AN ALTERNATIVE AND UNITED PROOF OF A DOUBLE INEQUALITY FOR BOUNDING THE ARITHMETIC-GEOMETRIC MEAN

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In the paper, we provide an alternative and united proof of a double inequality for bounding the arithmetic-geometric mean. Moreover we prove that the bounding constants of the double inequality are the best possible.

Keywords: alternative and united proof, double inequality, arithmetic-geometric mean, complete elliptic integral of the first kind, generalized logarithmic mean

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1. Introduction

The complete elliptic integral of the first kind was defined as

$$K(t)=\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{1-t^2\sin^2\theta}} \tag{1}$$

for $0<t<1$, see [2, p. 132, Definition 3.2.1]. It can also be defined in the following way: For positive numbers a and b,

$$K(a,b)=\int_0^{\pi/2}\frac{1}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}\,\mathrm{d}\theta. \tag{2}$$

For positive numbers $a=a_0$ and $b=b_0$, let

$$a_{k+1}=\frac{a_k+b_k}{2} \quad \text{and} \quad b_{k+1}=\sqrt{a_kb_k}\,. \tag{3}$$

In [2][p. 134, Definition 3.2.2] and [4], the common limit $M(a,b)$ of these two sequences $\{a_k\}_{k=0}^\infty$ and $\{b_k\}_{k=0}^\infty$ is called as the arithmetic-geometric mean. It was proved in [2][Theorem 3.2.3] and [4, Theorem 1] that

$$\frac{1}{M(a,b)}=\frac{2}{\pi}K(a,b). \tag{4}$$

For more information on the arithmetic-geometric mean and the complete elliptic integral of the first kind, please refer to [2, pp. 132–136], [4] and related references therein.

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In [4, Theorem 4] and [6], it was proved that the inequality

$$M(a,b) \geq L(a,b) \tag{5}$$

holds true for positive numbers a and b and that the inequality (5) becomes equality if and only if $a=b$, where

$$L(a,b) = \frac{b-a}{\ln b - \ln a} \tag{6}$$

stands for the logarithmic mean for positive numbers a and b with $a \neq b$.

In [16, Theorem 1.3], it was proved that

$$M(a,b) < I(a,b) \tag{7}$$

for positive numbers a and b with $a \neq b$, where

$$I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} \tag{8}$$

represents the exponential mean for for positive numbers a and b with $a \neq b$.

It is known that a generalization of the logarithmic mean $L(a,b)$ is the generalized logarithmic mean $L(p;a,b)$ of order $p \in \mathbb{R}$, which may be defined [5, p. 385] by

$$L(p;a,b) = \begin{cases} \left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & p \neq -1, 0 \\ L(a,b), & p=-1 \\ I(a,b), & p=0 \end{cases} \tag{9}$$

for positive numbers a and b with $a \neq b$, and that $L(p;a,b)$ is strictly increasing with respect to $p \in \mathbb{R}$. Therefore, one may naturally pose the following problem.

Problem 1. *What are the best constants $0 \geq \beta > \alpha \geq -1$ such that the double inequality*

$$L(\alpha;a,b) < M(a,b) < L(\beta;a,b) \tag{10}$$

holds for all positive numbers a and b with $a \neq b$? In other words, are the constants $\alpha=-1$ and $\beta=0$ the best possible in the inequality (10)?

It is easy to see that the complete elliptic integral $K(a,b)$ of the first kind tends to infinity as the ratio $\frac{b}{a}$ for $a>b>0$ tends to zero, equivalently, the arithmetic-geometric mean $M(a,b)$ tends to zero as $\frac{b}{a} \to 0^+$. As $\frac{b}{a} \to 0^+$, the logarithmic mean $L(a,b)$ also tends to 0. However, the exponential mean $I(a,b)$ does not tend to zero as $\frac{b}{a} \to 0^+$. These phenomena motivate us to put forward an alternative problem as follows.

Problem 2. *What are the best constants $\beta > \alpha \geq 1$ such that the double inequality*

$$\alpha L(a,b) < M(a,b) < \beta L(a,b) \tag{11}$$

holds for all positive numbers a and b with $a \neq b$?

In [15] and [16, Theorem 1.3], among others, the right-hand side inequality in 11 was verified to be valid for $\beta = \frac{\pi}{2}$.

The aim of this paper is to confirm and sharpen the inequality 11 alternatively and unitedly and to prove that the constants are the best possible.

Our main result may be recited as the following theorem.

