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# CHARACTERIZATION OF THE BOUNDEDNESS OF $\tau$-WIGNER TRANSFORM ON HARDY AND $BMO$ SPACES

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*One of the most popular families of time-frequency representations is the $\tau$ -Wigner transform. This paper is concerned with the boundedness of the $\tau$ -Wigner transform. Boundedness results for the $\tau$ -Wigner transform are obtained in both Hardy and BMO spaces. The Hardy and BMO-distance between two $\tau$ -Wigner transforms associated with different windows and different argument functions are then studied.*

**Keywords:** $\tau$-Wigner transform, Hardy space, $BMO$ space
**MSC2020:** 42B10, 47B38, 46E30

## 1. **Introduction**

The Wigner distribution which was introduced by the 1963 Nobel Prize Winner in Physics E. Wigner in 1932, [12], has a unique origin in quantum mechanics. Later it has been investigated by several authors as a tool for time-frequency signal analysis, and several applications have been recommended in different domains. Since the non-stationary signals cannot be analyzed completely by the Fourier analysis which is an effective tool for studying stationary signals, a complete analysis of non-stationary signals requires both time and frequency representations of signals. As the Wigner distribution provides a high-resolution representation in both time and frequency for non-stationary signals, it is the most popular time-frequency representation.

As a natural generalization of the Wigner distribution, depending on a parameter $\tau \in [0,1]$, another family of time-frequency representations was first introduced in [3]. This is called the $\tau$-Wigner distribution. The basic structures and properties of the $\tau$-Wigner distribution were discussed in some detail in [2, 3, 8], and the multilinear case of the $\tau$-Wigner distribution is defined and studied in detail in [9].

The present investigation is inspired by the papers of Chuong & Duong and Verma & Gupta, [4, 11]. The boundedness property of the wavelet integral operator on the Besov, BMO, and $H^1$ spaces was obtained in [4]. Verma and Gupta thereafter introduced in [11] a new class of continuous fractional wavelet transform and studied its properties in Hardy space and Morrey space.

This paper is also organized as follows. After this paragraph, we introduce the terminology used throughout this paper. In Section 2.**1**, we shall establish the $H^1$-boundedness of the $\tau$-Wigner transform for all $\tau \in [0,1]$. In Section 2.**2**, we obtained the boundedness property of the $\tau$-Wigner transform for all $\tau \in [0,1]$ on the space BMO as well. Further, the Hardy and $BMO$-distance between two $\tau$-Wigner transforms are studied.

We have compiled some basic facts as in follows:

We denote $\mathcal{S}(\mathbb{R}^d)$ as the space of complex-valued continuous functions on $\mathbb{R}^d$ rapidly decreasing at infinity. Let $f$ be a complex valued measurable function on $\mathbb{R}^d$. The operators $T_x f(t) = f(t-x)$ and $M_w f(t) = e^{2\pi i w t} f(t)$ are called translation and modulation

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operators for $x, w \in \mathbb{R}^d$, respectively. The compositions $T_x M_w f(t) = e^{2\pi i w\cdot(t-x)} f(t-x)$ or $M_w T_x f(t) = e^{2\pi i w\cdot t} f(t-x)$ are called time-frequency shifts (see [5, 6]) . We consider next the dilation operator $f \to f_\delta$ which is given by $f_\delta(x) = \delta^{-d} f\left(\frac{x}{\delta}\right)$, $\delta > 0$.

We write $(L^p(\mathbb{R}^d), \| \,.\, \|_p)$ as the Lebesgue spaces for $1 \leq p \leq \infty$. For $f \in L^1(\mathbb{R}^d)$ the Fourier transform $\widehat{f}$ (or $\mathcal{F}f$) is defined as

$$\widehat{f}(t) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot t} dx,$$

where $x \cdot t = \sum_{i=1}^d x_i t_i$ is the usual scalar product on $\mathbb{R}^d$.

Fix a function $g \neq 0$ (called the window function). The short-time Fourier transform (STFT) of a function $f$ with respect to $g$ is given by

$$V_g f(x, w) = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i t\cdot w} dt,$$

for all $x, w \in \mathbb{R}^d$. It is known that if $f, g \in L^2(\mathbb{R}^d)$, then $V_g f \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ and $V_g f$ is uniformly continuous (see [5, 6]).

Let define $V_g^\tau f$ as the function

$$V_g^\tau f\,(x, w) = V_g f\left(\frac{1}{1-\tau} x, \frac{1}{\tau} w\right)$$

for $\tau \in (0, 1)$ and $x, w \in \mathbb{R}^d$.

The cross-Wigner distribution of $f, g \in L^2(\mathbb{R}^d)$ is defined to be

$$W(f, g)(x, w) = \int_{\mathbb{R}^d} f(x + \frac{t}{2}) \overline{g(x - \frac{t}{2})} e^{-2\pi i t\cdot w} dt.$$

If $f = g$, then $W(f, f) = Wf$ is called the Wigner distribution of $f \in L^2(\mathbb{R}^d)$. For $\tau \in [0, 1]$ and $f, g \in \mathcal{S}(\mathbb{R}^d)$, the $\tau$-Wigner transform is defined as

$$W_\tau(f, g)(x, w) = \int_{\mathbb{R}^d} f(x + \tau t) \overline{g(x - (1-\tau)t)} e^{-2\pi i t\cdot w} dt.$$

If $\tau = \frac{1}{2}$, then the $\tau$-Wigner transform is the cross-Wigner distribution. For $\tau = 0$, $W_0$ is the Rihaczek transform, $W_0(f, g)(x, w) = e^{-2\pi i x\cdot w} f(x) \overline{\widehat{g}(w)}$, and the conjugate Rihaczek transform is $W_1(f, g)(x, w) = e^{2\pi i x\cdot w} \overline{g(x)} \widehat{f}(w)$, if $\tau = 1$, (see [2, 3]).

