STABILITY IN MEASURE FOR UNCERTAIN FRACTIONAL DIFFERENTIAL EQUATIONS WITH JUMPS

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Uncertain fractional differential equation with jump (UFDEJ) is an important mathematical model. Establishing the judgment of the stability is a basic problem of UFDEJs. This paper mainly investigates the stability in measure for the Caputo type of UFDEJs that the order is $0 < p \leq 1$. We first propose the concept of stability in measure of solutions to UFDEJs. Then, two sufficient conditions for stability in measure are obtained in different cases with order $0 < p \leq \frac{1}{2}$ and $\frac{1}{2} < p \leq 1$. In the end, some examples are given to illustrate the results.

Keywords: Liu process; V-jump process; Caputo fractional derivative; stability in measure

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1. Introduction

Uncertainty theory [1, 2] as one of the tools to deal with belief degrees were widely used in control [3] and game [4], prediction [5], variational inequality problems [6, 7] and other fields. Moreover, in the field of uncertain system, there have been many models [8, 9, 10] on uncertain differential equations (UDEs), and most scholars devoted his energies to UDEs and their improved forms driven by Liu process. For uncertain system, stability is the prerequisite to ensure the normal operation of the control system. Stability theory is one of the research hotspots in the application of UDEs, which has been extended into many widely used versions. The standard stability in measure of UDEs developed by Yao et al. [11] is the first one, which is suitable to nonlinear and linear uncertain system but is limited to a necessary condition for an UDE being stable. Recent years, researches on stability in measure for uncertain system are emerging one after another. For example, stability in measure for multifactor UDE [12], UDE with jumps [13], multi-dimensional UDE [14], uncertain heat equations [15], uncertain delay equation with jump[16] and uncertain fractional differential equations [17] were successively investigated.

V-jumps uncertain processes [18] were always used to characterize uncertain evolutionary phenomenon with jumps, whose uncertainty distribution function exists jump point, which could describe a sudden changes caused by emergencies, such as economics crisis, outbreaks of infectious diseases, earthquake, war, etc. Deng et al. [19] further studied uncertain differential equation with jump (UDEJ) under certain conditions. The UDEJ is expressed as follows

$$dZ_k = F(Z_k, k)dk + G(Z_k, k)dC_k + H(Z_k, k)dV_k,$$

where C_k is a Liu process, V_k is a V-jump process, k denotes time and F, G and H are some given functions.

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UDEJs play an important role in various fields, for instance optimal control and stability analysis. Some related references can be found in [16, 20, 21]. Among them, Ref [16] discussed stability in measure for UDEJ with delay. In addition, for the phenomena of uncertain complex systems, uncertain fractional differential equations (UFDEs) are the usable mathematical tools characterizing uncertain complex systems. Zhu [22] proposed two types of UFDEs in the one-dimensional case. After that, Zhu [23] extended the model of UFDEs to the multidimensional case again in the same year. For describing fractional differential system containing uncertainty and jumps more accurately, Jia et al. [24] proposed uncertain fractional differential system with jumps (UFDEJs) and considered the analytical solution and existence theorem. To the best of our knowledge, there are few results on UFDEJs, especially, the stability of UFDEJs.

The stability in measure of Ref [11, 16] were studied based on the integer order, which can't be applied by fractional order system. Compared with Ref [11, 16], we consider the fractional order and jump case. Compared with Ref [17], we consider the uncertain jump systems. In this paper, we consider two kinds of new sufficient conditions of the fractional order and jump case for UFDEJs driven by Liu processs and V-jumps processs, then two theorems of stability in measure for UFDEJs are verified. The contributions of this paper are: (1) to derive the first sufficient condition when $\frac{1}{2} < p \leq 1$; (2) to obtain the second sufficient condition when $0 < p \leq \frac{1}{2}$.

The structure of this paper is organized as follows. Section 2 recall two lemmas. Section 3 present the first sufficient condition and the second sufficient condition of stability in measure based on different condition for UFDEJs. In Section 4, we give some examples. In section 5, we give a conclusion.

2. Preliminaries

Lemma 2.1. [3] *Suppose that C_k is an l-dimensional Liu process, and Z_k is an integrable $n \times l$-dimensional uncertain process on $[u, v]$ with respect to time k. Then, the inequality*

$$\left\| \int_u^v Z_k(\gamma) dC_k(\gamma) \right\|_\infty \leq K_\gamma \int_u^v \|Z_k(\gamma)\|_\infty dk$$

holds, where $K(\gamma)$ is the Lipschitz constant of the sample path $C_k(\gamma)$ with the norm $\|\cdot\|_\infty$.

