NEW CLASSES OF EXPONENTIALLY GENERAL CONVEX FUNCTIONS

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In this paper, we define and introduce some new classes of the exponentially convex functions involving an arbitrary function. We investigate several properties of the exponentially general convex functions and discuss their relations with convex functions. Optimality conditions are characterized by a class of variational inequalities, which is called the exponentially general variational inequality. Several new results characterizing the exponentially general convex functions are obtained. The results obtained in this paper can be viewed as significant improvement of previously known results.

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1. Introduction

Convexity theory describes a broad spectrum of very interesting developments involving a link among various fields of mathematics, physics, economics and engineering sciences. The development of convexity theory can be viewed as the simultaneous pursuit of two different lines of research. On the one hand, it is related to integral inequalities. It has been shown that a function is a convex function, if and only if, it satisfies the Hermite-Hadamard type inequality [9, 10]. These inequalities help us to derive the upper and lower bounds of the integrals. On the other hand, the minimum of the differentiable convex functions on the convex set can be characterized by the variational inequalities. Variational inequalities [37], the origin of which can be traced back to Bernoullis brothers, Euler and Lagrange. Variational inequalities provide us a powerful tool to discuss the behaviour of solutions (regarding its existence, uniqueness and regularity) to important classes of problems. Variational inequality theory also enables us to develop highly efficient powerful new numerical methods to solve nonlinear problems, see [11, 12, 13, 14, 15, 16, 17, 21, 24, 29, 30, 31, 37].
In recent years, various extensions and generalizations of convex functions and convex sets have been considered and studied using innovative ideas and techniques. It is known that more accurate and inequalities can be obtained using the logarithmically convex functions than the convex functions. Closely related to the log-convex functions, we have the concept of exponentially convex(concave) functions, which have important applications in information theory, big data analysis, machine learning and statistic. Exponentially convex functions have appeared significant generalization of the convex functions, the origin of which can be traced back to Bernstein [6]. Avriel[3, 4] introduced the concept of r-convex functions, from which one can deduce the exponentially convex functions. Antczak [2] considered the (r, p) convex functions and discussed their applications in mathematical programming and optimization theory. Awan et al [5] and Pecaric et al.[26, 27, 28]also investigated some classes of

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exponentially convex functions. It is worth mentioning that all these classes of exponentially convex functions have important applications in information sciences, data mining and statistics, see, for example,[1, 2, 3, 4, 5, 6, 7, 15, 25, 26, 27, 28] and the references therein.
It is known that a set may not be convex set. However, a set can be made convex set with respect to an arbitrary function. Motivated by this fact, Youness [38] introduced the concept of general convex set involving an arbitrary function. Noor [15] proved that the minimum of the differentiable general convex function on the general convex set can be characterized by the general variational inequalities, which were introduced by Noor [13, 14] in 1988. The technique of the general variational inequalities can be used to consider the nonsymmetric, odd-order obstacle boundary values problems, which can not studied by the variational inequalities. For the formulation, applications, numerical methods, sensitivity analysis and other aspects of general variational inequalities, see [13, 14, 15, 15, 17, 24, 29, 30, 31, 34] and the references therein.
We would like to point out that the general convex functions and exponentially general convex functions are two distinct generalizations of the convex functions, which have played a crucial and significant role in the development of various branches of pure and applied sciences. It is natural to unify these concepts. Motivated by these facts and observations, we now introduce a new class of convex functions, which is called exponentially general convex functions in involving an arbitrary function. We discuss the basic properties of the exponentially general convex functions. It is has been shown that the exponentially general convex(concave) have nice properties which convex functions enjoy. Several new concepts have been introduced and investigated. We prove that the local minimum of the exponentially general convex functions is also the global minimum. The optimal conditions of the differentiable exponentially convex functions can be characterized by a class of variational inequalities, called the exponentially general variational inequality, which is itself an interesting problem. The ideas and techniques of this paper may be starting point for further research in these diversified areas

2. **Preliminary Results**

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ be the inner product and norm, respectively.

We recall the well known facts and basic concepts.