The theory of Hardy spaces has close connections to many branches of mathematics, including Fourier analysis, harmonic analysis, signal and image processing, control theory, singular integrals and operator theory. It is known that the Hardy space is much more suitable than the Lebesgue space for many questions in harmonic analysis. Recall that an equivalent definition of $H^1(\mathbb{R}^d)$ is given in terms of maximal functions $\mathcal{M}_\phi$ defined as follows: We fix an integrable smooth function $\phi$ on $\mathbb{R}^d$ supported in the unit ball such that $\int_{\mathbb{R}^d} \phi = 1$ and set $\phi_t(x) = t^{-d} \phi\left(\frac{x}{t}\right)$ for $t > 0$. The maximal operator $\mathcal{M}_\phi$ is defined by $\mathcal{M}_\phi f(x) = \sup_{t>0} |f * \phi_t(x)|$ for an integrable function $f$. The Hardy space $H^1(\mathbb{R}^d)$ is defined as the set of all $f \in L^1(\mathbb{R}^d)$ if, for some $\phi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \phi = 1$, the maximal function $\mathcal{M}_\phi f$ is in $L^1(\mathbb{R}^d)$. It is also a Banach space. Whenever $f \in H^1(\mathbb{R}^d)$, then both the translation operator $T_x f$ and the dilation operator $f_\delta$ are in $H^1(\mathbb{R}^d)$ with

$$\|T_x f\|_{H^1} = \|f\|_{H^1}, \qquad \|f_\delta\|_{H^1} = \|f\|_{H^1}. \tag{1}$$

The space of functions of bounded mean oscillation, or $BMO$ (it is also known as the John-Nirenberg space), arises as the class of functions whose deviation from their means over cubes is bounded. The space $BMO(\mathbb{R}^d)$ of functions of bounded mean oscillation was devised by John-Nirenberg in [7]. $BMO(\mathbb{R}^d)$ is the Banach space of all locally integrable functions $f$

on $\mathbb{R}^d$ for which $\|f\|_{BMO} = \sup\limits_{Q\subset\mathbb{R}^d} Q\left(|f - Q\left(f\right)|\right) < \infty$, where the supremum is taken over all cubes $Q$ with sides parallel to the coordinate axes, $|Q|$ is the Lebesgue measure of $Q$ and $Q\left(f\right)$ points out the mean of $f$ over the ball $Q$, that is

$$Q\left(f\right) = |Q|^{-1}\int_Q f\left(x\right)dx \le |Q|^{-1}\int_Q |f\left(x\right)|\,dx \le C < \infty. \tag{2}$$

The space $BMO(\mathbb{R}^d)$ is the dual of the Hardy space $H^1(\mathbb{R}^d)$. The Hardy space $H^1(\mathbb{R}^d)$ is a substitute for $L^1(\mathbb{R}^d)$ and the space $BMO(\mathbb{R}^d)$ is the corresponding natural substitute for the space $L^\infty(\mathbb{R}^d)$ of bounded functions on $\mathbb{R}^d$. An excellent references for these spaces are [1, 10].

## 2. $\tau$-Wigner transform on Hardy and $BMO$ spaces

### 2.1. Boundedness of $\tau$-Wigner transform on Hardy Space

In this section, we present $H^1(\mathbb{R}^d)$-boundedness of $\tau$-Wigner transform for all $\tau \in [0,1]$. Before stating our first theorem on boundedness, we need the following Lemma.

**Lemma 2.1.** *If $\tau \in [0,1]$, $f \in L^1(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, then $W_\tau\left(f,g\right)\left(\cdot,w\right) \in L^1(\mathbb{R}^d)$.*

*Proof.* Let $\tau \in (0,1)$. For a fixed $w \in \mathbb{R}^d$, $W_\tau\left(f,g\right)\left(x,w\right)$ is a function of $x$. Then we write by Lemma 6.2 in [3]

$$\begin{aligned}
\left|W_\tau\left(f,g\right)\left(x,w\right)\right| &= \left|\frac{1}{|\tau|^d}e^{2\pi i\frac{1}{\tau}x\cdot w}V_{A_\tau g}f\left(\frac{1}{1-\tau}x,\frac{1}{\tau}w\right)\right| \\
&\le \frac{1}{|\tau|^d}\int_{\mathbb{R}^d}|f\left(u\right)|\left|A_\tau g\left(u-\frac{1}{1-\tau}x\right)\right|du \\
&= \frac{1}{|\tau|^d}\int_{\mathbb{R}^d}\left|f\left(v+\frac{1}{1-\tau}x\right)\right|\left|A_\tau g\left(v\right)\right|dv
\end{aligned}$$

