EXTENDED TRIAL EQUATION METHOD AND APPLICATIONS TO SOME NONLINEAR PROBLEMS

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In this paper, we introduce the extended trial equation method for solving non-integrable partial differential equations in mathematical physics. Some exact solutions including soliton solutions, rational, elliptic integral function, Jacobi elliptic function solutions to the $(N+1)$-dimensional sine-cosine-Gordon equation and the KdV equation with dual power law nonlinearity are obtained by this method. Also, a more general version of the extended trial equation method is proposed.

Keywords: Extended trial equation method, sine-cosine-Gordon equation, KdV equation, elliptic integral function solutions.

MSC2000: 35Q51, 47J35, 74J35.

1. Introduction

Constructing exact solutions to partial differential equations is an important problem in nonlinear science. In order to obtain the exact solutions of nonlinear partial differential equations, various methods have been presented, such as Hirota method [1], tanh-coth method [2, 3], the solitary wave ansatz method [4], (G'/G)-expansion method [5, 6, 7], the trial equation method [8]-[16], Riccati equation method [17, 18], and so on. There are a lot of nonlinear evolution equations that are integrated using these and other mathematical methods. Soliton solutions, compactons, peakons, cuspons, stumpons, cnoidal waves, singular solitons and other solutions have been found [19, 20]. These types of solutions are very important and appear in various areas of physics, applied mathematics.

In Section 2, we give a new version of the trial equation method for nonlinear differential equations with generalized evolution. In Section 3, as applications, we obtain some exact solutions to two nonlinear problems with higher nonlinear terms such as the $(N+1)$-dimensional sine-cosine-Gordon equation [21]

$$\sum_{j=1}^{N} u_{x_j x_j} - u_{tt} - \alpha\cos(u) - \beta\sin(2u) = 0, \tag{1.1}$$

the KdV equation with dual power law nonlinearity [13]

$$u_t + \alpha u^p u_x + \beta u^{2p} u_x + \gamma u_{xxx} = 0, \tag{1.2}$$

In Discussion, we propose a more general trial equation method.

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2. **The extended trial equation method**

Step 1. We consider the most general form of the nonlinear partial differential equations

$$P(u, u_t, u_x, u_{xx}, \dots) = 0, \tag{2.1}$$

and use the wave transformation

$$u(x_1, x_1, \dots, x_N, t) = u(\eta), \quad \eta = \lambda \left(\sum_{j=1}^{N} x_j - ct \right), \tag{2.2}$$

where $\lambda \neq 0$ and $c \neq 0$. Substituting Eq. (2.2) into Eq. (2.1) yields the following nonlinear ordinary differential equation

$$N(u, u', u'', ...) = 0. \tag{2.3}$$

Step 2. We assume that the exact solutions to Eq. (2.3) can be obtained by

$$u = \sum_{i=0}^{\delta} \tau_i \Gamma^i, \tag{2.4}$$

where

$$(\Gamma')^2 = \Lambda(\Gamma) = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} = \frac{\xi_\theta \Gamma^\theta + ... + \xi_1 \Gamma + \xi_0}{\zeta_\epsilon \Gamma^\epsilon + ... + \zeta_1 \Gamma + \zeta_0}. \tag{2.5}$$

Using the relations (2.4) and (2.5), we can derive the terms $(u')^2$ and u'' as

$$(u')^2 = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} \left(\sum_{i=0}^{\delta} i\tau_i \Gamma^{i-1} \right)^2, \tag{2.6}$$

$$u'' = \frac{\Phi'(\Gamma)\Psi(\Gamma) - \Phi(\Gamma)\Psi'(\Gamma)}{2\Psi^2(\Gamma)} \left(\sum_{i=0}^{\delta} i\tau_i \Gamma^{i-1} \right) + \frac{\Phi(\Gamma)}{\Psi(\Gamma)} \left(\sum_{i=0}^{\delta} i(i-1)\tau_i \Gamma^{i-2} \right), \tag{2.7}$$

where $\Phi(\Gamma)$ and $\Psi(\Gamma)$ are polynomials of Γ. Substituting Eqs. (2.6) and (2.7) into Eq. (2.3) yields an algebraic equation of polynomial $\Omega(\Gamma)$ of Γ :

$$\Omega(\Gamma) = \varrho_s \Gamma^s + ... + \varrho_1 \Gamma + \varrho_0 = 0. \tag{2.8}$$

