WEAK CONVERGENCE OF A SELF-ADAPTIVE TSENG-TYPE ALGORITHM FOR SOLVING VARIATIONAL INCLUSION PROBLEMS

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In this paper, we discuss iterative algorithms for approximating a solution of a variational inclusion problem in a real Hilbert space. We propose a self-adaptive Tseng-type iterative sequence for finding a solution of a variational inclusion involved in plain monotone operators. We show the weak convergence of the sequence under some appropriate conditions.

Keywords: variational inclusion, Tseng-type algorithm, self-adaptive rule, monotone operator.

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1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Let $f : \mathcal{H} \to \mathcal{H}$ be a single-valued operator. Let $\varphi : \mathcal{H} \rightrightarrows 2^{\mathcal{H}}$ be a multi-valued operator. Recall that the variational inclusion problem aims to search a point $z^* \in \mathcal{H}$ such that

$$0 \in (f + \varphi)(z^*). \tag{1}$$

In the sequel, denote the solution set for the variational inclusion (1) by $(f + \varphi)^{-1}(0)$.

Now, it is well-known that the variational inclusion problem can be used to solve a large number of problems, such as, fixed point problems ([12, 14–16, 19]), variational inequality problems ([1, 11, 20, 22, 23, 25, 28, 30]), feasibility problems ([9, 10, 18, 24]), split problems ([4, 5, 32]), and equilibrium problems ([21, 31]). Especially, the variational inclusion is closely related to the following optimization problem ([6])

$$\min_{z^* \in \mathcal{H}} (\phi(z^*) + \psi(z^*)) \tag{2}$$

where $\phi, \psi : \mathcal{H} \to \mathbf{R} \cup \{+\infty\}$ are two proper, lower semicontinuous and convex functions. In fact, if ϕ is differentiable and ψ is subdifferentiable, then solving (2) equals to find a point $z^* \in \mathcal{H}$ such that $0 \in (\nabla\phi + \partial\psi)(z^*)$. There are many iterative algorithms for solving

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(1) in the literature, see [13, 17, 26, 27, 29]. An interesting way is to apply the following forward-backward method ([3, 7]) which defines a sequence iteratively by

$$x_0 \in H,\ x_{n+1} = (I + \delta_n \varphi)^{-1}(x_n - \delta_n f(x_n)),\ n \geq 0, \tag{3}$$

where φ is maximal monotone and f is (inverse) strongly monotone.

Note that the strong monotonicity condition imposed on f is a little strict. Very recently, Cholamjiak, Hieu and Cho [2] proposed the following relaxed forward-backward splitting algorithm for solving (1) in which f is a plain monotone operator.

Algorithm 1.1. *For given an initial point $x_0 \in \mathcal{H}$ and a positive constant λ_0, define an iterative sequence $\{x_n\}_{n\geq 1}$ by the following manner*

$$\begin{cases} y_n = (I + \lambda_n \varphi)^{-1}(x_n - \lambda_n f(x_n)),\ n \geq 0, \\ x_{n+1} = (1 - \theta_n)x_n + \theta_n y_n + \theta_n \lambda_n (f(x_n) - f(y_n)),\ n \geq 0, \\ \lambda_{n+1} = \min\left\{\lambda_n, \frac{\mu_n \|x_n - y_n\|}{\|f(x_n) - f(y_n)\|}\right\}. \end{cases}$$

It is obviously that the iterative steps in Algorithm 1.1 use Tseng method and self-adaptive rule. In this paper, we continue to investigate iterative algorithms for solving the variational inclusion (1). We use a different search rule to update the stepsize and suggest a Tseng-type algorithm for finding a solution of (1) in which f is plain monotone. We show the weak convergence of the sequence under some appropriate conditions.

2. **Preliminaries**

Throughout, we assume that $\mathcal{H}$ is a real Hilbert space. Then, we have

$$\|\gamma x + (1-\gamma)y\|^2 = \gamma\|x\|^2 + (1-\gamma)\|y\|^2 - \gamma(1-\gamma)\|x - y\|^2, \tag{4}$$

for all $x, y \in \mathcal{H}$ and $\forall \gamma \in \mathbf{R}$.