Definition 2.1. [12].*The set K in H is said to be a convex set, if*

$$u + t(v-u) \in K, \qquad \forall u, v \in K, t \in [0,1].$$

Definition 2.2. [12] *A function F is said to be convex function, if*

$$F((1-t)u+tv) \leq (1-t)F(u) + tF(v), \quad \forall u, v \in K, \quad t \in [0,1]. \tag{1}$$

It is well known that a function F is a convex functions, if and only if, it satisfies the inequality

$$F\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \quad \forall a, b \in I = [0,1], \tag{2}$$

which is known as the Hermite-Hadamard type inequality. Such type of the inequalities provide us with the upper and lower bounds for the mean value integral.
If the convex function F is differentiable, then $u \in K$ is the minimum of the F, if and only if, $u \in K$ satisfies the inequality

$$\langle F'(u), v-u \rangle \geq 0, \quad \forall v \in K. \tag{3}$$

The inequalities of the type (3) are called the variational inequalities, which were introduced and studied by Stampacchia [37] in 1964. For the applications, formulation, sensitivity, dynamical systems, generalizations, and other aspects of the variational inequalities, see [4, 5, 6, 7, 8, 11, 12, 13, 14, 15, 16, 24] and the references therein.

We now define the exponentially convex functions, which are mainly due to Noor and Noor [18, 19, 20, 21, 22, 23].

Definition 2.3. [18, 19, 20, 21] *A function F is said to be exponentially convex function, if*

$$e^{F((1-t)u+tv)} \leq (1-t)e^{F(u)} + te^{F(v)}, \quad \forall u, v \in K, \quad t \in [0,1].$$

It is worth mentioning that Avriel [3, 4] and Antczak [2] introduced the following concept.

Definition 2.4. [3, 4] *A function F is said to be exponentially convex function, if*

$$F((1-t)a + tb) \leq \log[(1-t)e^{F(a)} + te^{F(b)}], \quad \forall a, b \in K, \quad t \in [0,1], \tag{4}$$

Avriel [3, 4] and Antczak [2] discussed the application of the 1-convex functions in the mathematical programming. We note that the Definitions 2.3 and 2.4 are equivalent. A function is called the exponentially concave function f, if $-f$ is exponentially convex function. For the properties, generalizations and applications of the exponentially convex functions, see [1, 2, 3, 4, 5, 18, 19, 20, 21, 22, 23, 25, 28, 29, 30, 31, 32, 33, 34, 35, 36]

Definition 2.5. [38].*The set K_g in H is said to be general convex set, if there exists an arbitrary function g, such that*

$$g(u) + t(g(v) - g(u)) \in K_g, \quad \forall u, v \in H : g(u), g(v) \in K_g, t \in [0,1].$$

Note that, if $g = I$, the identity operator, then general convex set reduces to the classical convex set. Clearly every convex set is a general convex set, but the converse is not true.

For the sake of simplicity, we always assume that $\forall u, v \in H : g(u), g(v) \in K_g$, unless otherwise.

Definition 2.6. [38] *A function F is said to be general convex(g-convex) function, if there exists an arbitrary non-negative function g, such that*

$$F((1-t)g(u) + tg(v)) \leq (1-t)F(g(u)) + tF(g(v)), \quad \forall g(u), g(v) \in K_g, t \in [0,1]. \tag{5}$$

Remark 2.1. *It is known that the function $f(u) = e^u$ is a general convex function, but it is not a convex function. See also* [38] *for other examples for the importance of the arbitrary function g.*

Note that, if $g = I$, the identity operator, then the general convex function reduces to the classical convex function. Clearly every convex function is a general convex function, but the converse is not true, see [7].
Noor [15, 16] has shown that the minimum $u \in H : g(u) \in K_g$ of the differentiable general convex functions F can be characterized by the class of variational inequalities of the type:

$$\langle F'(g(u)), g(v) - g(u) \rangle \geq 0, \quad \forall u \in H : g(v) \in K_g, \tag{6}$$

which is known as general variational inequalities. For the applications of the general variational inequalities in various branches of pure and applied sciences, see [13, 14, 15, 16, 34] and the references therein.

We note that the exponentially convex functions and general convex functions are two distinct generalizations of the convex functions. It is natural to unify these concepts. Motivated by this fact, we now introduce some new concepts of exponentially general convex functions, which is the main motivation of this paper.

Definition 2.7. *A function F is said to be exponentially general convex function with respect to an arbitrary non-negative function g, if*

$$e^{F((1-t)g(u)+tg(v))} \leq (1-t)e^{F(g(u))} + te^{F(g(v))}, \quad \forall g(u), g(v) \in K_g, t \in [0,1]. \qquad (7)$$

or equivalently

Definition 2.8. *A function F is said to be exponentially general convex function with respect to an arbitrary non-negative function g, if,*

$$F((1-t)g(u)+tg(v)) \leq \log[(1-t)e^{F(g(u))} + te^{F(g(v))}], \quad \forall g(u), g(v) \in K_g, t \in [0,1]. \qquad (8)$$

A function is called the exponentially general concave function f, if $-f$ is exponentially general convex function.