According to the balance principle of this method, we can find a relation in determining the values of θ, ϵ, and δ. From here, we choose the appropriate values of θ, ϵ, and δ.

Step 3. Equating all coefficients of $\Omega(\Gamma)$ to zero yields a system of algebraic equations containing free parameters as follows:

$$\varrho_i = 0, \qquad i = 0, ..., s. \tag{2.9}$$

Solving the system (2.9) with the aid of Mathematica, we determine the values of $\xi_0, ..., \xi_\theta$; $\zeta_0, ..., \zeta_\epsilon$ and $\tau_0, ..., \tau_\delta$.

Step 4. Reduce Eq. (2.5) to the elementary integral form,

$$\pm(\eta - \eta_0) = \int \frac{d\Gamma}{\sqrt{\Lambda(\Gamma)}} = \int \sqrt{\frac{\Psi(\Gamma)}{\Phi(\Gamma)}} d\Gamma. \tag{2.10}$$

Using a complete discrimination system for polynomial to classify the roots of $\Phi(\Gamma)$, we solve the indefinite integral (2.10) and obtain the exact solutions to Eq. (2.3). Furthermore, we can write the exact traveling wave solutions to Eq. (2.1) respectively.

3. Applications

Now, we apply the extended trial equation method to the problems (1.1) and (1.2). Then, we compare the solutions with the exact solutions obtained in literature.

3.1. Application to the $(N+1)$-dimensional sine-cosine-Gordon equation

We consider the traveling wave transformation (2.2), and apply this to Eq. (1.1). Thus, we can write the following nonlinear ordinary differential equation

$$\lambda^2(N-c^2)u'' - \alpha\cos(u) - \beta\sin(2u) = 0, \tag{3.1}$$

where the prime denotes derivative with respect to η. Take the transformation of trigonometric function

$$u = 2\tan^{-1} v, \tag{3.2}$$

then we obtain the following relations, respectively:

$$u'' = \frac{2(v'' + v''v^2 - 2(v')^2 v)}{(1+v^2)^2}, \tag{3.3}$$

$$\cos(u) = \frac{1-v^2}{1+v^2}, \tag{3.4}$$

$$\sin(2u) = \frac{4v(1-v^2)}{(1+v^2)^2}. \tag{3.5}$$

Substituting Eqs. (3.3)-(3.5) in Eq. (3.1), we can get the nonlinear ordinary differential equation

$$2\lambda^2(N-c^2)(1+v^2)v'' - 4\lambda^2(N-c^2)v(v')^2 + (v^2-1)(\alpha v^2 + 4\beta v + \alpha) = 0. \tag{3.6}$$

Substituting Eqs. (2.6) and (2.7) into Eq. (3.6) and using the balance principle yield

$$\theta = \epsilon + \delta + 2.$$

If we take $\theta = 3$, $\epsilon = 0$ and $\delta = 1$, then

$$(v')^2 = \frac{\tau_1^2(\xi_3\Gamma^3 + \xi_2\Gamma^2 + \xi_1\Gamma + \xi_0)}{\zeta_0}, \tag{3.7}$$

$$v'' = \frac{\tau_1(3\xi_3\Gamma^2 + 2\xi_2\Gamma + \xi_1)}{2\zeta_0}, \tag{3.8}$$

where $\xi_3 \neq 0$, $\zeta_0 \neq 0$. The system of algebraic equations, which is obtained by substituting Eqs. (3.7)-(3.8) into Eq. (3.6), is solved by the Mathematica. Therefore, we respectively compute the following variables.