Theorem 1. *The double inequality 11 is valid for all positive numbers a and b with $a \neq b$ if and only if $\alpha \leq 1$ and $\beta \geq \frac{\pi}{2}$.*

Remark 1. Some inequalities were established in [3, 7, 8, 11, 12, 13, 14, 18] for bounding the complete elliptic integrals.

2. Lemmas

For proving our theorem alternatively and unitedly, we will employ the following lemmas.

Lemma 1 ([10])**.** *Let a_k and b_k for $k \in \mathbb{N}$ be real numbers and the power series*

$$A(x) = \sum_{k=1}^{\infty} a_k x^k \quad \text{and} \quad B(x) = \sum_{k=1}^{\infty} b_k x^k \tag{12}$$

be convergent on $(-R, R)$ for some $R > 0$. If $b_k > 0$ and the ratio $\frac{a_k}{b_k}$ is strictly increasing for $k \in \mathbb{N}$, then the function $\frac{A(x)}{B(x)}$ is also strictly increasing on $(0, R)$.

Lemma 2 ([17])**.** *For $n \in \mathbb{N}$,*

$$\int_0^{\pi/2} \sin^n x \, \mathrm{d}x = \frac{\sqrt{\pi}\,\Gamma((n+1)/2)}{n\Gamma(n/2)} = \begin{cases} \dfrac{\pi}{2} \cdot \dfrac{(n-1)!!}{n!!} & \text{for } n \text{ even}, \\ \dfrac{(n-1)!!}{n!!} & \text{for } n \text{ odd}, \end{cases} \tag{13}$$

where $n!!$ denotes a double factorial and $\Gamma(x)$ stands for the classical Euler's gamma function defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} \, \mathrm{d}t, \quad x > 0. \tag{14}$$

Lemma 3. *For $k \in \mathbb{N}$,*

$$\sum_{i=1}^{k} \binom{2i-2}{i-1} \frac{1}{i4^i} = \frac{1}{2} - \frac{2}{4^{k+1}} \binom{2k}{k}, \tag{15}$$

$$\sum_{i=1}^{k} \binom{2i-2}{i-1} \frac{1}{(k-i+1)4^i} = \left[2\ln 2 + \gamma + \psi\left(k + \frac{1}{2}\right)\right] \frac{\Gamma(k+1/2)}{4\sqrt{\pi}\,\Gamma(k+1)}, \tag{16}$$

where $\gamma = 0.57721566\cdots$ is Euler-Mascheroni's constant and $\psi(x)$ represents the logarithmic derivative $\frac{\Gamma'(x)}{\Gamma(x)}$ of $\Gamma(x)$.

Proof. For our own convenience, let us denote the two sequences in left-hand sides of (15) and (16) by $h(k)$ and $g(k)$ for $k \in \mathbb{N}$ respectively.

When $k = 1$, the identity (15) is valid clearly. Suppose the identity (15) holds for some $k > 1$, then it follows that

$$\begin{aligned} h(k+1) &= h(k) + \binom{2k}{k} \frac{1}{(k+1)4^{k+1}} \\ &= \frac{1}{2} - \frac{2}{4^{k+1}} \binom{2k}{k} + \binom{2k}{k} \frac{1}{(k+1)4^{k+1}} \end{aligned}$$

$$= \frac{1}{2} - \frac{2}{4^{k+2}}\binom{2k+2}{k+1}.$$

Therefore, by induction, the identity (15) is valid for all $k \in \mathbb{N}$.

Applying the Zeilberger algorithm (see [9, Chapter 6]) and (13) yields

$$2(k+1)g(k+1) - (2k+1)g(k) = \frac{\Gamma(k+1/2)}{2\sqrt{\pi}\,\Gamma(k+1)} = \frac{1}{2}\binom{2k}{k}\frac{1}{4^k} \tag{17}$$

for $k \in \mathbb{N}$, from which the identity (16) follows. □

Remark 2. The identities (15) and (16) can also be verified easily by the famous software packages Maple or MATHEMATICA.

3. An alternative and united proof of Theorem 1

Now we are in a position to alternatively and unitedly verify Theorem 1.