Definition 2.9. *A function F is said to be exponentially general affine convex function with respect to an arbitrary non-negative function g, if*

$$e^{F((1-t)g(u)+tg(v))} = (1-t)e^{F(g(u))} + te^{F(g(v))}, \quad , \forall g(u), g(v) \in K_g, t \in [0,1]. \qquad (9)$$

If $g = I$, the identity operator, then exponentially general convex functions reduce to the exponentially convex functions.

Definition 2.10. *The function F on the general convex set K_g is said to be exponentially general quasi convex, if*

$$e^{F(g(u)+t(g(v)-g(u)))} \leq \max\{e^{F(g(u))}, e^{F(g(v))}\}, \quad \forall g(u), g(v) \in K_g, t \in [0,1].$$

Definition 2.11. *The function F on the general convex set K_g is said to be exponentially general log-convex, if*

$$e^{F(g(u)+t(g(v)-g(u)))} \leq (e^{F(g(u))})^{1-t}(e^{F(g(v))})^{t}, \quad \forall g(u), g(v) \in K_g, t \in [0,1],$$

where $F(\cdot) > 0$.

From the above definitions, we have

$$\begin{aligned} e^{F(g(u)+t(g(v)-g(u))} &\leq (e^{F(g(u))})^{1-t}(e^{F(g(v))})^{t} \\ &\leq (1-t)e^{F(g(u))} + te^{F(g(v))}) \\ &\leq \max\{e^{F(g(u))}, e^{F((v))}\}. \end{aligned}$$

This shows that every exponentially general log-convex function is a exponentially general convex function and every exponentially general convex function is a exponentially general quasi-convex function. However, the converse is not true.
Let $K_g = I_g = [g(a), g(b)]$ be the interval. We now define the exponentially general convex functions on I_g.

Definition 2.12. *Let $I_g = [g(a), g(b)]$. Then F is exponentially general convex function, if and only if,*

$$\begin{vmatrix} 1 & 1 & 1 \\ g(a) & g(x) & g(b) \\ e^{F(g(a))} & e^{F(g(x))} & e^{F(g(b))} \end{vmatrix} \geq 0; \quad g(a) \leq g(x) \leq g(b).$$

One can easily show that the following are equivalent:

(1) F is exponentially general convex function.

(2) $e^{F(g(x))} \leq e^{F(g(a))} + \frac{e^{F(g(b))} - e^{F(g(a))}}{g(b)-g(a)}(g(x) - g(a))$.
(3) $\frac{e^{F(g(x))} - e^{F(g(a)}}{g(x)-g(a)} \leq \frac{e^{F(g(b))} - e^{F(g(a))}}{g(b)-g(a)}$.
(4) $(g(x) - g(b))e^{F(g(a))} + (g(b) - g(a))e^{F(g(x))} + (g(a) - g(x))e^{F(g(b))}) \geq 0$.
(5) $\frac{e^{F(g(a))}}{(g(b)-g(a))(g(a)-g(x))} + \frac{e^{F(g(x))}}{(g(x)-g(b))(g(a)-g(x))} + \frac{e^{F(g(b)}}{(g(b)-g(a))(g(x)-g(b))} \geq 0$,
where $g(x) = (1-t)g(a) + tg(b) \in [g(a), g(b)]$.

3. **Main Results**

In this section, we consider some basic properties of exponentially general convex functions.

Theorem 3.1. *Let F be a strictly exponentially general convex function. Then any local minimum of F is a global minimum.*

Proof. Let the strictly exponentially convex function F have a local minimum at $g(u) \in K_g$. Assume the contrary, that is, $F(g(v)) < F(g(u))$ for some $g(v) \in K_g$. Since F is strictly exponentially general convex function, so

$$e^{F(g(u)+t(g(v)-g(u)))} < te^{F(g(v))} + (1-t)e^{F(g(u))}, \quad \text{for} \quad 0 < t < 1.$$

Thus

$$e^{F(g(u)+t(g(v0-g(u)))} - e^{F(g(u))} < -t[e^{F(g(v))} - e^{F(g(u))}] < 0,$$

from which it follows that

$$e^{F(g(u)+t(g(v)-g(u0))} < e^{F(g(u))},$$

for arbitrary small $t > 0$, contradicting the local minimum. □

Theorem 3.2. *If the function F on the general convex set K_g is exponentially general convex, then the level set*

$$L_\alpha = \{g(u) \in K_g : e^{F(g(u))} \leq \alpha, \quad \alpha \in R\}$$

is a general convex set.