$$\xi_0 = \frac{\xi_2\tau_0(\alpha + \alpha\tau_0^2 + 2\beta\tau_0)}{\tau_1^2(3\alpha\tau_0 + 2\beta)}, \quad \xi_1 = \frac{\xi_2(\alpha + 3\alpha\tau_0^2 + 4\beta\tau_0)}{\tau_1(3\alpha\tau_0 + 2\beta)}, \tag{3.9}$$

$$\xi_3 = \frac{\alpha\xi_2\tau_1}{3\alpha\tau_0 + 2\beta}, \quad c = \pm\frac{1}{\lambda}\sqrt{\frac{\lambda^2\xi_2 N - 3\alpha\zeta_0\tau_0 - 2\beta\zeta_0}{\xi_2}}, \tag{3.10}$$

where ξ_2, τ_0, τ_1 and ζ_0 are free parameters. Substituting these results into Eqs. (2.5) and (2.10), we have

$$\pm(\eta - \eta_0) = \sqrt{\frac{\zeta_0(3\alpha\tau_0 + 2\beta)}{\alpha\xi_2\tau_1}} \int \frac{d\Gamma}{\sqrt{\Gamma^3 + \frac{3\alpha\tau_0 + 2\beta}{\alpha\tau_1}\Gamma^2 + \frac{\alpha + 3\alpha\tau_0^2 + 4\beta\tau_0}{\alpha\tau_1^2}\Gamma + \frac{\tau_0(\alpha + \alpha\tau_0^2 + 2\beta\tau_0)}{\alpha\tau_1^3}}} \tag{3.11}$$

Integrating Eq. (3.11), we obtain the solutions to the Eq. (1.1) as follows:

$$\pm(\eta-\eta_0) = -2\sqrt{A}\frac{1}{\sqrt{\Gamma-\alpha_1}}, \tag{3.12}$$

$$\pm(\eta-\eta_0) = 2\sqrt{\frac{A}{\alpha_2-\alpha_1}}\arctan\sqrt{\frac{\Gamma-\alpha_2}{\alpha_2-\alpha_1}}, \quad \alpha_2>\alpha_1, \tag{3.13}$$

$$\pm(\eta-\eta_0) = \sqrt{\frac{A}{\alpha_1-\alpha_2}}\ln\left|\frac{\sqrt{\Gamma-\alpha_2}-\sqrt{\alpha_1-\alpha_2}}{\sqrt{\Gamma-\alpha_2}+\sqrt{\alpha_1-\alpha_2}}\right|, \quad \alpha_1>\alpha_2, \tag{3.14}$$

$$\pm(\eta-\eta_0) = 2\sqrt{\frac{A}{\alpha_1-\alpha_3}}F(\varphi,l), \quad \alpha_1>\alpha_2>\alpha_3, \tag{3.15}$$

where

$$A=\frac{\zeta_0(3\alpha\tau_0+2\beta)}{\alpha\xi_2\tau_1}, \quad F(\varphi,l)=\int_0^{\varphi}\frac{d\psi}{\sqrt{1-l^2\sin^2\psi}}, \tag{3.16}$$

and

$$\varphi=\arcsin\sqrt{\frac{\Gamma-\alpha_3}{\alpha_2-\alpha_3}}, \quad l^2=\frac{\alpha_2-\alpha_3}{\alpha_1-\alpha_3}. \tag{3.17}$$

Here, α_1, α_2 and α_3 are the roots of the polynomial equation

$$\Gamma^3+\frac{\xi_2}{\xi_3}\Gamma^2+\frac{\xi_1}{\xi_3}\Gamma+\frac{\xi_0}{\xi_3}=0. \tag{3.18}$$