Lemma 2.2. [24] *Suppose that V_k is a l-dimensional V-jump process, and Z_k is an integrable $n \times l$-dimensional uncertain process on $[u, v]$ with respect to time k. Then, for any sample γ, the inequality*

$$\left\| \int_u^v Z_k(\gamma) dV_k(\gamma) \right\|_\infty \leq \int_u^v \|Z_k(\gamma)\|_\infty dk$$

holds,

3. Stability of UFDEJs

In this section, we will consider the Caputo type of UFDEJ, the form of which is as follows

$${}^cD^p Z_k = F(k, Z_k) + G(k, Z_k) \frac{\mathrm{d}C_k}{\mathrm{d}k} + H(k, Z_k) \frac{\mathrm{d}V_k}{\mathrm{d}k}, k > 0 \tag{1}$$

with initial value $Z_k|_{k=0} = Z_0 \in \Re^n$, where ${}^cD^p$ denotes the Caputo fractional derivative of order $0 < p \leq 1$ with the form

$${}^cD^p \delta(k) = \frac{1}{\Gamma(1-p)} \int_0^k (k-\mu)^{-p} \delta'(\mu) \mathrm{d}\mu$$

provided that $\delta(k)$ is differentiable on $k \in \Re^+$, and $F : [0, +\infty) \times \Re^n \to \Re^n$, $G : [0, +\infty) \times \Re^n \to \Re^{n\times l}$, $H : [0, +\infty) \times \Re^n \to \Re^{n\times l}$. $C_k = [C_{1k}, C_{2k}, \cdots, C_{lk}]^T, C_{ik}(i = 1, 2, \cdots, l)$ and $V_k = [V_{1k}, V_{2k}, \cdots, V_{lk}]^T, V_{ik}(i = 1, 2, \cdots, l)$ are Liu processes and V-jump processes,respectively.

In this paper, we denote

$$\|Z\| = \max_{1\le i\le n} |Z_i|, \|B\| = \max_{1\le i\le m} \left(\sum_{j=1}^{n} |b_{ij}|\right) \tag{2}$$

for an n-dimensional vector $Z = (Z_1, Z_2, \cdots, Z_n)^T$ and $m \times n$-matrix $B = [b_{ij}]_{m\times n}$, respectively. The UFDEJ (1) is equivalent with the integral equation

$$\begin{aligned} Z_k =& Z_0 + \frac{1}{\Gamma(p)} \int_0^k (k-\mu)^{p-1} F(\mu, Z_\mu)\, \mathrm{d}\mu + \frac{1}{\Gamma(p)} \int_0^k (k-\mu)^{p-1} G(\mu, Z_\mu)\, \mathrm{d}C_\mu \\ &+ \frac{1}{\Gamma(p)} \int_0^k (k-\mu)^{p-1} H(\mu, Z_\mu)\, \mathrm{d}V_\mu. \end{aligned} \tag{3}$$

Jia et al. [24] proved the existence and uniqueness theorems of UFDEJ (1). Similar to Refs [11, 12, 13, 14, 15, 16], we first give the concept of the stability in measure for UFDEJs.

Definition 3.1. *An UFDEJ* (1) *is said to be stable in measure, if for any solutions Z_k and U_k with initial values Z_0 and U_0, respectively, we have*

$$\lim_{\|Z_0-U_0\|\to 0} \mathcal{M}\left\{\sup_{k\ge 0} \|Z_k - U_k\| \le \varepsilon\right\} = 1, \tag{4}$$

for any given number $\epsilon > 0$.

Example 3.1. *Assume that $a, b, c \in \Re$ are constants and $a < 0$, Z_k and U_k are the solutions to the linear UFDEJ,*

$$^cD^p Z_k = aZ_k + b\frac{dC_k}{\mathrm{d}k} + c\frac{dV_k}{\mathrm{d}k} \tag{5}$$

with initial values Z_0 and U_0, respectively. By Ref [24]*, it holds that*

$$\begin{aligned} Z_k(\gamma) =& Z_0 E_{p,1}(ak^p) + \int_0^k (k-\mu)^{p-1} bE_{p,p}(a(k-\mu)^p)\, \mathrm{d}C_\mu(\gamma) \\ &+ \int_0^k (k-\mu)^{p-1} cE_{p,p}(a(k-\mu)^p)\, \mathrm{d}V_\mu(\gamma) \\ U_k(\gamma) =& U_0 E_{p,1}(ak^p) + \int_0^k (k-\mu)^{p-1} bE_{p,p}(a(k-\mu)^p)\, dC_\mu(\gamma) \\ &+ \int_0^k (k-\mu)^{p-1} cE_{p,p}(a(k-\mu)^p)\, dV_\mu(\gamma), \end{aligned}$$

where $E_{i,j}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(mi+j)}$ is the Mittag-Leffler function. By Ref [25]*, we know that $E_{i,j}(z) \le \frac{Q}{1+|z|}$ for $i \in (0, 2)$, the constant $Q > 0$, argument $arg(z) = \pi$ and $|z| \ge 0$. Then $\forall \gamma \in \Gamma$, $k \ge 0$, it holds that*

$$|Z_k(\gamma) - U_k(\gamma)| = |Z_0 - U_0|\, E_{p,1}(ak^p) \le |Z_0 - U_0| \frac{Q}{1 + |a|k^p}.$$