Proof. Let $g(u), g(v) \in L_\alpha$. Then
$e^{F(g(u))} \leq \alpha$ and $e^{F(g(v))} \leq \alpha$. Now, $\forall t \in (0,1), \quad g(w) = g(v) + t(g(u) - g(v)) \in K_g$, since K_g is a convex set. Thus, by the exponentially general convexity of F, we have

$$\begin{aligned} Fe^{(g(v)+t(g(u)-g(v))} &\leq (1-t)e^{F(g(v))} + te^{F(g(u))} \\ &\leq (1-t)\alpha + t\alpha = \alpha, \end{aligned}$$

from which it follows that $g(v) + t(g(u) - g(v)) \in L_\alpha$ Hence L_α is a general convex set. □

Theorem 3.3. *The function F is exponentially general convex function, if and only if*

$$epi(F) = \{(g(u), \alpha) : g(u) \in K_g : e^{F(g(u))} \leq \alpha, \alpha \in R\}$$

is a general convex set.

Proof. Assume that$F$ is exponentially general convex function. Let

$$(g(u), \alpha), \quad (g(v), \beta) \in epi(F).$$

Then it follows that $e^{F(g(u))} \leq \alpha$ and $e^{F(g(v))} \leq \beta$. Hence, we have

$$e^{F(g(u)+t(g(v)-g(u)))} \leq (1-t)e^{F(g(u))} + te^{F(g(v))} \leq (1-t)\alpha + t\beta,$$

which implies that

$$((1-t)g(u) + tg(v)), (1-t)\alpha + t\beta) \in epi(F).$$

Thus $epi(F)$ is a general convex set. Conversely, let $epi(F)$ be a general convex set. Let $g(u), g(v) \in K_g$. Then $(g(u), e^{F(g(u)})) \in epi(F)$ and $(g(v, e^{F(g(v))}) \in epi(F)$. Since $epi(F)$ is a general convex set, we must have

$$(g(u) + t(g(v) - g(u), (1-t)e^{F(g(u))} + te^{F(g(v))} \in epi(F),$$

which implies that

$$e^{F((1-t)g(u)+tg(v))} \leq (1-t)e^{F(g(u))} + te^{F(g(u))}.$$

This shows that F is an exponentially general convex function. □

Theorem 3.4. *The function F is exponentially general quasi convex, if and only if, the level set*

$$L_\alpha = \{g(u) \in K_g, \alpha \in R : e^{F(g(u))} \leq \alpha\}$$

is a general convex set.

Proof. Let $g(u), g(v) \in L_\alpha$. Then $g(u), g(v) \in K_g$ and $\max(e^{F(g(u))}, e^{F(g(v))}) \leq \alpha$.
Now for $t \in (0,1), g(w) = g(u) + t(g(v) - g()u) \in K_g$. We have to prove that $g(u) + t(g(v) - g(u)) \in L_\alpha$. By the exponentially general convexity of F, we have

$$e^{F(g(u)+t(g(v)-g(u)))} \leq \max\left(e^{F(g(u))}, e^{F(g(v))}\right) \leq \alpha,$$

which implies that $g(u) + t(g(v) - g(u)) \in L_\alpha$, showing that the level set L_α is indeed a general convex set.

Conversely, assume that L_α is a general convex set. Then, $\forall \quad g(u), g(v) \in L_\alpha, t \in [0,1]$, $g(u) + t(g(v) - g(u)) \in L_\alpha$. Let $g(u), g(v) \in L_\alpha$ for

$$\alpha = max(e^{F(g(u))}, e^{F(g(v))}) \quad \text{and} \quad e^{F(g(v))} \leq e^{F(g(u))}.$$

Then, from the definition of the level set L_α, it follows that

$$e^{F(g(u)+t(g(v)-g(u))} \leq \max\left(e^{F(g(u))}, e^{F(g(v))}\right) \leq \alpha.$$

Thus F is an exponentially general quasi convex function. This completes the proof. □

Theorem 3.5. *Let F be an exponentially general convex function. Let $\mu = \inf_{u \in K} F(u)$. Then the set*

$$E = \{g(u) \in K_g : e^{F(g(u))} = \mu\}$$

is a general convex set of K_g. If F is strictly exponentially general convex function , then E is a singleton.