Substituting the solutions (3.12-3.15) into (2.4) and (3.2), we have

$$u(x_1,x_2,\ldots,x_N,t)=2\arctan\left\{\tau_0+\tau_1\alpha_1+\frac{4\tau_1 A}{\lambda^2\left(\sum_{j=1}^{N}x_j-vt-\frac{\eta_0}{\lambda}\right)^2}\right\}, \tag{3.19}$$

$$u(x_1,x_2,\ldots,x_N,t)=2\arctan\left\{\tau_0+\tau_1\alpha_2-A_1\tanh^2\left[\pm B_1\left(\sum_{j=1}^{N}x_j-vt-\frac{\eta_0}{\lambda}\right)\right]\right\}, \tag{3.20}$$

$$u(x_1,x_2,\ldots,x_N,t)=2\arctan\left\{\tau_0+\tau_1\alpha_1+A_2\text{cosech}^2\left[B_1\left(\sum_{j=1}^{N}x_j-vt\right)\right]\right\}, \tag{3.21}$$

$$u(x_1,x_2,\ldots,x_N,t)=2\arctan\left\{\tau_0+\tau_1\alpha_1+A_3\text{sn}^2\left[\mp B_2\left(\sum_{j=1}^{N}x_j-vt-\frac{\eta_0}{\lambda}\right),l^2\right]\right\}, \tag{3.22}$$

where $A_1=\tau_1(\alpha_2-\alpha_1)$, $A_2=\tau_1(\alpha_1-\alpha_2)$, $A_3=\tau_1(\alpha_2-\alpha_3)$, $B_1=\frac{\lambda}{2}\sqrt{\frac{\alpha_1-\alpha_2}{A}}$, $B_2=\frac{\lambda}{2}\sqrt{\frac{\alpha_1-\alpha_3}{A}}$ and $v=\pm\frac{1}{\lambda}\sqrt{\frac{\lambda^2\xi_2N-3\alpha\zeta_0\tau_0-2\beta\zeta_0}{\xi_2}}$. If we choose $\tau_0=-\tau_1\alpha_1$ and $\eta_0=0$, then the solutions (3.19)-(3.21) can be reduced to rational function solution

$$u(x_1,x_2,\ldots,x_N,t)=2\arctan\left\{\frac{4\tau_1 A}{\lambda^2\left(\sum_{j=1}^{N}x_j-vt\right)^2}\right\}, \tag{3.23}$$

1-soliton solution

$$u(x_1,x_2,\ldots,x_N,t)=2\arctan\left\{\frac{A_1}{\cosh^2\left[\mp B_1\left(\sum_{j=1}^{N}x_j-vt\right)\right]}\right\},\tag{3.24}$$

and singular soliton solution

$$u(x_1,x_2,\ldots,x_N,t)=2\arctan\left\{\frac{A_2}{\sinh^2\left[B_1\left(\sum_{j=1}^{N}x_j-vt\right)\right]}\right\}.\tag{3.25}$$

Here, A_1 and A_2 are respectively the amplitudes of 1-soliton and singular soliton, while v is the velocity of these solitons and B_1 is the inverse width of the solitons. Thus, we can say that the solitons exist for $\tau_1>0$.
Furthermore, when $\tau_0=-\tau_1\alpha_3$ and $\eta_0=0$, then the Jacobi elliptic function solution(3.22) can be simplified as

$$u(x_1,x_2,\ldots,x_N,t)=2\arctan\left\{A_3\mathrm{sn}^2\left[\mp B_2\left(\sum_{j=1}^{N}x_j-vt\right),l^2\right]\right\}.\tag{3.26}$$

Remark 3.1. The traveling wave solutions (3.23)-(3.26) found by the extended trial equation method for Eq. (1.1) have been checked by Mathematica. To our knowledge, the rational function solution and the Jacobi elliptic function solutions obtained in this paper are not given in the previous literature. These are new traveling wave solutions of Eq. (1.1).

Remark 3.2. If the modulus $l\to 1$, the solution (3.26) can be reduced to the following exact solutions of the $(N+1)$-dimensional sine-cosine-Gordon equation

$$u(x_1,x_2,\ldots,x_N,t)=2\arctan\left\{A_3\tanh^2\left[\mp B_2\left(\sum_{j=1}^{N}x_j-vt\right)\right]\right\},\tag{3.27}$$

where $\alpha_1=\alpha_2$.