Making use of the power series expansion

$$\frac{1}{\sqrt{1-s}} = \sum_{i=0}^{\infty} \frac{(2i)!}{4^i (i!)^2} s^i, \quad 0 < s < 1,$$

it is obtained that

$$\frac{1}{\sqrt{1-s^2\sin^2\theta}} = \sum_{i=0}^{\infty} \frac{(2i)!}{4^i (i!)^2} s^{2i}\sin^{2i}\theta, \quad 0 < s < 1. \tag{18}$$

From the celebrated Wallis sine formula (13) in Lemma 2, it is obtained that

$$\int_0^{\pi/2} \sin^{2i}\theta \,\mathrm{d}\theta = \frac{1}{4^i}\binom{2i}{i}\frac{\pi}{2} = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma(i+1/2)}{\Gamma(i+1)}, \quad i \in \mathbb{N}. \tag{19}$$

Integrating on both sides of (18) with respect to θ from 0 to $\frac{\pi}{2}$ and using the identity (19) yield

$$\frac{2}{\pi}\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-s^2\sin^2\theta}} = \sum_{i=0}^{\infty}\frac{1}{2^{4i}}\binom{2i}{i}^2 s^{2i} = \frac{1}{\pi}\sum_{i=0}^{\infty}\left[\frac{\Gamma(i+1/2)}{\Gamma(i+1)}\right]^2 s^{2i} \tag{20}$$

for $0 < s < 1$. Letting $s^2 = 1 - t^2$ in (20) yields

$$\frac{2}{\pi}\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-(1-t^2)\sin^2\theta}} = \sum_{i=0}^{\infty}\frac{1}{2^{4i}}\binom{2i}{i}^2 \big(1-t^2\big)^i \triangleq \sum_{k=0}^{\infty} b_k\big(1-t^2\big)^k \tag{21}$$

for $0 < t < 1$.

It is easy to see that

$$\left(\frac{1}{t}\right)^{(k)} = \frac{(-1)^k k!}{t^{k+1}} \quad \text{and} \quad \big(\sqrt{t}\,\big)^{(k)} = \frac{(-1)^{k+1}(2k-1)!!}{2^k(2k-1)t^{k-1/2}} \tag{22}$$

for $k \in \mathbb{N}$. Then, by Leibniz's Theorem [1, p. 12, 3.38] for differentiation of a product, we gain that

$$\big[\big(1+\sqrt{t}\,\big)\ln t\big]^{(k)} = \big(1+\sqrt{t}\,\big)\left(\frac{1}{t}\right)^{(k-1)} + \big(1+\sqrt{t}\,\big)^{(k)}\ln t$$

$$+\sum_{i=1}^{k-1}\binom{k}{i}\bigl(1+\sqrt{t}\,\bigr)^{(i)}\biggl(\frac{1}{t}\biggr)^{(k-i-1)}=\bigl(1+\sqrt{t}\,\bigr)\frac{(-1)^{k-1}(k-1)!}{t^k}$$
$$+\bigl(\sqrt{t}\,\bigr)^{(k)}\ln t+\frac{(-1)^k k!}{t^{k-1/2}}\sum_{i=1}^{k-1}\frac{(2i-1)!!}{(2i)!!(2i-1)(k-i)},$$

where, and elsewhere in this paper, an empty sum is understood to be nil. Thus,

$$\bigl[\bigl(1+\sqrt{t}\,\bigr)\ln t\bigr]^{(k)}\Bigr|_{t=1}=(-1)^k k!\left[\sum_{i=1}^{k-1}\frac{(2i-1)!!}{(2i)!!(2i-1)(k-i)}-\frac{2}{k}\right] \tag{23}$$

for $k\in\mathbb{N}$. Hence,

$$\bigl(1+\sqrt{t}\,\bigr)\ln t=\sum_{k=1}^{\infty}\left[\sum_{i=1}^{k-1}\frac{(2i-1)!!}{(2i)!!(2i-1)(k-i)}-\frac{2}{k}\right](1-t)^k \tag{24}$$

which can be reduced by replacing $\sqrt{t}$ by t to

$$(1+t)\ln t=\sum_{k=1}^{\infty}\left[\frac{1}{2}\sum_{i=1}^{k-1}\frac{(2i-1)!!}{(2i)!!(2i-1)(k-i)}-\frac{1}{k}\right]\bigl(1-t^2\bigr)^k, \tag{25}$$

and so

$$\begin{aligned}\frac{\ln t}{t-1}&=\frac{(1+t)\ln t}{t^2-1}=\sum_{k=1}^{\infty}\left[\frac{1}{k}-\frac{1}{2}\sum_{i=1}^{k-1}\frac{(2i-1)!!}{(2i)!!(2i-1)(k-i)}\right]\bigl(1-t^2\bigr)^{k-1}\\&=\sum_{k=0}^{\infty}\left[\frac{1}{k+1}-\frac{1}{2}\sum_{i=1}^{k}\frac{(2i-1)!!}{(2i)!!(2i-1)(k-i+1)}\right]\bigl(1-t^2\bigr)^k\triangleq\sum_{k=0}^{\infty}a_k\bigl(1-t^2\bigr)^k\end{aligned}$$

for $0<t<1$.