Proof. Let $g(u), g(v) \in E$. For $0 < t < 1$, let $g(w) = g(u) + t(g(v) - g(u))$. Since F is a exponentially general convex function, then

$$\begin{aligned} F(g(w)) &= e^{F(g(u)+t(g(v)-g(u))} \\ &\leq (1-t)e^{F(g(u))} + te^{F(g(v))} = t\mu + (1-t)\mu = \mu, \end{aligned}$$

which implies $g(w) \in E$. and hence E is a general convex set. For the second part, assume to the contrary that $F(g(u)) = F(g(v)) = \mu$. Since K is a general convex set, then for $0 < t < 1, g(u) + t(g(v) - g(u)) \in K_g$. Since F is strictly exponentially general convex function, so

$$e^{F(g(u)+t(g(v)-g(u)))} \quad < \quad (1-t)e^{F(g(u))} + te^{F(g(v))} = (1-t)\mu + t\mu = \mu.$$

This contradicts the fact that $\mu = \inf_{g(u) \in K_g} F(u)$ and hence the result follows. □

Theorem 3.6. *If the function F is exponentially general convex such that*

$$e^{F(g(v))} < e^{F(g(u))}, \forall g(u), g(v) \in K_g,$$

then F is a strictly exponentially general quasi function.

Proof. By the exponentially general convexity of the function F, we have

$$\begin{aligned} e^{F(g(u)+t(g(v)-g(u)))} &\leq (1-t)e^{F(g(u))} + te^{F(g(v))}, \forall g(u), g(v) \in K_g, t \in [0,1] \\ &< e^F(g(u)), \end{aligned}$$

since $e^{F(g(v))} < e^{F(g(u))}$, which shows that the function F is strictly exponentially general quasi convex. □

We now show that the difference of exponentially convex function and exponentially affine convex function is again an exponentially general convex function.

Theorem 3.7. *Let f be a exponentially general affine convex function. Then F is a exponentially general convex function, if and only if, $H = F-$ is a exponentially convex function.*

Proof. Let f be exponentially general affine convex function. Then

$$e^{f((1-t)g(u)+tg(v))} = (1-t)e^{f(g(u))} + te^{f(g(v))}, \quad \forall g(u), g(v) \in K_g, t \in [0,1]. \tag{10}$$

From the exponentially general convexity of F, we have

$$e^{F((1-t)g(u)+tg(v))} \leq (1-t)e^{F(g(u))} + te^{F(g(v))}, \quad \forall g(u), g(v) \in K_g, t \in [0,1]. \tag{11}$$

From (10) and (11), we have

$$\begin{aligned} & e^{F((1-t)g(u)+tg(v))} - e^{f((1-t)g(u)+tg(v))} \\ & \leq (1-t)(e^{F(g(u))} - e^{f(g(u))}) + t(e^{F(g(v))} - e^{f(g(v))}), \end{aligned} \tag{12}$$

from which it follows that

$$\begin{aligned} e^{H((1-t)g(u)+tg(v))} &= e^{F((1-t)g(u)+tg(v)))} - e^{f((1-t)f(g(u))+tf(g(v))} \\ &\leq (1-t)(e^{F(g(u))} - e^{f(g(u))}) + t(e^{F(g(v))} - e^{f(g(v))}), \end{aligned}$$

which show that $H = F - f$ is an exponentially general convex function.
The inverse implication is obvious. □

Definition 3.1. *A function F is said to be a exponentially general pseudo convex function, if there exists a strictly positive bifunction $B(.,.)$, such that*

$$\begin{aligned} & e^{F(g(v))} < e^{F(g(u))} \\ & \Rightarrow \\ & e^{F(g(u)+t(g(v)-g(u)))} < e^{F(g(u))} + t(t-1)B(g(v), g(u)), \quad \forall g(u), g(v) \in K_g, t \in [0,1]. \end{aligned}$$

Theorem 3.8. *If the function F is exponentially general convex function such that*

$$e^{F(g(v))} < e^{F(g(u))},$$

then the function F is an exponentially general pseudo convex function.

Proof. Since $e^{F(g(v))} < e^{F(g(u))}$ and F is exponentially general convex function, then $\forall g(u), g(v) \in K_g, \quad t \in [0,1]$, we have

$$\begin{aligned} e^{F(g(u)+(1-t)(g(v)-g(u)))} &\leq e^{F(g(u))} + t(e^{F(g(v))} - e^{F(g(u))}) \\ &< e^{F(g(u))} + t(1-t)(e^{F(g(v))} - e^{F(g(u))}) \\ &= e^{F(g(u))} + t(t-1)(e^{F(g(u))} - e^{F(g(v))})) \\ &< e^{F(g(u))} + t(t-1)B(g(u), g(v)), \end{aligned}$$

where $B(g(u), g(v)) = e^{F(g(u))} - e^{F(g(v))} > 0$. This shows that the function F is exponentially general pseudo convex function. □

We now study some properties of the differentiable exponentially general convex functions.