Then,

$$\sup_{k\ge 0} |Z_k(\gamma) - U_k(\gamma)| \le M\, |Z_0 - U_0| \to 0$$

as $|Z_0 - U_0| \to 0$, *that means* $\lim_{\|Z_0-U_0\|\to 0} \mathcal{M}\left\{\sup_{k\geq 0} \|Z_k - U_k\| \leq \varepsilon\right\} = 1$ *holds for any* $\epsilon > 0$. *By Definition 3.1, the linear UFDEJ* (5) *is stable in measure.*

For the need of the stability criteria of UFDEJ (1), we talk two cases about fractional order p, that is, $\frac{1}{2} < p \leq 1$ and $0 < p \leq \frac{1}{2}$, meanwhile, we need the following assumption.

Condition 1 $\forall z, u \in \Re^n$ and $k \in [0, T)$, the coefficients $F(k,z)$, $G(k,z)$ and $H(k,z)$ satisfy the following Lipschitz condition

$$\begin{aligned}&\|F(k,z) - F(k,u)\| \leq N_{1k}\|z-u\|, \quad \|G(k,z) - G(k,u)\| \leq N_{2k}||z-u||,\\&\|H(k,z) - H(k,u)\| \leq N_{3k}||z-u||,\end{aligned} \tag{6}$$

where $T \leq +\infty$, N_{1k}, N_{2k} and N_{3k} are positive functions.

3.1. Case 1: $\frac{1}{2} < p \leq 1$

Theorem 3.1. *We assume that condition 1 holds and* N_{1k}, N_{2k} *and* N_{3k} *satisfy*

$$\int_0^{+\infty} e^{2\mu}(N_{1\mu} + N_{3\mu})^2 \,\mathrm{d}\mu < \infty, \int_0^{+\infty} \mathrm{e}^{2\mu} N_{2\mu}^2 \,\mathrm{d}\mu < \infty, \tag{7}$$

then the UFDEJ (1) *is stable in measure.*

Proof. Assume that Z_k and U_k are two solutions of UFDEJ (1) with initial values Z_0 and U_0, respectively. Then we have

$$\begin{aligned}Z_k(\gamma) =& Z_0 + \frac{1}{\Gamma(p)}\int_0^k (k-\mu)^{p-1} F\left(\mu, Z_\mu(\gamma)\right) \mathrm{d}\mu + \frac{1}{\Gamma(p)}\int_0^k (k-\mu)^{p-1} G\left(\mu, Z_\mu(\gamma)\right) \mathrm{d}C_\mu(\gamma)\\&+ \frac{1}{\Gamma(p)}\int_0^k (k-\mu)^{p-1} H\left(\mu, Z_\mu(\gamma)\right) \mathrm{d}V_\mu(\gamma),\end{aligned} \tag{8}$$

$$\begin{aligned}U_k(\gamma) =& U_0 + \frac{1}{\Gamma(p)}\int_0^k (k-\mu)^{p-1} F\left(\mu, U_\mu(\gamma)\right) \mathrm{d}\mu + \frac{1}{\Gamma(p)}\int_0^k (k-\mu)^{p-1} G\left(\mu, U_\mu(\gamma)\right) \mathrm{d}C_\mu(\gamma)\\&+ \frac{1}{\Gamma(p)}\int_0^k (k-\mu)^{p-1} H\left(\mu, U_\mu(\gamma)\right) \mathrm{d}V_\mu(\gamma).\end{aligned} \tag{9}$$

It follows from Assumption 1, Lemma 2.1, Lemma 2.2, (8) and (9) that

$$\begin{aligned}&\|Z_k(\gamma) - U_k(\gamma)\|\\&\leq \|Z_0 - U_0\| + \frac{1}{\Gamma(p)}\int_0^k (k-\mu)^{p-1}\|F\left(\mu, Z_\mu(\gamma)\right) - F\left(\mu, U_\mu(\gamma)\right)\|\mathrm{d}\mu\\&\quad+ \frac{1}{\Gamma(p)}\int_0^k (k-\mu)^{p-1}\left\|G\left(\mu, Z_\mu(\gamma)\right) - G\left(\mu, U_\mu(\gamma)\right)\right\| \|dC_\mu(\gamma)\|\\&\quad+ \frac{1}{\Gamma(p)}\int_0^k (k-\mu)^{p-1}\left\|H\left(\mu, Z_\mu(\gamma)\right) - H\left(\mu, U_\mu(\gamma)\right)\right\| \|dV_\mu(\gamma)\|\\&\leq \|Z_0 - U_0\| + \frac{1}{\Gamma(p)}\int_0^k (k-\mu)^{p-1}(N_{1\mu} + N_{3\mu})\left\|Z_\mu(\gamma) - U_\mu(\gamma)\right\| d\mu\\&\quad+ \frac{K_\gamma}{\Gamma(p)}\int_0^k (k-\mu)^{p-1} N_{2\mu}\left\|Z_\mu(\gamma) - U_\mu(\gamma)\right\| \mathrm{d}\mu,\end{aligned} \tag{10}$$

where K_γ is the Lipschitz constant of $C_k(\gamma)$, and by Theorem 2 of Ref [11], we know that

$$\lim_{z\to+\infty} \mathcal{M}\{\gamma \in \Gamma | K_\gamma \leq z\} = 1. \tag{11}$$