by changing variable $u - \frac{1}{1-\tau}x = v$. Hence we have

$$\begin{aligned}
\left\|W_\tau\left(f,g\right)\left(\cdot,w\right)\right\|_1 &\le \frac{1}{|\tau|^d}\int_{\mathbb{R}^d}\left|A_\tau g\left(v\right)\right|\left(\int_{\mathbb{R}^d}\left|\left(1-\tau\right)^d\left(T_{-v}f\right)_{(1-\tau)}\left(x\right)\right|dx\right)dv \\
&= \frac{|1-\tau|^d}{|\tau|^d}\left\|A_\tau g\right\|_1\left\|T_{-v}f\right\|_1
\end{aligned}$$

by the dilation invariance of $T_{-v}f$ in $L^1(\mathbb{R}^d)$. Also since the space $L^1(\mathbb{R}^d)$ is strongly translation invariant and by the equality (6.3) in [3], we obtain

$$\left\|W_\tau\left(f,g\right)\left(\cdot,w\right)\right\|_1 = \frac{|1-\tau|^d}{|\tau|^d}\frac{|\tau|^d}{|1-\tau|^d}\|g\|_1\|f\|_1 = \|g\|_1\|f\|_1.$$

Hence, $W_\tau\left(f,g\right)\left(\cdot,w\right) \in L^1(\mathbb{R}^d)$ for $\tau \in (0,1)$. If $\tau = 0$, we have

$$\left\|W_0\left(f,g\right)\left(\cdot,w\right)\right\|_1 \le \int_{\mathbb{R}^d}|f\left(x\right)|\left(\int_{\mathbb{R}^d}|g\left(x-t\right)|\,dt\right)dx = \|g\|_1\|f\|_1$$

and so $W_0\left(f,g\right)\left(\cdot,w\right) \in L^1(\mathbb{R}^d)$. Similarly, it is proved that $W_1\left(f,g\right)\left(\cdot,w\right) \in L^1(\mathbb{R}^d)$. □

**Theorem 2.1.** (i) *Let $\tau \in [0,1)$ and $g \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$. Then the operator $W_\tau(\cdot,g) : H^1(\mathbb{R}^d) \to H^1(\mathbb{R}^d)$ defined by $f \to W_\tau(f,g)(\cdot,w)$ is bounded. In particular,*

$$\left\|W_\tau\left(f,g\right)\left(\cdot,w\right)\right\|_{H^1} \le \|g\|_1\|f\|_{H^1}.$$

(ii) *Let* $\tau = 1$ *and* $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$*. Then the operator* $W_1(f, \cdot) : H^1(\mathbb{R}^d) \to H^1(\mathbb{R}^d)$ *defined by* $g \to W_1(f, g)(\cdot, w)$ *is bounded. In particular,*

$$\left\| W_1 \left( f, g \right) \left( \cdot, w \right) \right\|_{H^1} \leq \left\| f \right\|_1 \left\| g \right\|_{H^1}.$$

*Proof.* (i) Let $\tau \in (0, 1)$. Making the substitution $u - \frac{1}{1-\tau}x = v$ in Lemma 6.2 in [3], we write

$$\begin{aligned} W_\tau \left( f, g \right) \left( x, w \right) &= \frac{1}{\left| \tau \right|^d} e^{2\pi i \frac{1}{\tau} x \cdot w} \int_{\mathbb{R}^d} f \left( u \right) \overline{A_\tau g \left( u - \frac{1}{1-\tau} x \right)} e^{-2\pi i u \cdot \frac{w}{\tau}} du \qquad (3) \\ &= \frac{1}{\left| \tau \right|^d} e^{2\pi i \frac{1}{\tau} x \cdot w} \int_{\mathbb{R}^d} f \left( v + \frac{1}{1-\tau} x \right) \overline{A_\tau g \left( v \right)} e^{-2\pi i \left( v + \frac{1}{1-\tau} x \right) \cdot \frac{w}{\tau}} dv. \end{aligned}$$

Applying the Fubini's Theorem, we have

$$\begin{aligned} &\left( W_\tau \left( f, g \right) \left( \cdot, w \right) * \phi_t \right) \left( x \right) \qquad (4) \\ &= \int_{\mathbb{R}^d} W_\tau \left( f, g \right) \left( x - y, w \right) \phi_t \left( y \right) dy \\ &= \frac{1}{\left| \tau \right|^d} \int_{\mathbb{R}^d} e^{2\pi i \frac{1}{\tau} (x-y) \cdot w} \overline{A_\tau g \left( v \right)} \left( \int_{\mathbb{R}^d} \left( 1 - \tau \right)^d \left( T_{-v} M_{-\frac{w}{\tau}} f \right)_{(1-\tau)} \left( x - y \right) \phi_t \left( y \right) dy \right) dv \\ &= \frac{\left| 1 - \tau \right|^d}{\left| \tau \right|^d} \int_{\mathbb{R}^d} e^{2\pi i \frac{1}{\tau} (x-y) \cdot w} \overline{A_\tau g \left( v \right)} \left( \left( T_{-v} M_{-\frac{w}{\tau}} f \right)_{(1-\tau)} * \phi_t \right) \left( x \right) dv. \end{aligned}$$