Theorem 3.9. *Let F be a differentiable function on the general convex set K_g. Then the function F is exponentially general convex function, if and only if,*

$$e^{F(g(v))} - e^{F(g(u))} \geq \langle e^{F(g(u))} F'(g(u)), g(v) - g(u) \rangle, \quad \forall g(v), g(u) \in K_g. \tag{13}$$

Proof. Let F be a exponentially general convex function. Then

$$e^{F(g(u)+t(g(v)-g(u)))} \leq (1-t)e^{F(g(u))} + te^{F(g(v))}, \qquad \forall g(u), g(v) \in K_g,$$

which can be written as

$$e^{F(g(v))} - e^{F(g(u))} \geq \{\frac{e^{F(g(u)+t(g(v)-g(u)))} - e^{F(g(u))}}{t}\}.$$

Taking the limit in the above inequality as $t \to 0$, we have

$$e^{F(g(v))} - e^{F(g(u))} \geq \langle e^{F(g(u))} F'(g(u)), g(v) - g(u)) \rangle,$$

which is (13), the required result.

Conversely, let (13) hold. Then $\forall g(u), g(v) \in K_g, t \in [0, 1]$,

$$g(v_t) = g(u) + t(g(v) - g(u)) \in K_g,$$

we have

$$\begin{aligned} e^{F(g(v))} - e^{F(g(v_t))} &\geq \langle e^{F(g(v_t))} F'(g(v_t)), g(v) - g(v_t)) \rangle \\ &= (1-t)\langle e^{F(g(v_t))} F'(g(v_t)), g(v) - g(u) \rangle. \end{aligned} \tag{14}$$

In a similar way, we have

$$\begin{aligned} e^{F(g(u))} - e^{F(g(v_t))} &\geq \langle e^{F(g(v_t))} F'(g(v_t)), g(u) - g(v_t)) \rangle \\ &= -t\langle e^{F(g(v_t))} F'(g(v_t)), g(v) - g(u) \rangle. \end{aligned} \tag{15}$$

Multiplying (14) by t and (15) by $(1-t)$ and adding the resultant, we have

$$e^{F(g(u)+t(g(v)-g(u)))} \leq (1-t)e^{F(g(u))} + te^{F(g(v))},$$

showing that F is a exponentially general convex function. □

Remark 3.1. *From* (13)*, we have*

$$e^{F(g(v))-F(g(u))} - 1 \geq \langle F'(g(u)), g(v) - g(u) \rangle, \quad \forall g(v), g(u) \in K_g,$$

which can be written as

$$F(g(v)) - F(g(u)) \geq \log\{1 + \langle F'(g(u)), g(v) - g(u) \rangle, \} \quad \forall g(v), g(u) \in K_g, \tag{16}$$

Changing the role of u and v in (16)*, we also*

$$F(g(u)) - F(g(v)) \geq \log\{1 + \langle F'(g(v)), g(u) - g(v) \rangle, \} \quad \forall g(v), g(u) \in K_g, \tag{17}$$

Adding (16) *and* (17)*, we have*

$$\langle F'(g(u)) - F'(g(v)), g(u) - g(v) \rangle \geq (\langle F'(g(u)), g(u) - g(v) \rangle)(F'(g(v)), g(u) - g(v) \rangle)$$

which express the monotonicity of the differential $F'(.)$ of the exponentially general convex function.

Theorem 3.9 enables us to introduce the concept of the exponentially monotone operators, which appears to be new ones.

Definition 3.2. *The differential $F'(.)$ is said to be exponentially general monotone, if*

$$\langle e^{F(g(u))}F'(g(u)) - e^{F(g(v))}F'(g(v)), g(u) - g(v)\rangle \geq 0, \quad \forall u, v \in H.$$

Definition 3.3. *The differential $F'(.)$ is said to be exponentially general pseudo-monotone, if*

$$\begin{aligned} &\langle e^{F(g(u))}F'(g(u)), g(v) - g(u)\rangle \geq 0, \\ &\quad \Rightarrow \\ &\langle e^{F(g(v))}F'(g(v)), g(v) - g(u)\rangle \geq 0, \quad \forall u, v \in H. \end{aligned}$$

From these definitions, it follows that exponentially general monotonicity implies exponentially general pseudo-monotonicity, but the converse is not true.