Let $f\colon \mathcal{H} \to \mathcal{H}$ be a single-valued operator. f is said to be

(i) λ-Lipschitz if $\|f(x) - f(x^\dagger)\| \leq \lambda\|x - x^\dagger\|$, $\forall x, x^\dagger \in \mathcal{H}$, where $\lambda > 0$ is some constant.

(ii) strongly monotone if $\langle f(x) - f(x^\dagger), x - x^\dagger\rangle \geq \alpha\|x - x^\dagger\|^2$, $\forall x, x^\dagger \in \mathcal{H}$, where $\alpha > 0$.

(iii) (plain) monotone if $\langle f(x) - f(x^\dagger), x - x^\dagger\rangle \geq 0$, $\forall x, x^\dagger \in \mathcal{H}$.

Let $\varphi\colon \mathcal{H} \rightrightarrows 2^{\mathcal{H}}$ be a multi-valued operator. φ is said to be monotone if and only if

$$\langle x - y, p - q\rangle \geq 0,\ \forall x, y \in \mathcal{H} \text{ and } p \in \varphi(x) \text{ and } q \in \varphi(y).$$

A monotone operator φ is said to be maximal monotone if and only if its graph is not strictly contained in the graph of any other monotone operator.

Lemma 2.1 ([8]). *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $\{x_n\}$ be a sequence in $\mathcal{H}$. Suppose that (i) $\forall p \in C$, $\lim_{n\to\infty}\|x_n - p\|$ exists; (ii) $\omega_w(x_n) \subset C$, where $\omega_w(x_n) := \{z \in \mathcal{H}$: there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z$ as $i \to +\infty\}$. Then the sequence $\{x_n\}$ converges weakly to some point in C.*

3. Main results

Suppose that H is a real Hilbert space. Suppose that (i) $\varphi : H \rightrightarrows 2^H$ is a maximal monotone operator; (ii) $f : H \to H$ is a λ-Lipschitz monotone operator. In the sequel, we assume that $(f+\varphi)^{-1}(0) \neq \emptyset$. Suppose that τ, β and L are three positive constants in $(0,1)$ and $\{\gamma_n\}$ is a real number sequence such that $0 < c_1 \leq \gamma_n \leq c_2 < 1$ for all $n \geq 0$.

Now, we put forward an iterative algorithm for solving (1).

Algorithm 3.1. *For given an initial point $x_0 \in H$ and a positive constant δ_0, define an iterative sequence $\{x_n\}_{n\geq 1}$ by the following manner*

$$y_n = (I + \beta\delta_n\varphi)^{-1}(x_n - \beta\delta_n f(x_n)),\ n \geq 0, \tag{5}$$

and

$$x_{n+1} = (1-\gamma_n)x_n + \gamma_n[y_n - \beta\delta_n(f(y_n) - f(x_n))],\ n \geq 0, \tag{6}$$

where $\delta_n = \max\{1, \tau, \tau^2, \cdots\}$ satisfies

$$\beta\delta_n\|f(x_n) - f(y_n)\| \leq L\|x_n - y_n\|. \tag{7}$$

Remark 3.1. *Noting that $\|f(x_n) - f(y_n)\| \leq \lambda\|x_n - y_n\|$, there is $\delta_n \in \{1, \tau, \tau^2, \cdots\}$ such that*

$$\delta_n \leq \frac{L}{\lambda\beta}. \tag{8}$$

Moreover, there exists a positive constant $\rho = \tau^m$ (for some positive integer m) such that (7) is well-defined. Otherwise, for all n, we have $L\|x_n - y_n\| < \beta\delta_n\|f(x_n) - f(y_n)\|$. This together with $\|f(x_n) - f(y_n)\| \leq \lambda\|x_n - y_n\|$ implies that $\delta_n > \frac{L}{\lambda\beta}$, which contradicts (8).