Cauchy-Schwartz inequality implies that

$$\|Z_k(\gamma) - U_k(\gamma)\|$$

$$\leq \|Z_0 - U_0\| + \frac{1}{\Gamma(p)} \int_0^k (k-\mu)^{p-1} e^{-\mu} e^{\mu} (N_{1\mu} + N_{3\mu}) \|Z_\mu(\gamma) - U_\mu(\gamma)\| \,\mathrm{d}\mu$$

$$+ \frac{K_\gamma}{\Gamma(p)} \int_0^k (k-\mu)^{p-1} e^{-\mu} e^{\mu} N_{2\mu} \|Z_\mu(\gamma) - U_\mu(\gamma)\| \, d\mu$$

$$\leq \|Z_0 - U_0\| + \frac{1}{\Gamma(p)} \left(\int_0^k (k-\mu)^{2p-2} e^{-2} \mathrm{d}\mu \right)^{\frac{1}{2}}$$

$$\left(\int_0^k e^{2\mu} (N_{1\mu} + N_{3\mu})^2 \|Z_\mu(\gamma) - U_\mu(\gamma)\|^2 \ \mathrm{d}\mu \right)^{\frac{1}{2}}$$

$$+ \frac{K_\gamma}{\Gamma(p)} \left(\int_0^k (k-\mu)^{2p-2} e^{-2\mu} \ \mathrm{d}\mu \right)^{\frac{1}{2}} \left(\int_0^k e^{2\mu} N_{2\mu}^2 \|Z_\mu(\gamma) - U_\mu(\gamma)\|^2 \mathrm{d}\mu \right)^{\frac{1}{2}}. \tag{12}$$

Obviously, for $k \in [0, 1]$, continuity makes the inequality $\int_0^k (k-\mu)^{2p-2} e^{-2} \mathrm{d}\mu \leq 2Q_1$ true, where Q_1 is a positive constant. For $k > 1$, owing to $\frac{1}{2} < p \leq 1$, it holds that

$$\begin{aligned}
\int_0^k (k-\mu)^{2p-2} e^{-2\mu} d\mu &= \int_0^k r^{2p-2} e^{-2(k-r)} dr (\text{ let } r = k-\mu) \\
&= \int_0^k r^{2p-2} e^{2r} dr \cdot e^{-2k} = e^{-2k} \left[\int_0^1 r^{2p-2} e^{2r} dr + \int_1^k r^{2p-2} e^{2r} dr \right] \\
&\leq e^{-2k} \left[\int_0^1 r^{2p-2} e^2 \ \mathrm{d}r + \int_1^k e^{2r} \ \mathrm{d}r \right] = e^{-2k} \left[\frac{e^2}{2p-1} + \frac{e^{2k} - e^2}{2} \right] \\
&\leq e^{-2k} \cdot \frac{2e^{2k}}{2p-1} = \frac{2}{2p-1}.
\end{aligned}$$

Let $Q = \max\{Q_1, \frac{1}{2p-1}\}$, it holds that

$$\int_0^k (k-\mu)^{2p-2} e^{-2\mu} \ \mathrm{d}\mu \leq 2Q, \forall k \geq 0. \tag{13}$$

Substituting (13) into (12) yields

$$\begin{aligned}
\|Z_k(\gamma) - U_k(\gamma)\| \leq \|Z_0 - U_0\| &+ \frac{1}{\Gamma(p)} (2Q)^{\frac{1}{2}} \left(\int_0^k e^{2\mu} (N_{1\mu} + N_{3\mu})^2 \|Z_\mu(\gamma) - U_\mu(\gamma)\|^2 \ \mathrm{d}\mu \right)^{\frac{1}{2}} \\
&+ \frac{K_\gamma}{\Gamma(p)} (2Q)^{\frac{1}{2}} \left(\int_0^k e^{2\mu} N_{2\mu}^2 \|Z_\mu(\gamma) - U_\mu(\gamma)\|^2 \mathrm{d}\mu \right)^{\frac{1}{2}}
\end{aligned}$$

for $k \geq 0$.