Theorem 3.10. *Let F be differentiable exponentially general convex function. Then,* (13) *holds, if and only if, $F'(.)$ satisfies*

$$\langle e^{F(g(u))}F'(g(u)) - e^{F(g(v))}F'(g(v)), g(u) - g(v)\rangle \geq 0, \quad \forall g(u), g(v) \in K_g. \tag{18}$$

Proof. Let F be a exponentially general convex function. Then, from Theorem 3.9, we have

$$e^{F(g(v))} - e^{F(g(u))} \geq \langle e^{F(g(u))}F'(g(u)), g(v) - g(u)\rangle, \quad \forall g(u), g(v) \in K_g. \tag{19}$$

Changing the role of u and v in (19), we have

$$e^{F(g(u)} - e^{F(g(v))} \geq \langle e^{F(g(v))}F'(g(v)), g(u) - g(v))\rangle, \quad \forall g(u), g(v) \in K_g. \tag{20}$$

Adding (19) and (20), we have

$$\langle e^{F(g(u))}F'(g(u)) - e^{F(g(v))}F'(g(v)), g(u) - g(v)\rangle \geq 0,$$

which shows that F' is exponentially general monotone.

Conversely, from (18), we have

$$\langle e^{F(g(v0)}F'(g(v)), g(u) - g(v)\rangle \leq \langle e^{F(g(u))}F'(g(u)), g(u) - g(v))\rangle. \tag{21}$$

Since K_g is a general convex set, $\forall g(u), g(v) \in K_g, \quad t \in [0, 1]$,

$$g(v_t) = g(u) + t(g(v) - g(u)) \in K_g.$$

Taking $g(v) = g(v_t)$ in (21), we have

$$\begin{aligned} \langle e^{F(g(v_t))}F'(g(v_t)), g(u) - g(v_t)\rangle &\leq \langle e^{F(g(u))}F'(g(u)), g(u) - g(v_t)\rangle \\ &= -t\langle e^{F(g(u))}F'(g(u)), g(v) - g(u)\rangle, \end{aligned}$$

which implies that

$$\langle e^{F(g(v_t))}F'(g(v_t)), g(v) - g(u)\rangle \geq \langle e^{F(g(u))}F'(g(u)), g(v) - g(u)\rangle. \tag{22}$$

Consider the auxiliary function

$$\xi(t) = e^{F(g(u)+t(g(v)-g(u))},$$

from which, we have

$$\xi(1) = e^{F(g(v))}, \quad \xi(0) = e^{F(g(u))}.$$

Then, from (22), we have

$$\begin{aligned} \xi'(t) &= \langle e^{F(g(v_t))}F'(g(v_t)), g(v) - g(u)\rangle \\ &\geq \langle e^{F(g(u))}F'(g(u)), g(v) - g(u)\rangle. \end{aligned} \tag{23}$$

Integrating (23) between 0 and 1, we have

$$\xi(1) - \xi(0) = \int_0^1 g'(t)dt \geq \langle e^{F(g(u))}F'(g(u)), g(v) - g(u)\rangle.$$

Thus it follows that

$$e^{F(g(v))} - e^{F(g(u))} \geq \langle e^{F(g(u))}F'(g(u)), g(v) - g(u)\rangle,$$

which is the required (13). □

We now give a necessary condition for exponentially general pseudo-convex function.

Theorem 3.11. *Let $F'(.)$ be exponentially general pseudomonotone. Then F is a exponentially general pseudo-convex function.*

Proof. Let F' be a exponentially general pseudomonotone. Then, $\forall g(u), g(v) \in K_g$,

$$\langle e^{F(g(u))}F'(g(u)), g(v) - g(u)\rangle \geq 0.$$

implies that

$$\langle e^{F(g(v))}F'(g(v)), g(v) - g(u)\rangle \geq 0. \tag{24}$$

Since K_g is a general convex set, $\forall g(u), g(v) \in K_g, \quad t \in [0, 1]$,

$$g(v_t) = g(u) + t(g(v) - g(u)) \in K_g.$$

Taking $g(v) = g(v_t)$ in (24), we have

$$\langle e^{F(g(v_t))}F'(g(v_t)), g(v) - g(u)\rangle \geq 0. \tag{25}$$

Consider the auxiliary function $\xi(t) = e^{F(g(u)+t(g(v)-g(u)))} = e^{F(g(v_t))}, \quad \forall g(u), g(v) \in K_g, t \in [0, 1]$, which is differentiable, since F is differentiable function. Then, using (25), we have $\xi'(t) = \langle e^{F(g(v_t))}F'(g(v_t)), g(v) - g(u))\rangle \geq 0$. Integrating the above relation between 0 to 1, we have $\xi(1) - \xi(0) = \int_0^1 g'(t)dt \geq 0$, that is, $e^{F(g(v))} - e^{F(g(u))} \geq 0$, showing that F is a exponentially general pseudo-convex function. □