3.2. The KdV equation with dual power law nonlinearity

To find the wave solutions to Eq. (1.2), we use the traveling wave transformation $u(x,t)=u(\eta),\eta=x-ct$, where c is an arbitrary constant. Integrating this nonlinear ordinary differential equation once and equating the integration constant to zero, we obtain

$$-cu+\frac{\alpha}{p+1}u^{p+1}+\frac{\beta}{2p+1}u^{2p+1}+\gamma u''=0,\tag{3.28}$$

where p is a positive integer and α,β,γ are free parameters.
Eq. (3.28), with the transformation

$$u=v^{\frac{1}{p}},\tag{3.29}$$

reduces to

$$Mvv''+N(v')^2-cPv^2+Rv^3+Tv^4=0,\tag{3.30}$$

where $M=\gamma p(p+1)(2p+1)$, $N=\gamma(1-p^2)(2p+1)$, $P=p^2(p+1)(2p+1)$, $R=\alpha p^2(2p+1)$ and $T=\beta p^2(p+1)$. Substituting Eqs. (2.6) and (2.7) into Eq. (3.30) and using the balance procedure yields

$$\theta=\epsilon+2\delta+2.$$

If we get $\theta=4$, $\epsilon=0$ and $\delta=1$, then

$$(v')^2=\frac{\tau_1^2(\xi_4\Gamma^4+\xi_3\Gamma^3+\xi_2\Gamma^2+\xi_1\Gamma+\xi_0)}{\zeta_0}, \tag{3.31}$$

$$v''=\frac{\tau_1(4\xi_4\Gamma^3+3\xi_3\Gamma^2+2\xi_2\Gamma+\xi_1)}{2\zeta_0}, \tag{3.32}$$

where $\xi_4\neq 0$, $\zeta_0\neq 0$. Substituting Eqs. (3.31) and (3.32) into Eq. (3.30), we have a system of algebraic equations. Then, we solve this system by using of the Mathematica and compute the following results:

$$\xi_0=\frac{\tau_0^2}{\tau_1^4}\left[\xi_2\tau_1^2-\xi_4\tau_0\left(5\tau_0+\frac{4R(2M+N)}{T(3M+2N)}\right)\right],\ \xi_1=\frac{2\tau_0}{\tau_1^3}\left[\xi_2\tau_1^2-\xi_4\tau_0\left(4\tau_0+\frac{3R(2M+N)}{T(3M+2N)}\right)\right],$$

$$\xi_3=\frac{2\xi_4(2MR+NR+6\tau_0MT+4\tau_0NT)}{\tau_1T(3M+2N)},\quad \zeta_0=-\frac{\xi_4(2M+N)}{\tau_1^2T},$$

$$c=\frac{M+N}{P}\left(\frac{6\tau_0R}{3M+2N}+\frac{T(6\xi_4\tau_0^2-\xi_2\tau_1^2)}{\xi_4(2M+N)}\right),$$

where ξ_2, ξ_4, τ_0 and τ_1 are free parameters. Substituting these results into Eq. (2.5) and Eq. (2.10), we have

$$\pm(\eta-\eta_0)=\sqrt{-\frac{2M+N}{\tau_1^2T}}\int\frac{d\Gamma}{\sqrt{\Gamma^4+\frac{\xi_3}{\xi_4}\Gamma^3+\frac{\xi_2}{\xi_4}\Gamma^2+\frac{\xi_1}{\xi_4}\Gamma+\frac{\xi_0}{\xi_4}}} \tag{3.33}$$

Integrating Eq. (3.33), we obtain the solutions to Eq. (1.2), respectively:

When $\Lambda(\Gamma)=(\Gamma-\alpha_1)^4$, then

$$\pm(\eta-\eta_0)=-\frac{B}{\Gamma-\alpha_1}, \tag{3.34}$$

If we take $\Lambda(\Gamma)=(\Gamma-\alpha_1)^3(\Gamma-\alpha_2)$ and $\alpha_2>\alpha_1$, then

$$\pm(\eta-\eta_0)=\frac{2B}{\alpha_1-\alpha_2}\sqrt{\frac{\Gamma-\alpha_2}{\Gamma-\alpha_1}}, \tag{3.35}$$

If we choose $\Lambda(\Gamma)=(\Gamma-\alpha_1)^2(\Gamma-\alpha_2)^2$ and $\alpha_1>\alpha_2$, then

$$\pm(\eta-\eta_0)=\frac{B}{\alpha_1-\alpha_2}\ln\left|\frac{\Gamma-\alpha_1}{\Gamma-\alpha_2}\right|, \tag{3.36}$$

When $\Lambda(\Gamma)=(\Gamma-\alpha_1)^2(\Gamma-\alpha_2)(\Gamma-\alpha_3)$ and $\alpha_1>\alpha_2>\alpha_3$, then

$$\pm(\eta-\eta_0)=\frac{B}{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}\ln\left|\frac{\sqrt{(\Gamma-\alpha_2)(\alpha_1-\alpha_3)}-\sqrt{(\Gamma-\alpha_3)(\alpha_1-\alpha_2)}}{\sqrt{(\Gamma-\alpha_2)(\alpha_1-\alpha_3)}+\sqrt{(\Gamma-\alpha_3)(\alpha_1-\alpha_2)}}\right|, \tag{3.37}$$

When $\Lambda(\Gamma)=(\Gamma-\alpha_1)(\Gamma-\alpha_2)(\Gamma-\alpha_3)(\Gamma-\alpha_4)$ and $\alpha_1>\alpha_2>\alpha_3>\alpha_4$, then

$$\pm(\eta-\eta_0)=2\sqrt{\frac{B}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}F(\varphi,l), \tag{3.38}$$

where

$$B=\sqrt{-\frac{2M+N}{\tau_1^2 T}},\quad F(\varphi,l)=\int_0^{\varphi}\frac{d\psi}{\sqrt{1-l^2\sin^2\psi}},\tag{3.39}$$

and

$$\varphi=\arcsin\sqrt{\frac{(\Gamma-\alpha_1)(\alpha_2-\alpha_4)}{(\Gamma-\alpha_2)(\alpha_1-\alpha_4)}},\quad l^2=\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}.\tag{3.40}$$

Also α_1, α_2, α_3 and α_4 are the roots of the polynomial equation

$$\Gamma^4+\frac{\xi_3}{\xi_4}\Gamma^3+\frac{\xi_2}{\xi_4}\Gamma^2+\frac{\xi_1}{\xi_4}\Gamma+\frac{\xi_0}{\xi_4}=0.\tag{3.41}$$

Substituting the solutions (3.34-3.38) into (2.4) and (3.29), we have

$$u(x,t)=\left\{\tau_0+\tau_1\alpha_1+\frac{\tau_1 B^2}{(x-ct-\eta_0)^2}\right\}^{\frac{1}{p}},\tag{3.42}$$

$$u(x,t)=\left\{\tau_0+\tau_1\alpha_1+\frac{4B^2(\alpha_2-\alpha_1)\tau_1}{4B^2-[(\alpha_1-\alpha_2)(x-ct-\eta_0)]^2}\right\}^{\frac{1}{p}},\tag{3.43}$$

$$u(x,t)=\left\{\tau_0+\tau_1\alpha_2+\frac{(\alpha_2-\alpha_1)\tau_1}{\exp\left[B_1(x-ct-\eta_0)\right]-1}\right\}^{\frac{1}{p}},\tag{3.44}$$

$$u(x,t)=\left\{\tau_0+\tau_1\alpha_1+\frac{(\alpha_1-\alpha_2)\tau_1}{\exp\left[B_1(x-ct-\eta_0)\right]-1}\right\}^{\frac{1}{p}},\tag{3.45}$$