Let $n \in N, \eta > 1$ and $Z_i \in \Re^+, i = 1, 2, \cdots, n$, we have

$$\left(\sum_{i=1}^n Z_i \right)^\eta \leq n^{\eta-1} \sum_{i=1}^n Z_i^\eta. \tag{14}$$

For (14), let $n = 3, \eta = 2$, it holds that

$$\|Z_k(\gamma) - U_k(\gamma)\|^2 \leq 3 \|Z_0 - U_0\|^2 + \left(\frac{1}{\Gamma(p)} \right)^2 6Q \int_0^k e^{2\mu} (N_{1\mu} + N_{3\mu})^2 \|Z_\mu(\gamma) - U_\mu(\gamma)\|^2 \ \mathrm{d}\mu$$

$$+\left(\frac{K_\gamma}{\Gamma(p)}\right)^2 6Q\int_0^k e^{2\mu}N_{2\mu}^2\left\|Z_\mu(\gamma)-U_\mu(\gamma)\right\|^2 d\mu.$$

Gronwall inequality implies that

$$\begin{aligned}&\|Z_k(\gamma)-U_k(\gamma)\|^2\\ \leq&3\left\|Z_0-U_0\right\|^2\exp\left(2Q'\int_0^k e^{2\mu}(N_{1\mu}+N_{3\mu})^2\mathrm{d}\mu\right)\exp\left(2Q'K_\gamma^2\int_0^k \mathrm{e}^{2\mu}N_{2\mu}^2\ \mathrm{d}\mu\right)\\ \leq&3\left\|Z_0-U_0\right\|^2\exp\left(2Q'\int_0^{+\infty} e^{2\mu}(N_{1\mu}+N_{3\mu})^2\ \mathrm{d}\mu\right)\exp\left(2Q'K_\gamma^2\int_0^{+\infty} e^{2\mu}N_{2\mu}^2\ \mathrm{d}\mu\right)\end{aligned}$$

where $Q'=3Q\cdot(\frac{1}{\Gamma(p)})^2$. It follows from (11) that for any given $\epsilon>0$, there exists a positive number L_ϵ such that

$$\mathcal{M}\{\gamma\in\Gamma|K_\gamma\leq L_\epsilon\}>1-\epsilon.$$

So, $\forall\gamma\in\Gamma$, it holds that

$$\begin{aligned}&\|Z_k(\gamma)-U_k(\gamma)\|^2\\ \leq&3\left\|Z_0-U_0\right\|^2\exp\left(2Q'\int_0^{+\infty} e^{2\mu}(N_{1\mu}+N_{3\mu})^2\mathrm{d}\mu\right)\exp\left(2Q'L_\epsilon^2\int_0^{+\infty} e^{2\mu}N_{2\mu}^2\ \mathrm{d}\mu\right).\end{aligned}$$

Taking $\delta=\frac{\varepsilon}{\sqrt{3}}\exp(-Q'(\int_0^{+\infty}e^{2\mu}(N_{1\mu}+N_{3\mu})^2\mathrm{d}\mu+L_\epsilon^2\int_0^{+\infty}e^{2\mu}N_{2\mu}^2\ \mathrm{d}\mu))$, we obtain that $\sup_{k\geq0}\|Z_k(\gamma)-U_k(\gamma)\|\leq\epsilon$ holds when $\|Z_0-U_0\|\leq\delta$, which implies

$$\{\gamma\in\Gamma\mid K_\gamma\leq L_\epsilon\}\subset\left\{\gamma\in\Gamma\mid\sup_{k\geq0}\|Z_k(\gamma)-U_k(\gamma)\|\leq\varepsilon\right\}$$

provided that $\|Z_0-U_0\|\leq\delta$. That is $\lim_{\|Z_0-U_0\|\to0}\mathcal{M}\left\{\gamma\in\Gamma\mid\sup_{k\geq0}\|Z_k-U_k\|\leq\varepsilon\right\}=1$ holds for any $\varepsilon>0$. By Definition 3.1, the UFDEJ (1) is stable in measure.

3.2. Case 2: $0<p\leq\frac{1}{2}$

Theorem 3.2. *We assume that condition 1 holds and N_{1k}, N_{2k} and N_{3k} satisfy*

$$\int_0^{+\infty}e^{\omega\mu}(N_{1\mu}+N_{3\mu})^\omega\mathrm{d}\mu<\infty,\int_0^{+\infty}e^{\omega\mu}N_{2\mu}^\omega\mathrm{d}\mu<\infty,\tag{15}$$

where $\omega=1+\frac{1}{p}$, then the UFDEJ (1) is stable in measure.