So, we obtain

$$\begin{aligned} &\left\| W_\tau \left( f, g \right) \left( \cdot, w \right) \right\|_{H^1} \\ &= \int_{\mathbb{R}^d} \sup_{t > 0} \left| \left( W_\tau \left( f, g \right) \left( \cdot, w \right) * \phi_t \right) \left( x \right) \right| dx \\ &\leq \frac{\left| 1 - \tau \right|^d}{\left| \tau \right|^d} \int_{\mathbb{R}^d} \left| A_\tau g \left( v \right) \right| \left( \int_{\mathbb{R}^d} \sup_{t > 0} \left| \left( \left( T_{-v} M_{-\frac{w}{\tau}} f \right)_{(1-\tau)} * \phi_t \right) \left( x \right) \right| dx \right) dv \\ &= \frac{\left| 1 - \tau \right|^d}{\left| \tau \right|^d} \left\| A_\tau g \right\|_1 \left\| \left( T_{-v} M_{-\frac{w}{\tau}} f \right)_{(1-\tau)} \right\|_{H^1}. \end{aligned}$$

By using the equality (6.3) in [3] and the translation and the dilation invariant of Hardy space (see equations (1)), we get for $\tau \in (0, 1)$

$$\left\| W_\tau \left( f, g \right) \left( \cdot, w \right) \right\|_{H^1} \leq \frac{\left| 1 - \tau \right|^d}{\left| \tau \right|^d} \frac{\left| \tau \right|^d}{\left| 1 - \tau \right|^d} \left\| g \right\|_1 \left\| f \right\|_{H^1} = \left\| g \right\|_1 \left\| f \right\|_{H^1}.$$

If $\tau = 0$, we obtain

$$\begin{aligned} \left\| W_0 \left( f, g \right) \left( \cdot, w \right) \right\|_{H^1} &= \int_{\mathbb{R}^d} \sup_{t > 0} \left| \int_{\mathbb{R}^d} W_0 \left( f, g \right) \left( x - y, w \right) \phi_t \left( y \right) dy \right| dx \\ &= \int_{\mathbb{R}^d} \sup_{t > 0} \left| \int_{\mathbb{R}^d} e^{-2\pi i (x-y) \cdot w} f \left( x - y \right) \overline{\widehat{g} \left( w \right)} \phi_t \left( y \right) dy \right| dx \\ &\leq \left| \widehat{g} \left( w \right) \right| \int_{\mathbb{R}^d} \sup_{t > 0} \left( \left| f \right| * \left| \phi_t \right| \right) \left( x \right) dx \leq \left\| g \right\|_1 \left\| f \right\|_{H^1}. \end{aligned}$$

(ii) Let $\tau = 1$. Recalling that $W_1(f,g)(x,w) = e^{2\pi i x\cdot w}\overline{g(x)}\widehat{f}(w)$, we have

$$\begin{aligned}\|W_1(f,g)(\cdot,w)\|_{H^1} &= \int_{\mathbb{R}^d}\sup_{t>0}\left|\int_{\mathbb{R}^d} e^{2\pi i(x-y)\cdot w}\widehat{f}(w)\overline{g(x-y)}\phi_t(y)\,dy\right|dx\\ &\le \left|\widehat{f}(w)\right|\int_{\mathbb{R}^d}\sup_{t>0}(|g|*|\phi_t|)(x)\,dx \le \|f\|_1\|g\|_{H^1}.\end{aligned}$$

This completes the proof.

□

Now, we will give the $H^1(\mathbb{R}^d)$-distance of two $\tau-$Wigner transforms associated with different window functions and different argument functions.

**Theorem 2.2.** (i) *Let $\tau \in [0,1)$ and $g_1, g_2 \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$. If $f,h \in H^1(\mathbb{R}^d)$, then we have*

$$\|W_\tau(f,g_1)(\cdot,w) - W_\tau(h,g_2)(\cdot,w)\|_{H^1} \le \|g_1-g_2\|_1\|f\|_{H^1} + \|g_2\|_1\|f-h\|_{H^1}.$$

(ii) *Let $\tau = 1$ and $f,h \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$. If $g_1,g_2 \in H^1(\mathbb{R}^d)$, then*

$$\|W_1(f,g_1)(\cdot,w) - W_1(h,g_2)(\cdot,w)\|_{H^1} \le \|g_1-g_2\|_{H^1}\|f\|_1 + \|g_2\|_{H^1}\|f-h\|_1.$$

*Proof.* (i) Let $\tau \in (0,1)$. By Lemma 6.2 in [3], (3) and (4), we write

$$\begin{aligned}&((W_\tau(f,g_1)(\cdot,w) - W_\tau(f,g_2)(\cdot,w)) * \phi_t)(x)\\ &= \frac{|1-\tau|^d}{|\tau|^d}\int_{\mathbb{R}^d} e^{2\pi i\frac{1}{\tau}(x-y)\cdot w}\overline{A_\tau(g_1-g_2)(v)}\left(\left(T_{-v}M_{-\frac{w}{\tau}}f\right)_{(1-\tau)} * \phi_t\right)(x)\,dv.\end{aligned}$$

Then by Theorem 2.1 i), we have

$$\|W_\tau(f,g_1)(\cdot,w) - W_\tau(f,g_2)(\cdot,w)\|_{H^1} \le \|g_1-g_2\|_1\|f\|_{H^1}. \tag{5}$$

Also since

$$\begin{aligned}&((W_\tau(f,g_2)(\cdot,w) - W_\tau(h,g_2)(\cdot,w)) * \phi_t)(x)\\ &= \frac{|1-\tau|^d}{|\tau|^d}\int_{\mathbb{R}^d} e^{2\pi i\frac{1}{\tau}(x-y)\cdot w}\overline{A_\tau g_2(v)}\left(\left(T_{-v}M_{-\frac{w}{\tau}}(f-h)\right)_{(1-\tau)} * \phi_t\right)(x)\,dv,\end{aligned}$$

by Theorem 2.1 i), we write

$$\|W_\tau(f,g_2)(\cdot,w) - W_\tau(h,g_2)(\cdot,w)\|_{H^1} \le \|g_2\|_1\|f-h\|_{H^1}. \tag{6}$$

