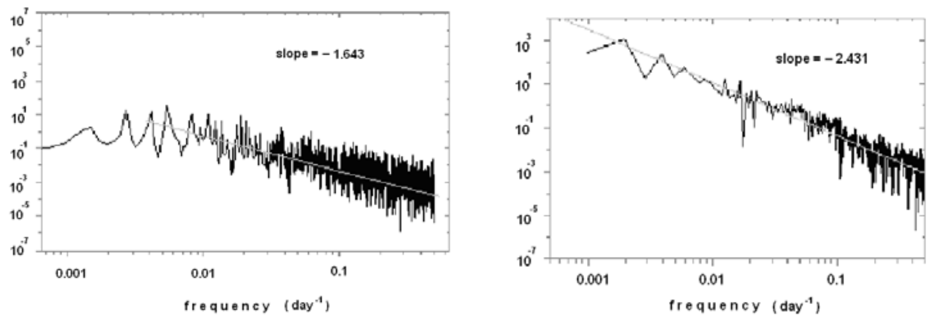


a. b.

Fig.4 Insufficient number of samples in the series might provoke the lack of clear saturation: correlation dimension for ROL/USD exchange rates 1990-1993 (a) and 1990-1996 (b)

The greater the correlation dimension, the more complex the economic system and the monetary policy of the Central Bank. This means that, in the first period, the largest part of the resources are spent for qualitative changes, the economic aggregates diminishing their outputs[25].

On the other side, as shown in Fig.5, the spectral analysis shows different slopes of the linear regression in the power spectra for the investigated periods. Both of them have absolute values greater than unity, indicating the existence of the so called "coloured" chaos[26].



a. **b.**

Fig.5 Power spectrum (arbitrary units) for ROL-USD exchange rate: structural changes (a), steady state (b)

Conclusions

From the historical point of view, the analysts are considering that the structural transition from the command economy to the free market economy came to an end with the year 2000. Our study partially confirms that by revealing quite perceivable distinctions along the time evolution of the ROL-USD exchange rate for the intervals 1990-2001 and 2002-2005. This allows us to perform the analysis separately for each of the intervals. However, we considered here that a twelve years period is the best approximation for the transition time length.

The novel approach in the present work is the analysis of a non-stationary evolution, and it seems to furnish reliable results. Positive Lyapunov exponents were found for both intervals. The smaller exponent characterizing the structural changes period indicates smaller sensitivity to the initial condition, but a non-randomly evolution toward a more complex system characterizing the steady state regime. The power spectra reveal coloured chaotic behaviour for the whole period.

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