The two identities in Lemma 3 and the equality in the right-hand side of the inequality (19) give

$$\begin{aligned}\frac{1}{k+1}-a_k&=\frac{1}{2}\sum_{i=1}^{k}\frac{(2i-1)!}{2^{i-1}(i-1)!2^i i!(2i-1)(k-i+1)}\\&=\sum_{i=1}^{k}\binom{2i-2}{i-1}\frac{1}{2^{2i}i(k-i+1)}\\&=\frac{1}{k+1}\sum_{i=1}^{k}\binom{2i-2}{i-1}\frac{1}{4^i}\biggl(\frac{1}{i}+\frac{1}{k-i+1}\biggr)\\&=\frac{1}{k+1}\sum_{i=1}^{k}\binom{2i-2}{i-1}\frac{1}{4^i i}+\frac{1}{k+1}\sum_{i=1}^{k}\binom{2i-2}{i-1}\frac{1}{4^i(k-i+1)}\\&=\frac{1}{k+1}\left[\frac{1}{2}-\frac{2}{4^{k+1}}\binom{2k}{k}\right]+\frac{1}{k+1}\sum_{i=1}^{k}\binom{2i-2}{i-1}\frac{1}{4^i(k-i+1)}\\&=\frac{1}{k+1}\left\{\frac{1}{2}-\frac{2}{4^{k+1}}\binom{2k}{k}+\frac{[2\ln 2+\gamma+\psi(k+1/2)]\Gamma(k+1/2)}{4\sqrt{\pi}\,\Gamma(k+1)}\right\}\end{aligned}$$

$$= \frac{1}{k+1}\left[\frac{1}{2} - \frac{2-2\ln 2-\gamma-\psi(k+1/2)}{4\sqrt{\pi}} \cdot \frac{\Gamma(k+1/2)}{\Gamma(k+1)}\right],$$

that is,

$$a_k = \frac{1}{k+1}\left[\frac{1}{2} + \frac{2-2\ln 2-\gamma-\psi(k+1/2)}{4\sqrt{\pi}} \cdot \frac{\Gamma(k+1/2)}{\Gamma(k+1)}\right], \quad k \in \mathbb{N}.$$

It is listed in [1, p. 258, 6.3.4] that

$$\psi\left(n+\frac{1}{2}\right) = -\gamma - 2\ln 2 + 2\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right), \quad n \ge 1. \tag{26}$$

Hence,

$$a_k = \frac{1}{2(k+1)}\left[1 - \frac{\Gamma(k+1/2)}{\sqrt{\pi}\,\Gamma(k+1)}\left(\sum_{i=1}^{k}\frac{1}{2i-1} - 1\right)\right], \quad k \in \mathbb{N}. \tag{27}$$

Now let us discuss the increasingly monotonic property of the ratio $\frac{a_k}{b_k}$ for $k \in \mathbb{N}$. It is clear that

$$\frac{a_k}{b_k} \le \frac{a_{k+1}}{b_{k+1}} \iff \frac{b_{k+1}}{b_k} \le \frac{a_{k+1}}{a_k} \iff \left(\frac{2k+1}{2k+2}\right)^2 \le \frac{a_{k+1}}{a_k} \tag{28}$$

which is equivalent to

$$\left[\sum_{i=2}^{k+1}\frac{1}{2i-1} - \frac{(k+1/2)(k+2)}{(k+1)^2}\sum_{i=2}^{k}\frac{1}{2i-1}\right]\frac{\Gamma(k+3/2)}{\sqrt{\pi}\,\Gamma(k+2)} \le 1 - \frac{(k+1/2)^2(k+2)}{(k+1)^3} \tag{29}$$

for $k \ge 2$. Furthermore, easy calculation gives

$$\begin{aligned} &\sum_{i=2}^{k+1}\frac{1}{2i-1} - \frac{(k+1/2)(k+2)}{(k+1)^2}\sum_{i=2}^{k}\frac{1}{2i-1} \\ &= \frac{1}{2k+1} + \left[1 - \frac{(k+1/2)(k+2)}{(k+1)^2}\right]\sum_{i=2}^{k}\frac{1}{2i-1} \\ &= \frac{1}{2k+1} - \frac{k}{2(k+1)^2}\sum_{i=2}^{k}\frac{1}{2i-1} \\ &= \frac{k}{2(k+1)^2}\left[\frac{2(k+1)^2}{k(2k+1)} - \sum_{i=2}^{k}\frac{1}{2i-1}\right]. \end{aligned}$$