Then by (5) and (6), we obtain

$$\begin{aligned}&\|W_\tau(f,g_1)(\cdot,w) - W_\tau(h,g_2)(\cdot,w)\|_{H^1}\\ &\le \|W_\tau(f,g_1)(\cdot,w) - W_\tau(f,g_2)(\cdot,w)\|_{H^1}\\ &\quad + \|W_\tau(f,g_2)(\cdot,w) - W_\tau(h,g_2)(\cdot,w)\|_{H^1}\\ &\le \|g_1-g_2\|_1\|f\|_{H^1} + \|g_2\|_1\|f-h\|_{H^1}.\end{aligned}$$

Now, let $\tau = 0$. Because of the definition of $W_0(f,g)$, we write

$$\begin{aligned}W_0(f,g_1)(x,w) - W_0(f,g_2)(x,w) &= e^{-2\pi i x\cdot w}f(x)\int_{\mathbb{R}^d}\overline{(g_1-g_2)(u)\,e^{-2\pi i u\cdot w}}du\\ &= e^{-2\pi i x\cdot w}f(x)\overline{(g_1-g_2)^{\wedge}(w)}\end{aligned}$$

and hence, we obtain

$$
\begin{aligned}
&\left\| W_0\left(f,g_1\right)(\cdot,w)-W_0\left(f,g_2\right)(\cdot,w)\right\|_{H^1} \qquad (7)\\
&=\int_{\mathbb{R}^d}\sup_{t>0}\left|\int_{\mathbb{R}^d}\left(W_0\left(f,g_1\right)-W_0\left(f,g_2\right)\right)(x-y,w)\,\phi_t(y)\,dy\right|dx\\
&=\int_{\mathbb{R}^d}\sup_{t>0}\left|\int_{\mathbb{R}^d}e^{-2\pi i(x-y)\cdot w}f(x-y)\,\overline{\left(g_1-g_2\right)^{\wedge}(w)}\phi_t(y)\,dy\right|dx\\
&\le\left|\left(g_1-g_2\right)^{\wedge}(w)\right|\int_{\mathbb{R}^d}\sup_{t>0}\left(|f|*|\phi_t|\right)(x)\,dx\le\left\|g_1-g_2\right\|_1\|f\|_{H^1}.
\end{aligned}
$$

Moreover, we have

$$W_0\left(f,g_2\right)(x,w)-W_0\left(h,g_2\right)(x,w)=e^{-2\pi ix\cdot w}\,(f-h)(x)\,\overline{\widehat{g_2}(w)}$$

and similarly to (7), we write

$$\left\|W_0\left(f,g_2\right)(\cdot,w)-W_0\left(h,g_2\right)(\cdot,w)\right\|_{H^1}\le\left\|g_2\right\|_1\|f-h\|_{H^1}. \qquad (8)$$

From the inequalities (7) and (8), we get

$$
\begin{aligned}
&\left\|W_0\left(f,g_1\right)(\cdot,w)-W_0\left(h,g_2\right)(\cdot,w)\right\|_{H^1}\\
&\le\left\|W_0\left(f,g_1\right)(\cdot,w)-W_0\left(f,g_2\right)(\cdot,w)\right\|_{H^1}\\
&\quad+\left\|W_0\left(f,g_2\right)(\cdot,w)-W_0\left(h,g_2\right)(\cdot,w)\right\|_{H^1}\\
&\le\left\|g_1-g_2\right\|_1\|f\|_{H^1}+\left\|g_2\right\|_1\|f-h\|_{H^1}.
\end{aligned}
$$

So, the first part of the Theorem is proved.

(ii) The same reasoning for $\tau=1$ in Theorem 2.1 and for $\tau=0$ in Theorem 2.2 i) applies to the case $\tau=1$.

□

### 2.2. Boundedness of $\tau$-Wigner transform on BMO space

In this part, we will discuss the $BMO$-boundedness of $\tau$-Wigner transform. To facilitate the proof of the boundedness of $\tau$-Wigner transform on $BMO(\mathbb{R}^d)$, we need the following Lemma related to the space $BMO(\mathbb{R}^d)$.

**Lemma 2.2.** *The space $BMO(\mathbb{R}^d)$ is invariant under time-frequency shifts.*

*Proof.* Let $f\in BMO(\mathbb{R}^d)$, $Q$ be an arbitrary ball in $\mathbb{R}^d$ and $x,w\in\mathbb{R}^d$. Then we write

$$
\begin{aligned}
&\|M_wT_xf\|_{BMO}\\
&=\sup_{Q\subset\mathbb{R}^d}|Q|^{-1}\int_Q\left|M_wT_xf(t)-|Q|^{-1}\int_Q M_wT_xf(z)\,dz\right|dt\\
&\le\sup_{Q\subset\mathbb{R}^d}|Q|^{-1}\int_Q\left|e^{2\pi iw\cdot t}f(t-x)-|Q|^{-1}e^{2\pi iw\cdot t}\int_Q f(z-x)\,dz\right|dt\\
&\quad+\sup_{Q\subset\mathbb{R}^d}|Q|^{-1}\int_Q\left||Q|^{-1}e^{2\pi iw\cdot t}\int_Q f(z-x)\,dz\right|dt\\
&\quad+\sup_{Q\subset\mathbb{R}^d}|Q|^{-1}\int_Q\left||Q|^{-1}\int_Q e^{2\pi iw\cdot z}f(z-x)\,dz\right|dt,
\end{aligned}
$$