Definition 3.4. *The function F is said to be sharply exponentially general pseudo convex, if*

$$\langle e^{F(g(u))}F'(g(u)), g(v) - g(u)\rangle \geq 0 \Rightarrow F(g(v)) \geq e^{F(g(v)+t(g(u)-g(v)))}, \forall g(u), g(v) \in K_g, t \in [0, 1].$$

Theorem 3.12. *Let F be a sharply exponentially general pseudo convex function. Then*

$$\langle e^{F(g(v))}F'(g(v)), g(v) - g(u)\rangle \geq 0, \quad \forall g(u), g(v) \in K_g.$$

Proof. Let F be a sharply exponentially general pseudo convex function. Then $e^{F(g(v))} \geq e^{F(g(v)+t(g(u)-g(v)))}, \quad \forall g(u), g(v) \in K_g, t \in [0, 1]$, from which we have

$$0 \leq \lim_{t\to 0}\{\frac{e^{F(g(v)+t(g(u)-g(v)))} - e^{F(g(v))}}{t}\} = \langle e^{F(g(v))}F'(g(v)), g(v) - g(u)\rangle,$$

the required result. □

We now discuss the optimality condition for the differentiable exponentially convex functions, which is the main motivation of our next result.

Theorem 3.13. *Let F be a differentiable general exponentially convex function. Then $u \in H : g(u) \in K_g$ is the minimum of the function F, if and only if, $u \in H : g(u) \in K_g$ satisfies the inequality*

$$\langle e^{F(g(u))}F'(g(u)), g(v) - g(u)\rangle \geq 0, \quad \forall g(u), g(v) \in K_g. \tag{26}$$

Proof. Let $u \in H : g(u) \in K_g$ be a minimum of the function F. Then $F(g(u)) \leq F(g(v)), \forall v \in H : g(v) \in K_g$. from which, we have

$$e^{F(g(u))} \leq e^{F(g(v))}, \forall g(v) \in K_g. \tag{27}$$

Since K_g is a general convex set, so, $\forall g(u), g(v) \in K_g, \quad t \in [0,1], g(v_t) = (1-t)g(u) + tg(v) \in K_g$. Taking $g(v) = g(v_t)$ in (27), we have

$$\begin{aligned} 0 &\leq \lim_{t\to 0}\{\frac{e^{F(g(u)+t(g(v)-g(u)))} - e^{F(g(u))}}{t}\} \\ &= \langle e^{F(g(u))}F'(g(u)), g(v) - g()u\rangle. \end{aligned} \tag{28}$$

Since F is an exponentially general convex function, so $e^{F(g(u)+t(g(v)-g(u)))} \leq e^{F(g(u))} + t(e^{F(g(v))} - e^{F(g(u))}, g(u), g(v) \in K_g, t \in [0,1]$, from which, using (28), we have

$$e^{F(g(v))} - e^{F(g(u))} \geq \lim_{t\to 0}\{\frac{e^{F(g(u)+t(g(v)-g(u)))} - e^{F(g(u))}}{t}\} = \langle e^{F(g(u))}F'(g(u)), g(v) - g(u)\rangle \geq 0.$$

This implies that $e^{F(g(v))} - e^{F(g(u))} \geq 0$, from which, we have $F(g(u)) \leq F(g(v))$. This shows that $u \in H : g(u) \in K_g$ is the minimum of the differentiable exponentially general convex function, the required result. □

Remark 3.2. *Find $u \in H : g(u) \in K_g$, such that*

$$\langle e^{F(g(u))}F'(g(u)), g(v) - g(u)\rangle \geq 0, \forall g(v(\in K_g \tag{29}$$

is called the exponentially general variational inequality and appears to be a new one. Using the technique and ideas of Noor [15, 16] *and Noor et al.* [24]*, one can develop some iterative methods for solving the exponentially general variational inequalities of the type (29). It is an open problem to study the applications in various fields of pure and applied sciences.*

Conclusion

In this paper, we have introduced and studied a new class of convex functions which is called the exponentially general convex function. It have been shown that exponentially general convex functions enjoy several properties which convex functions have. We have shown that the minimum of the expedientially differentiable general convex functions can be characterized by a new class of variational inequalities, which is called the exponentially general variational inequality. To develop the numerical methods for solving exponentially strongly general variational inequalities need further efforts. This is an interesting problem for future research. This may stimulate further research.

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