$$u(x,t)=\left\{\tau_0+\tau_1\alpha_1-\frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\tau_1}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh\left[B_2(x-ct)\right]}\right\}^{\frac{1}{p}},\tag{3.46}$$

$$u(x,t)=\left\{\tau_0+\tau_1\alpha_2+\frac{(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)\tau_1}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)\mathrm{sn}^2\left[\mp B_3(x-ct-\eta_0),l^2\right]}\right\}^{\frac{1}{p}},\tag{3.47}$$

where

$$B_1=\frac{\alpha_1-\alpha_2}{B},\quad B_2=\frac{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}{B},\quad B_3=\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2B},$$

and

$$c=\frac{M+N}{P}\left(\frac{6\tau_0 R}{3M+2N}+\frac{T(6\xi_4\tau_0^2-\xi_2\tau_1^2)}{\xi_4(2M+N)}\right),\quad l^2=\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}.$$

If we take $\tau_0=-\tau_1\alpha_1$ and $\eta_0=0$, then the solutions (3.42)-(3.46) can reduce to rational function solutions

$$u(x,t)=\left[\frac{B\sqrt{\tau_1}}{x-ct}\right]^{\frac{2}{p}},\tag{3.48}$$

$$u(x,t)=\left\{\frac{4B^2(\alpha_2-\alpha_1)\tau_1}{4B^2-[(\alpha_1-\alpha_2)(x-ct)]^2}\right\}^{\frac{1}{p}},\tag{3.49}$$

traveling wave solutions

$$u(x,t)=\left\{\frac{(\alpha_2-\alpha_1)\tau_1}{2}\left\{1\mp\coth\left[\frac{\alpha_1-\alpha_2}{2B}(x-ct)\right]\right\}\right\}^{\frac{1}{p}},\tag{3.50}$$

and soliton solution

$$u(x,t)=\frac{A_3}{\left(D+\cosh\left[B_2(x-ct)\right]\right)^{\frac{1}{p}}},\tag{3.51}$$

where $A_3 = \left(\frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\tau_1}{\alpha_3-\alpha_2}\right)^{\frac{1}{p}}$, $D = \frac{2\alpha_1-\alpha_2-\alpha_3}{\alpha_3-\alpha_2}$. Here, A_3 and c are respectively the amplitude and velocity of the soliton, while B_2 is the inverse width of the soliton. Thus, we can say that the solitons exist for $\tau_1 < 0$. Furthermore, for $\tau_0 = -\tau_1\alpha_2$ and $\eta_0 = 0$, the Jacobi elliptic function solution (3.47) can be reduced to the form

$$u(x,t) = \frac{A_4}{\left(D_1 + \mathrm{sn}^2\left[\mp B_3(x-ct), \frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}\right]\right)^{\frac{1}{p}}} \tag{3.52}$$

where $A_4 = \left(\frac{\tau_1(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_1-\alpha_4}\right)^{\frac{1}{p}}$ and $D_1 = \frac{\alpha_4-\alpha_2}{\alpha_1-\alpha_4}$.

Remark 3.3. All exact solutions obtained for Eq. (1.2) have been checked by Mathematica. Also, for the corresponding values of some parameters, the soliton solution (3.51) is in full agree with the solution obtained in Ref. [13]. The exact solutions Eqs. (3.48)-(3.50) and (3.52) are not shown in the previous literature.

Remark 3.4. If we choose $l \to 1$, $\alpha_3 = \alpha_4$ and $l \to 0$, $\alpha_2 = \alpha_3$, the Jacobi elliptic function solutions can be written as follows, respectively:

$$u(x,t) = \frac{A_4}{\left(D_1 + \tanh^2\left[\mp B_3(x-ct)\right]\right)^{\frac{1}{p}}} \tag{3.53}$$

and

$$u(x,t) = \frac{A_4}{\left(D_1 + \sin^2\left[\mp B_3(x-ct)\right]\right)^{\frac{1}{p}}}. \tag{3.54}$$

4. **Discussion**

Now, we discuss a more general form of the extended trial equation method in order to solve the nonlinear partial differential equations as follows.