and so, we have

$$\begin{aligned}\|M_wT_xf\|_{BMO} &\le \sup_{Q\subset\mathbb{R}^d}|Q|^{-1}\int_Q\left|f(t-x)-|Q|^{-1}\int_Q f(z-x)\,dz\right|dt\\ &+2\sup_{Q\subset\mathbb{R}^d}|Q|^{-1}\int_Q\left(|Q|^{-1}\int_Q|f(z-x)|\,dz\right)dt.\end{aligned}$$

Now, let us say $P=Q-x$ for $x\in\mathbb{R}^d$. We then obtain by the inequality (2)

$$\begin{aligned}\|M_wT_xf\|_{BMO} &\le \sup_{P\subset\mathbb{R}^d}|P|^{-1}\int_P\left|f(u)-|P|^{-1}\int_P f(v)\,dv\right|du\\ &+2\sup_{P\subset\mathbb{R}^d}|P|^{-1}\int_P\left(|P|^{-1}\int_P|f(v)|\,dv\right)du\\ &=\sup_{P\subset\mathbb{R}^d}|P|^{-1}\int_P|f(u)-P(f)|\,du+2\sup_{P\subset\mathbb{R}^d}|P|^{-1}\int_P P(|f|)\,du\\ &=\|f\|_{BMO}+2|P|^{-1}C|P|=\|f\|_{BMO}+2C.\end{aligned}$$

Note that the space $BMO$ is invariant under modulation if $x=0$, and the space $BMO$ is invariant under translation if $w=0$. □

We shall need the following Lemma for the next Theorems.

**Lemma 2.3.** (i) *Let $\tau\in(0,1)$ and $g\in L^1(\mathbb{R}^d)$ be a compactly supported. If $f\in L^1_{loc}(\mathbb{R}^d)$, then $W_\tau(f,g)(\cdot,w)$ is in $L^1_{loc}(\mathbb{R}^d)$.*

(ii) *Let $\tau=0$. If $g\in L^1(\mathbb{R}^d)$ and $f\in L^1_{loc}(\mathbb{R}^d)$, then $W_0(f,g)(\cdot,w)$ is in $L^1_{loc}(\mathbb{R}^d)$.*

(iii) *Let $\tau=1$. If $f\in L^1(\mathbb{R}^d)$ and $g\in L^1_{loc}(\mathbb{R}^d)$, then $W_1(f,g)(\cdot,w)$ is in $L^1_{loc}(\mathbb{R}^d)$.*

*Proof.* (i) Let $\tau\in(0,1)$. Recall that $W_\tau(f,g)(x,w)$ is a function of $x$. Also we know that

$$|W_\tau(f,g)(x,w)|\le\frac{1}{|\tau|^d}\int_{\mathbb{R}^d}\left|f\left(v+\frac{1}{1-\tau}x\right)\right||A_\tau g(v)|\,dv$$

from the proof of Lemma 2.1. Then we write for any ball $Q\subset\mathbb{R}^d$

$$\int_Q|W_\tau(f,g)(x,w)|\,dx\le\frac{1}{|\tau|^d}\int_{\mathbb{R}^d}|A_\tau g(v)|\left(\int_Q\left|f\left(v+\frac{1}{1-\tau}x\right)\right|dx\right)dv.$$

Let us say $K=v+\frac{1}{1-\tau}Q$. Since $K\subset \operatorname{sup}pg+\frac{1}{1-\tau}Q$ is a compact set in $\mathbb{R}^d$ and $f\in L^1_{loc}(\mathbb{R}^d)$, we have by the equality (6.3) in [3]

$$\begin{aligned}\int_Q|W_\tau(f,g)(x,w)|\,dx&\le\frac{|1-\tau|^d}{|\tau|^d}\int_{\mathbb{R}^d}|A_\tau g(v)|\left(\int_K|f(u)|\,du\right)dv\\ &=M\frac{|1-\tau|^d}{|\tau|^d}\|A_\tau g\|_1=M\|g\|_1<\infty.\end{aligned}$$

Hence $W_\tau(f,g)(\cdot,w)\in L^1_{loc}(\mathbb{R}^d)$.

(ii) Let $\tau=0$, $f\in L^1_{loc}(\mathbb{R}^d)$ and $K\subset\mathbb{R}^d$ be a compact set. By the definition of $W_0(f,g)(\cdot,w)$, we write

$$\int_K|W_0(f,g)(x,w)|\,dx=|\widehat{g}(w)|\int_K|f(x)|\,dx\le M\|g\|_1<\infty,$$

and so $W_0(f,g)(\cdot,w)\in L^1_{loc}(\mathbb{R}^d)$.

(iii) Similarly, let $\tau = 1$, $g \in L^1_{loc}(\mathbb{R}^d)$ and $K \subset \mathbb{R}^d$ be a compact set. By the definition of $W_1(f,g)(\cdot,w)$, we get

$$\int_K |W_1(f,g)(x,w)|\,dx = \left|\widehat{f}(w)\right| \int_K |g(x)|\,dx \le L\,\|f\|_1 < \infty,$$

which proves iii).