Proof. It follows from (10) and the Hölder inequality that

$$\begin{aligned}&\|Z_k(\gamma)-U_k(\gamma)\|\\ \leq&\|Z_0-U_0\|+\frac{1}{\Gamma(p)}\int_0^k(k-\mu)^{p-1}e^{-\mu}e^{\mu}(N_{1\mu}+N_{3\mu})\left\|Z_\mu(\gamma)-U_\mu(\gamma)\right\|\mathrm{d}\mu\\ &+\frac{K_\gamma}{\Gamma(p)}\int_0^k(k-\mu)^{p-1}e^{-\mu}e^{\mu}N_{2\mu}\left\|Z_\mu(\gamma)-U_\mu(\gamma)\right\|d\mu\leq\|Z_0-U_0\|\\ &+\frac{1}{\Gamma(p)}\left(\int_0^k(k-\mu)^{\varpi p-\varpi}e^{-\varpi\mu}\mathrm{d}\mu\right)^{\frac{1}{\varpi}}\left(\int_0^k e^{\omega\mu}(N_{1\mu}+N_{3\mu})^\omega\left\|Z_\mu(\gamma)-U_\mu(\gamma)\right\|^\omega\ \mathrm{d}\mu\right)^{\frac{1}{\omega}}\\ &+\frac{K_\gamma}{\Gamma(p)}\left(\int_0^k(k-\mu)^{\varpi p-\varpi}e^{-\varpi\mu}\ \mathrm{d}\mu\right)^{\frac{1}{\varpi}}\left(\int_0^k e^{\omega\mu}N_{2\mu}^\omega\left\|Z_\mu(\gamma)-U_\mu(\gamma)\right\|^\omega\mathrm{d}\mu\right)^{\frac{1}{\omega}}\end{aligned}\tag{16}$$

where $\varpi = 1 + p$, $\omega = 1 + \frac{1}{p}$ and $\frac{1}{\varpi} + \frac{1}{\omega} = 1$. Because of that $0 < p \le \frac{1}{2}$ and $\varpi(p-1) = p^2 - 1 \in (-1, -3/4]$, $\int_0^k (k-\mu)^{\varpi p - \varpi} e^{-\varpi\mu}\, \mathrm{d}\mu$ is continuous for $k \in [0,1]$, and then there exists a number $Q_2 > 0$ such that $\int_0^k (k-\mu)^{\varpi p-\varpi} e^{-\varpi\mu}\, \mathrm{d}\mu \le Q_2$. For $k \ge 1$, it holds that

$$\begin{aligned}\int_0^k (k-\mu)^{\varpi p-\varpi} e^{-\varpi\mu} d\mu &= \int_0^k r^{\varpi p-\varpi} e^{-\varpi(k-r)} du (\text{let}\, r = k-\mu) = e^{-\varpi k}\int_0^k \mu^{\varpi p-\varpi} e^{\varpi\mu} d\mu \\ &= e^{-\varpi k}\left[\int_0^1 r^{\varpi p-\varpi} e^{\varpi r} d\mu + \int_1^k r^{\varpi p-\varpi} e^{\varpi\mu} d\mu\right] \\ &\le e^{-\varpi k}\left[\int_0^1 r^{\varpi p-\varpi} e^{\varpi} dr + \int_1^k e^{\varpi r} dr\right] \\ &= e^{-\varpi k}\left[\frac{e^{\varpi}}{\varpi p - \varpi + 1} + \frac{1}{\varpi} e^{\varpi k} - \frac{1}{\varpi} e^{\varpi}\right] \le \frac{2}{p^2}.\end{aligned}$$

Taking $Q_2' = \max\{Q_2, \frac{2}{p^2}\}$ yields

$$\int_0^k (k-\mu)^{\varpi p-\varpi} e^{-\varpi\mu}\, \mathrm{d}\mu \le Q_2', \forall k \ge 0. \tag{17}$$

Substituting (17) into (16), we obtain

$$\begin{aligned}\|Z_k(\gamma) - U_k(\gamma)\| \le \|Z_0 - U_0\| &+ \frac{1}{\Gamma(p)} Q_2'^{\frac{1}{\varpi}} \left(\int_0^k e^{\omega\mu} (N_{1\mu} + N_{3\mu})^{\omega} \|Z_\mu(\gamma) - U_\mu(\gamma)\|^{\omega}\, \mathrm{d}\mu\right)^{\frac{1}{\omega}} \\ &+ \frac{K_\gamma}{\Gamma(p)} Q_2'^{\frac{1}{\varpi}} \left(\int_0^k e^{\omega\mu} N_{2\mu}^{\omega} \|Z_\mu(\gamma) - U_\mu(\gamma)\|^{\omega}\, \mathrm{d}\mu\right)^{\frac{1}{\omega}}\end{aligned}$$

By (14), we let $n = 3$ and $\eta = \omega$, it holds that

$$\begin{aligned}&\|Z_k(\gamma) - U_k(\gamma)\|^{\omega} \\ \le& 3^{\omega-1}\Bigg[||Z_0 - U_0\|^{\omega} + \left(\frac{1}{\Gamma(p)}\right)^{\omega} Q_2'^{\frac{\omega}{\varpi}} \int_0^k e^{\omega\mu} (N_{1\mu} + N_{3\mu})^{\omega} \|Z_\mu(\gamma) - U_\mu(\gamma)\|^{\omega} \mathrm{d}\mu \\ &+ \left(\frac{K_\gamma}{\Gamma(p)}\right)^{\omega} Q_2'^{\frac{\omega}{\varpi}} \int_0^k e^{\omega\mu} N_{2\mu}^{\omega} \|Z_\mu(\gamma) - U_\mu(\gamma)\|^{\omega}\, \mathrm{d}\mu\Bigg] \\ =& 3^{\omega-1} \|Z_0 - U_0\|^3 + \hat{Q}_2 \int_0^k e^{\omega\mu} (N_{1\mu} + N_{3\mu})^{\omega} \|Z_\mu(\gamma) - U_\mu(\gamma)\|^{\omega}\, \mathrm{d}\mu \\ &+ \hat{Q}_2 K_\gamma^{\omega} \int_0^k e^{\omega\mu} N_{2\mu}^{\omega} \|Z_\mu(\gamma) - U_\mu(\gamma)\|^{\omega}\, \mathrm{d}\mu\end{aligned}$$