Next, we firstly prove several propositions. In what follows, choose any $z^\dagger \in (f+\varphi)^{-1}(0)$.

Proposition 3.1. $\lim_{n\to+\infty}\|x_n - z^\dagger\|$ *exists.*

Proof. Taking into account

$$\|y_n - z^\dagger + \beta\delta_n(f(x_n) - f(y_n))\|^2 = \|y_n - z^\dagger\|^2 + \beta^2\delta_n^2\|f(x_n) - f(y_n)\|^2 + 2\beta\delta_n\langle f(x_n) - f(y_n), y_n - z^\dagger\rangle,$$

and

$$\|y_n - z^\dagger\|^2 = \|x_n - z^\dagger\|^2 - \|y_n - x_n\|^2 + 2\langle y_n - x_n, y_n - z^\dagger\rangle,$$

we obtain

$$\begin{aligned}\|y_n - z^\dagger + \beta\delta_n(f(x_n) - f(y_n))\|^2 &= \|x_n - z^\dagger\|^2 + 2\beta\delta_n\langle f(x_n) - f(y_n), y_n - z^\dagger\rangle \\ &\quad - \|y_n - x_n\|^2 + 2\langle y_n - x_n, y_n - z^\dagger\rangle + \beta^2\delta_n^2\|f(x_n) - f(y_n)\|^2 \\ &= \|x_n - z^\dagger\|^2 + \beta^2\delta_n^2\|f(x_n) - f(y_n)\|^2 \\ &\quad + 2\langle y_n - x_n + \beta\delta_n(f(x_n) - f(y_n)), y_n - z^\dagger\rangle - \|y_n - x_n\|^2.\end{aligned} \tag{9}$$

Thanks to (5), we have

$$x_n - \beta\delta_n f(x_n) \in (I + \beta\delta_n\varphi)y_n. \tag{10}$$

Thus,

$$x_n - y_n - \beta\delta_n(f(x_n) - f(y_n)) \in \beta\delta_n(f + \varphi)y_n. \tag{11}$$

Observe that $0 \in \beta\delta_n(f + \varphi)z^\dagger$ and $\beta\delta_n(f + \varphi)$ is monotone. Based on (11), we get

$$\langle y_n - x_n + \beta\delta_n(f(x_n) - f(y_n)), y_n - z^\dagger\rangle \leq 0. \tag{12}$$

Furthermore, by (7),

$$\beta^2\delta_n^2\|f(x_n) - f(y_n)\|^2 \leq L^2\|x_n - y_n\|^2.$$

This together with (9) and (12) implies that

$$\|y_n - z^\dagger + \beta\delta_n(f(x_n) - f(y_n))\|^2 \leq \|x_n - z^\dagger\|^2 - (1 - L^2)\|y_n - x_n\|^2. \tag{13}$$

Applying (4) to (6) to derive

$$\begin{aligned} \|x_{n+1} - z^\dagger\|^2 = (1 - \gamma_n)\|x_n - z^\dagger\|^2 + \gamma_n\|y_n - z^\dagger + \beta\delta_n(f(x_n) - f(y_n))\|^2 \\ - (1 - \gamma_n)\gamma_n\|y_n - x_n + \beta\delta_n(f(x_n) - f(y_n))\|^2. \end{aligned} \tag{14}$$

Combining (13) and (14), we have

$$\begin{aligned} \|x_{n+1} - z^\dagger\|^2 &\leq \|x_n - z^\dagger\|^2 - \gamma_n(1 - L^2)\|y_n - x_n\|^2 \\ &\quad - (1 - \gamma_n)\gamma_n\|y_n - x_n + \beta\delta_n(f(x_n) - f(y_n))\|^2 \\ &\leq \|x_n - z^\dagger\|^2, \end{aligned} \tag{15}$$

which implies that $\lim_{n\to+\infty}\|x_n - z^\dagger\|$ exists. □

Proposition 3.2. $\lim_{n\to+\infty}\|y_n - x_n\| = 0$ *and* $\lim_{n\to+\infty}\|x_{n+1} - x_n\| = 0$.