□

Now we will state the $BMO$-boundedness of the $\tau$-Wigner transform.

**Theorem 2.3.** (i) *Let $\tau \in (0,1)$ and $g \in L^1(\mathbb{R}^d)$ be a compactly supported. Then the operator $W_\tau(\cdot,g) : BMO(\mathbb{R}^d) \to BMO(\mathbb{R}^d)$ defined by $f \to W_\tau(f,g)(\cdot,w)$ is bounded. In particular,*

$$\|W_\tau(f,g)(\cdot,w)\|_{BMO} \le \|g\|_1 \left(\|f\|_{BMO} + 4C\right).$$

(ii) *Let $\tau = 0$ and $g \in L^1(\mathbb{R}^d)$. Then the operator $W_0(\cdot,g) : BMO(\mathbb{R}^d) \to BMO(\mathbb{R}^d)$ defined by $f \to W_0(f,g)(\cdot,w)$ is bounded. Moreover, we have,*

$$\|W_0(f,g)(\cdot,w)\|_{BMO} \le \|g\|_1 \left(\|f\|_{BMO} + 2C\right).$$

(iii) *Let $\tau = 1$ and $f \in L^1(\mathbb{R}^d)$. Then the operator $W_1(f,\cdot) : BMO(\mathbb{R}^d) \to BMO(\mathbb{R}^d)$ defined by $g \to W_1(f,g)(\cdot,w)$ is bounded. Moreover, we have,*

$$\|W_1(f,g)(\cdot,w)\|_{BMO} \le \|f\|_1 \left(\|g\|_{BMO} + 2C\right).$$

*Proof.* (i) Let $\tau \in (0,1)$, $Q$ be an arbitrary ball in $\mathbb{R}^d$ and $f \in BMO(\mathbb{R}^d)$. Then $f \in L^1_{loc}(\mathbb{R}^d)$ and so $W_\tau(f,g)(\cdot,w) \in L^1_{loc}(\mathbb{R}^d)$ by Lemma 2.3 i). By using Fubini Theorem, we have

$$\begin{aligned} Q\left(W_\tau(f,g)\right) &= |Q|^{-1} \int_Q W_\tau(f,g)(z,w)\,dz \\ &= \frac{1}{|\tau|^d} \int_{\mathbb{R}^d} \overline{A_\tau g(v)} \left( |Q|^{-1} \int_Q M_{-\frac{w}{\tau}} f\left(v + \frac{1}{1-\tau} z\right) e^{2\pi i \frac{1}{\tau} z\cdot w} dz \right) dv \\ &= \frac{|1-\tau|^d}{|\tau|^d} \int_{\mathbb{R}^d} \overline{A_\tau g(v)}\, Q\left( M_{-\frac{w}{\tau}} \left(T_{-v} M_{-\frac{w}{\tau}} f\right)_{(1-\tau)} \right) dv. \end{aligned}$$

Thus we get

$$\begin{aligned} &\|W_\tau(f,g)(\cdot,w)\|_{BMO} \\ &= \sup_{Q\subset\mathbb{R}^d} |Q|^{-1} \int_Q \left|W_\tau(f,g)(x,w) - Q\left(W_\tau(f,g)\right)\right| dx \\ &\le \frac{|1-\tau|^d}{|\tau|^d} \int_{\mathbb{R}^d} |A_\tau g(v)| \\ &\left( \sup_{Q\subset\mathbb{R}^d} |Q|^{-1} \int_Q \left| M_{-\frac{w}{\tau}} \left(T_{-v} M_{-\frac{w}{\tau}} f\right)_{(1-\tau)}(x) - Q\left( M_{-\frac{w}{\tau}} \left(T_{-v} M_{-\frac{w}{\tau}} f\right)_{(1-\tau)} \right) \right| dx \right) dv \\ &= \frac{|1-\tau|^d}{|\tau|^d} \int_{\mathbb{R}^d} |A_\tau g(v)| \left\| M_{-\frac{w}{\tau}} \left(T_{-v} M_{-\frac{w}{\tau}} f\right)_{(1-\tau)} \right\|_{BMO} dv, \end{aligned}$$

also by using Lemma 2.2, the dilation invariant property of the space $BMO$ and the equality (6.3) in [3], we obtain

$$\begin{aligned} \|W_\tau(f,g)(\cdot,w)\|_{BMO} &\le \frac{|1-\tau|^d}{|\tau|^d} \int_{\mathbb{R}^d} |A_\tau g(v)| \left(\|f\|_{BMO} + 4C\right) dv \\ &= \|g\|_1 \left(\|f\|_{BMO} + 4C\right). \end{aligned}$$

(ii) Let $\tau = 0$, $Q$ be an arbitrary ball in $\mathbb{R}^d$ and $f \in BMO(\mathbb{R}^d)$. Then $f \in L^1_{loc}(\mathbb{R}^d)$ and so $W_0(f,g)(\cdot,w) \in L^1_{loc}(\mathbb{R}^d)$ by Lemma 2.3 ii). Then we have

$$Q(W_0(f,g)) = |Q|^{-1}\int_Q e^{-2\pi iz\cdot w} f(z)\overline{\widehat{g}(w)}dz = \overline{\widehat{g}(w)}Q(M_{-w}f).$$