where $\hat{Q}_2 = 3^{\omega-1} Q_2'^{\frac{\omega}{\varpi}} \cdot \left(\frac{1}{\Gamma(p)}\right)^{\omega}$. Gronwall inequality implies that

$$\begin{aligned}&\|Z_k(\gamma) - U_k(\gamma)\|^{\omega} \\ \le& 3^{\omega-1} \|Z_0 - U_0\|^{\omega} \exp\left(\hat{Q}_2 \int_0^k e^{\omega\mu} (N_{1\mu} + N_{3\mu})^{\omega} \mathrm{d}\mu\right) \exp\left(\hat{Q}_2 K_\gamma^{\omega} \int_0^k e^{\omega\mu} N_{2\mu}^{\omega} \mathrm{d}\mu\right) \\ \le& 3^{\omega-1} \|Z_0 - U_0\|^{\omega} \exp\left(\hat{Q}_2 \int_0^{+\infty} e^{\omega\mu} (N_{1\mu} + N_{3\mu})^{\omega} \mathrm{d}\mu\right) \exp\left(\hat{Q}_2 K_\gamma^{\omega} \int_0^{+\infty} e^{\omega\mu} N_{2\mu}^{\omega} \mathrm{d}\mu\right)\end{aligned}$$

For each $\gamma \in \{\gamma | K_\gamma \le L_\epsilon\}$, we have

$$\|Z_k(\gamma) - U_k(\gamma)\| \le 3^{\frac{\omega-1}{\omega}} \|Z_0 - U_0\| \exp\left[\frac{1}{\omega}\left(\hat{Q}_2 \int_0^{+\infty} e^{\omega\mu}(N_{1\mu} + N_{3\mu})^\omega \mathrm{d}\mu\right)\right] \exp\left[\frac{1}{\omega}\left(\hat{Q}_2 L_\epsilon^\omega \int_0^{+\infty} e^{\omega\mu} N_{2\mu}^\omega \mathrm{d}\mu\right)\right] \le \varepsilon.$$

provided that $\|Z_0 - U_0\| \le \hat{\delta}$, where

$$\delta = 3^{\frac{1-\omega}{\omega}} \exp\left[-\frac{1}{\omega}\left(\hat{Q}_2 \int_0^{+\infty} e^{\omega\mu}(N_{1\mu} + N_{3\mu})^\omega \mathrm{d}\mu + \hat{Q}_2 L_\epsilon^\omega \int_0^{+\infty} e^{\omega\mu} N_{2\mu}^\omega \mathrm{d}\mu\right)\right] \epsilon.$$

That means

$$\{\gamma \mid K_\gamma \le L_\epsilon\} \subset \left\{\sup_{k\ge 0} \|Z_k(\gamma) - U_k(\gamma)\| \le \varepsilon\right\}.$$

By (11), we know that there exists a positive number L_ϵ for any given $\epsilon > 0$ satisfy

$$\mathcal{M}\{\gamma \in \Gamma | K_\gamma \le L_\epsilon\} > 1 - \epsilon.$$

Moreover, for any $\epsilon > 0$, it holds that

$$\mathcal{M}\left\{\sup_{k\ge 0} \|Z_k(\gamma) - U_k(\gamma)\| \le \varepsilon\right\} \ge \mathcal{M}\{\gamma \mid K_\gamma \le L_\epsilon\} > 1 - \epsilon$$

provided that $\|Z_0 - U_0\| \le \hat{\delta}$. So we obtain $\lim_{\|Z_0 - U_0\| \to 0} \mathcal{M}\left\{\sup_{k\ge 0} \|Z_k(\gamma) - U_k(\gamma)\| \le \varepsilon\right\} = 1$ for any $\epsilon > 0$. Thus, the UFDEJ (1) is the stability in measure.