Step 1. The new trial equation can be given in the more general form

$$u = \frac{A(\Gamma)}{B(\Gamma)} = \frac{\sum_{i=0}^{\delta} \tau_i \Gamma^i}{\sum_{j=0}^{\mu} \omega_j \Gamma^j}, \tag{4.1}$$

where

$$(\Gamma')^2 = \Lambda(\Gamma) = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} = \frac{\xi_\theta \Gamma^\theta + ... + \xi_1 \Gamma + \xi_0}{\zeta_\epsilon \Gamma^\epsilon + ... + \zeta_1 \Gamma + \zeta_0}. \tag{4.2}$$

Here, τ_i $(i = 0, ..., \delta)$, ω_j $(j = 0, ..., \mu)$, ξ_ς $(\varsigma = 0, ..., \theta)$ and ζ_σ $(\sigma = 0, ..., \epsilon)$ are the constants of the above equations.

Step 2. Using Eqs. (4.1) and (4.2), we have

$$(u')^2 = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} \frac{\left(A'(\Gamma)B(\Gamma) - A(\Gamma)B'(\Gamma)\right)^2}{B^4(\Gamma)}, \tag{4.3}$$

$$\begin{aligned} u'' &= \frac{\left(A'(\Gamma)B(\Gamma) - A(\Gamma)B'(\Gamma)\right)\left\{(\Phi'(\Gamma)\Psi(\Gamma) - \Phi(\Gamma)\Psi'(\Gamma))B(\Gamma) - 4\Phi(\Gamma)\Psi(\Gamma)B'(\Gamma)\right\}}{2B^3(\Gamma)\Psi^2(\Gamma)} \\ &+ \frac{2\Phi(\Gamma)\Psi(\Gamma)B(\Gamma)(A''(\Gamma)B(\Gamma) - A(\Gamma)B''(\Gamma))}{2B^3(\Gamma)\Psi^2(\Gamma)} \end{aligned} \tag{4.4}$$

and other derivation terms such as u''', and so on.

STEP 3. By substituting $(u')^2$, u'' and other derivation terms into Eq. (2.3), the polynomial algebraic equation can be obtained as

$$\Omega(\Gamma) = \varrho_s \Gamma^s + ... + \varrho_1 \Gamma + \varrho_0 = 0. \tag{4.5}$$

A relationship among the values θ, ϵ, δ and μ can be determined by the balance procedure.

STEP 4. Equating the coefficients of $\Omega(\Gamma)$ to zero yields a system of algebraic equations $\varrho_i = 0$ $(i = 0, ..., s)$. Solving this system by using of the Mathematica, Matlab, and so on, we can determine the values $\tau_0, ...\tau_\delta$; $\omega_0, ..., \omega_\mu$; $\xi_0 ..., \xi_\theta$ and $\zeta_0, ..., \zeta_\epsilon$.

STEP 5. Substituting the values computed in the previous step into Eq. (4.2) and integrating Eq. (4.2), we can classify the traveling wave solutions of Eq. (2.1).

5. Conclusion

In this article, we studied the extended trial equation method as an alternative approach to obtain the exact solutions of nonlinear partial differential equations arising in mathematical physics. We use this method aided with symbolic computation to construct the soliton solutions, the elliptic integral function, Jacobi elliptic function and rational function solutions for the $(N+1)$-dimensional sine-cosine-Gordon equation and the KdV equation with dual power law nonlinearity. The elliptic integral function and Jacobi elliptic function solutions obtained by the present approach are new exact solutions. Also, we propose a more general trial equation method in discussion. We think that the methods presented and proposed in this paper can also be applied to other generalized nonlinear differential equations.

Acknowledgements The research has been supported by the Yozgat University Foundation.

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