Hence we obtain by Lemma 2.2

$$\begin{aligned}\|W_0(f,g)(\cdot,w)\|_{BMO} &= \sup_{Q\subset\mathbb{R}^d}|Q|^{-1}\int_Q |W_0(f,g)(x,w) - Q(W_0(f,g))|\,dx \\ &= \sup_{Q\subset\mathbb{R}^d}|Q|^{-1}\int_Q \left|e^{-2\pi ix\cdot w} f(x)\overline{\widehat{g}(w)} - \overline{\widehat{g}(w)}Q(M_{-w}f)\right| dx \\ &= |\widehat{g}(w)| \sup_{Q\subset\mathbb{R}^d}|Q|^{-1}\int_Q |(M_{-w}f)(x) - Q(M_{-w}f)|\,dx \\ &= |\widehat{g}(w)|\,\|M_{-w}f\|_{BMO} \le \|g\|_1(\|f\|_{BMO} + 2C)\end{aligned}$$

which proves ii).

(iii) The proof is omitted as it is similar to the proof of ii).

□

Now, we will give the $BMO(\mathbb{R}^d)$-distance of two $\tau$-Wigner transforms.

**Theorem 2.4.** (i) *Let $\tau \in (0,1)$ and $g_1, g_2 \in L^1(\mathbb{R}^d)$ be a compactly supported. If $f, h \in BMO(\mathbb{R}^d)$, then*

$$\begin{aligned}&\|W_\tau(f,g_1)(\cdot,w) - W_\tau(h,g_2)(\cdot,w)\|_{BMO} \\ &\le \|g_1 - g_2\|_1(\|f\|_{BMO} + 4C) + \|g_2\|_1(\|f-h\|_{BMO} + 4C).\end{aligned}$$

(ii) *Let $\tau = 0$ and $g_1, g_2 \in L^1(\mathbb{R}^d)$. If $f, h \in BMO(\mathbb{R}^d)$, then*

$$\begin{aligned}&\|W_0(f,g_1)(\cdot,w) - W_0(h,g_2)(\cdot,w)\|_{BMO} \\ &\le \|g_1 - g_2\|_1(\|f\|_{BMO} + 2C) + \|g_2\|_1(\|f-h\|_{BMO} + 2C).\end{aligned}$$

(iii) *Let $\tau = 1$ and $f, h \in L^1(\mathbb{R}^d)$. If $g_1, g_2 \in BMO(\mathbb{R}^d)$, then*

$$\begin{aligned}&\|W_1(f,g_1)(\cdot,w) - W_1(h,g_2)(\cdot,w)\|_{BMO} \\ &\le \|f\|_1(\|g_1 - g_2\|_{BMO} + 2C) + \|f-h\|_1(\|g_2\|_{BMO} + 2C).\end{aligned}$$

*Proof.* (i) Let $\tau \in (0,1)$, $g_1, g_2 \in L^1(\mathbb{R}^d)$ be a compactly supported and $f, h \in BMO(\mathbb{R}^d)$. Then $f, h \in L^1_{loc}(\mathbb{R}^d)$ and so, $W_\tau(f,g_1)(\cdot,w)$, $W_\tau(f,g_2)(\cdot,w)$ and $W_\tau(h,g_2)(\cdot,w) \in L^1_{loc}(\mathbb{R}^d)$ by Lemma 2.3 i). Then we obtain by Theorem 2.3 i)

$$\begin{aligned}&\|W_\tau(f,g_1)(\cdot,w) - W_\tau(h,g_2)(\cdot,w)\|_{BMO} \\ &\le \|W_\tau(f,g_1)(\cdot,w) - W_\tau(f,g_2)(\cdot,w)\|_{BMO} \\ &\quad + \|W_\tau(f,g_2)(\cdot,w) - W_\tau(h,g_2)(\cdot,w)\|_{BMO} \\ &= \|W_\tau(f,g_1-g_2)(\cdot,w)\|_{BMO} + \|W_\tau(f-h,g_2)(\cdot,w)\|_{BMO} \\ &\le \|g_1 - g_2\|_1(\|f\|_{BMO} + 4C) + \|g_2\|_1(\|f-h\|_{BMO} + 4C)\end{aligned}$$

which proves i).

(ii) Let $\tau = 0$, $g_1, g_2 \in L^1(\mathbb{R}^d)$ and $f, h \in BMO(\mathbb{R}^d)$. Then $W_0(f, g_1)(\cdot, w)$, $W_0(f, g_2)(\cdot, w)$ and $W_0(h, g_2)(\cdot, w) \in L^1_{loc}(\mathbb{R}^d)$ by Lemma 2.3 ii). Thus we have by Theorem 2.3 ii)

$$\begin{aligned}
&\|W_0(f, g_1)(\cdot, w) - W_0(h, g_2)(\cdot, w)\|_{BMO} \\
&\le \|W_0(f, g_1)(\cdot, w) - W_0(f, g_2)(\cdot, w)\|_{BMO} \\
&\quad + \|W_0(f, g_2)(\cdot, w) - W_0(h, g_2)(\cdot, w)\|_{BMO} \\
&= \|W_0(f, g_1 - g_2)(\cdot, w)\|_{BMO} + \|W_0(f - h, g_2)(\cdot, w)\|_{BMO} \\
&\le \|g_1 - g_2\|_1 (\|f\|_{BMO} + 2C) + \|g_2\|_1 (\|f - h\|_{BMO} + 2C).
\end{aligned}$$

This is the desired result.

(iii) The proof is similar in spirit to ii).

□

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