Since the sequence $\frac{2(k+1)^2}{k(2k+1)}$ for $k \ge 2$ is strictly decreasing and tends to 1 as $k \to \infty$ and the sequence $\sum_{i=2}^{k}\frac{1}{2i-1}$ for $k \ge 2$ is strictly increasing and diverges to ∞, the sequence

$$S_k \triangleq \frac{2(k+1)^2}{k(2k+1)} - \sum_{i=2}^{k}\frac{1}{2i-1} \tag{30}$$

for $k \ge 2$ is strictly decreasing and diverges to $-\infty$. As a result, there exists an integer $k_0 \ge 2$ such that the sequence S_k is negative for all $k \ge k_0$. From the fact that $S_9 = 0.01\cdots$ and $S_{10} = -0.04\cdots$, it follows that $k_0 = 10$. Therefore, considering the facts that

$$\frac{(k+1/2)^2(k+2)}{(k+1)^3} \le 1$$

and

$$\frac{\Gamma(k+3/2)}{\sqrt{\pi}\,\Gamma(k+2)} > 0$$

for $k \ge 2$, it readily follows that the inequality (29) holds for all $k \ge 10$.

Straightforward computations reveal that

k	1	2	3	4	5	6	7	8	9	10
a_k	$\frac{1}{4}$	$\frac{7}{48}$	$\frac{5}{48}$	$\frac{313}{3840}$	$\frac{43}{640}$	$\frac{12317}{215040}$	$\frac{10751}{215040}$	$\frac{183349}{4128768}$	$\frac{206329}{5160960}$	$\frac{66087019}{1816657920}$

and that

k	1	2	3
$\frac{a_{k+1}}{a_k}$	$\frac{7}{12} = 0.583\cdots$	$\frac{5}{7} = 0.714\cdots$	$\frac{313}{400} = 0.782\cdots$
$\left(\frac{2k+1}{2k+2}\right)^2$	$\frac{9}{16} = 0.562\cdots$	$\frac{25}{36} = 0.694\cdots$	$\frac{49}{64} = 0.765\cdots$
k	4	5	6
$\frac{a_{k+1}}{a_k}$	$\frac{258}{313} = 0.824\cdots$	$\frac{12317}{14448} = 0.852\cdots$	$\frac{10751}{12317} = 0.872\cdots$
$\left(\frac{2k+1}{2k+2}\right)^2$	$\frac{81}{100} = 0.810\cdots$	$\frac{121}{144} = 0.840\cdots$	$\frac{169}{196} = 0.862\cdots$
k	7	8	9
$\frac{a_{k+1}}{a_k}$	$\frac{916745}{1032096} = 0.888\cdots$	$\frac{825316}{916745} = 0.900\cdots$	$\frac{66087019}{72627808} = 0.909\cdots$
$\left(\frac{2k+1}{2k+2}\right)^2$	$\frac{225}{256} = 0.878\cdots$	$\frac{289}{324} = 0.892\cdots$	$\frac{361}{400} = 0.902\cdots$

Consequently, the inequality (29) holds for all $k \in \mathbb{N}$. This demonstrates that the ratio $\frac{a_k}{b_k}$ for $k \in \mathbb{N}$ is strictly increasing. By virtue of Lemma 1, it is obtained that the function

$$\frac{(\ln t)/(t-1)}{(2/\pi)\int_0^{\pi/2} 1\Big/\sqrt{1-(1-t^2)\sin^2\theta}\,\mathrm{d}\theta} \tag{31}$$

is strictly decreasing in $t \in (0,1)$. The well-known L'Hôpital's rule yields that the limits of the function (31) are $\frac{\pi}{2}$ and 1 as t tends to 0^+ and 1^- respectively. Hence, the double inequality

$$\frac{2}{\pi}\cdot\frac{\ln t}{t-1} < \frac{2}{\pi}\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{1-(1-t^2)\sin^2\theta}} < \frac{\ln t}{t-1} \tag{32}$$

for $t \in (0,1)$ is valid and sharp. Letting $t = \frac{a}{b}$ for $b > a > 0$ in (32) leads to

$$\frac{2}{\pi}\cdot\frac{\ln a - \ln b}{a-b} < \frac{2}{\pi}\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} < \frac{\ln a - \ln b}{a-b}. \tag{33}$$

It is easy to see that the inequality (33) is valid for all positive numbers a and b with $a \ne b$. This implies that the double inequality (11) is valid for all positive numbers a and b with $a \ne b$ if and only if $\alpha \le \frac{2}{\pi}$ and $\beta \ge 1$. The proof of Theorem 1 is complete.

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