4. **Examples**

Example 4.1. *Let $\frac{1}{2} < p \le 1$. Consider the nonlinear UFDEJ*

$$ {}^cD^p \begin{bmatrix} Z_{1k} \\ Z_{2k} \end{bmatrix} = \begin{bmatrix} e^{-\frac{k^2}{2}} & 0 \\ 0 & e^{-k^2} \end{bmatrix} \begin{bmatrix} Z_{1k} \\ Z_{2k} \end{bmatrix} + \begin{bmatrix} \exp\left(-3k - Z_{2k}^2\right) \\ e^{-2k} Z_{1k} \end{bmatrix} \frac{dC_k}{dk} + \begin{bmatrix} e^{-\frac{k^2}{2}} & 0 \\ 0 & e^{-k^2} \end{bmatrix} \begin{bmatrix} Z_{1k} \\ Z_{2k} \end{bmatrix} \frac{dV_k}{dk} \tag{18}$$

with initial value $Z_k|_{k=0} = Z_0 \in \Re^2$. Since the coefficients $F(k,z) = \begin{bmatrix} e^{-\frac{k^2}{2}} & 0 \\ 0 & e^{-k^2} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$, $G(k,z) = \begin{bmatrix} \exp\left(-3k - Z_2^2\right) \\ e^{-2k} Z_1 \end{bmatrix}$, and $H(k,z) = \begin{bmatrix} 2e^{-\frac{k^2}{2}} & 0 \\ 0 & 2e^{-k^2} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$. Thus, we can obtain that

$$\|F(k,z) - F(k,u)\| \le \exp\left(-\frac{1}{2}k^2\right) \|z - u\|, \|G(k,z) - G(k,u)\| \quad \le \exp(-2k)||z - u\|$$

$$\|H(k,z) - H(k,u)\| \le 2\exp\left(-\frac{1}{2}k^2\right) \|z - u\|$$

for any $z, u \in \Re^2$ and $k \ge 0$, which implies $N_{1k} = \exp(-\frac{k^2}{2})$, $N_{2k} = \exp(-2k)$, and $N_{3k} = 2\exp(-\frac{k^2}{2})$. Then it holds that

$$\int_0^{+\infty} e^{2\mu}(N_{1\mu} + N_{3\mu})^2 \ \mathrm{d}\mu = \int_0^{+\infty} e^{2\mu} 9 e^{-\mu^2} \ \mathrm{d}\mu = 9e \cdot \int_0^{+\infty} e^{-\mu^2 + 2\mu - 1} \ \mathrm{d}\mu = 9e \cdot \int_{-1}^{+\infty} e^{-r^2} \ \mathrm{d}r \ \left(\textit{let } r = \mu - 1\right) \le 9e \cdot \int_{-\infty}^{+\infty} e^{-r^2} \ \mathrm{d}r = 9e\sqrt{\pi}$$

$$\int_0^{+\infty} e^{2\mu} N_{2\mu}^2 \, \mathrm{d}\mu = \int_0^{+\infty} e^{2\mu} e^{-4\mu} \, \mathrm{d}\mu = \frac{1}{2}.$$

By Theorem 3.1, the system (18) *is stable in measure.*

Example 4.2. *Consider the Caputo type of UFDEJ of order* $p = \frac{1}{2}$

$$^{c}D^{p} Z_k = e^{-2k} \sin Z_k + \frac{\exp(-k)}{1+k^2} Z_k \frac{dC_k}{\mathrm{d}k} + e^{-2k} Z_k \frac{dV_k}{\mathrm{d}k} \tag{19}$$

with initial value $Z_k|_{k=0} = Z_0$. *The coefficients* $F(k,z) = e^{-2k} sinz$, $G(k,z) = \frac{e^{-k}}{1+k^2} z$ *and* $H(k,z) = e^{-2k} z$ *satisfy*

$$\|F(k,z) - F(k,u)\| = e^{-2k} \|\sin z - \sin u\| \leq e^{-2k} ||z-u\|$$

$$\|G(k,z) - G(k,u)\| = \frac{e^{-k}}{1+k^2} \|z-u\|, \|H(k,z) - H(k,u)\| = e^{-2k} \|z-u\|$$

for any $z, u \in \Re$ *and* $k \geq 0$. *Clearly,* $N_{1k} = e^{-2k}$, $N_{2k} = \frac{e^{-k}}{1+k^2}$, *and* $N_{3k} = e^{-2k}$. *Then it holds that*

$$\int_0^{\infty} e^{3\mu} (N_{1\mu} + N_{3\mu})^3 \, \mathrm{d}\mu = \int_0^{\infty} 8e^{3\mu} e^{-6\mu} \, \mathrm{d}\mu = \int_0^{\infty} e^{-3\mu} \, \mathrm{d}\mu = \frac{8}{3}$$

$$\int_0^{\infty} e^{3\mu} N_{2\mu}^3 \, \mathrm{d}\mu = \int_0^{\infty} e^{3\mu} \frac{e^{-3\mu}}{(1+\mu^2)^3} \mathrm{d}\mu \leq \int_0^{\infty} \frac{1}{1+\mu^2} \, \mathrm{d}\mu = \frac{\pi}{2}.$$

By Theorem 3.2, the UFDE(19) *is stable in measure.*

5. Conclusion

For dealing with the stability problems of fractional differential system including both uncertainty and jumps, this paper considered he stability in measure for the Caputo type of UFDEJs that the order is $0 < p \leq 1$ via uncertain theory. Considering the importance of the oscillation of differential equations, such as [26, 27, 28, 29], the authors are intending to continue investigating the oscillation of differential equations for uncertain fractional jump systems in the future.

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