Proof. By (15), we deduce that the sequence $\{x_n\}$ is bounded and

$$\gamma_n(1 - L^2)\|y_n - x_n\|^2 \leq \|x_n - z^\dagger\|^2 - \|x_{n+1} - z^\dagger\|^2 \to 0.$$

It results in that

$$\lim_{n\to+\infty}\|y_n - x_n\| = 0. \tag{16}$$

From (6), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\gamma_n[y_n - x_n - \beta\delta_n(f(y_n) - f(x_n))]\| \\ &\leq \gamma_n(1 + \beta\lambda\delta_n)\|y_n - x_n\|. \end{aligned}$$

This together with (16) implies that

$$\lim_{n\to+\infty}\|x_{n+1} - x_n\| = 0. \tag{17}$$

□

Proposition 3.3. $\omega_w(x_n) \subset (f + \varphi)^{-1}(0)$.

Proof. Letting $\hat{u} \in \omega_w(x_n)$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ satisfying $x_{n_i} \rightharpoonup \hat{u}$ as $i \to \infty$. Next, we will prove $\hat{u} \in (f+\varphi)^{-1}(0)$.

Let $(v^\dagger, \hat{b}) \in \text{graph}(f+\varphi)$. Then, $\hat{b} - f(v^\dagger) \in \varphi(v^\dagger)$. By virtue of (5), we have $y_{n_i} = (I + \beta\delta_{n_i}\varphi)^{-1}(x_{n_i} - \beta\delta_{n_i} f(x_{n_i}))$ which yields that

$$\frac{x_{n_i} - y_{n_i}}{\beta\delta_{n_i}} - f(x_{n_i}) \in \varphi(y_{n_i}). \tag{18}$$

According to the monotonicity of φ and (18), we derive

$$\langle \hat{b} - f(v^\dagger) - (\frac{x_{n_i} - y_{n_i}}{\beta\delta_{n_i}} - f(x_{n_i})), v^\dagger - y_{n_i}\rangle \geq 0.$$

It follows that

$$\begin{aligned}\langle \hat{b}, v^\dagger - y_{n_i}\rangle &\geq \langle f(v^\dagger) - f(x_{n_i}) + \frac{x_{n_i} - y_{n_i}}{\beta\delta_{n_i}}, v^\dagger - y_{n_i}\rangle \\ &= \langle f(v^\dagger) - f(y_{n_i}), v^\dagger - y_{n_i}\rangle + \langle f(y_{n_i}) - f(x_{n_i}), v^\dagger - y_{n_i}\rangle \\ &\quad + \frac{1}{\beta\delta_{n_i}}\langle x_{n_i} - y_{n_i}, v^\dagger - y_{n_i}\rangle.\end{aligned} \tag{19}$$

As a result of $\langle f(v^\dagger) - f(y_{n_i}), v^\dagger - y_{n_i}\rangle \geq 0$, from (19), we attain

$$\langle \hat{b}, v^\dagger - y_{n_i}\rangle \geq \langle f(y_{n_i}) - f(x_{n_i}), v^\dagger - y_{n_i}\rangle + \frac{1}{\beta\delta_{n_i}}\langle x_{n_i} - y_{n_i}, v^\dagger - y_{n_i}\rangle. \tag{20}$$

Since $x_{n_i} \rightharpoonup \hat{u}$, from Proposition 3.2, we conclude that $y_{n_i} \rightharpoonup \hat{u}$. So, by (20), we receive $\langle \hat{b}, v^\dagger - \hat{u}\rangle \geq 0$ for all $(v^\dagger, \hat{b}) \in \text{graph}(f+\varphi)$. Hence, $\hat{u} \in (f+\varphi)^{-1}(0)$ which implies that $\omega_w(x_n) \subset (f+\varphi)^{-1}(0)$. □

Next, we state our main convergence theorem.

Theorem 3.1. *The sequence $\{x_n\}$ generated by Algorithm 3.1 converges weakly to some point in $(f+\varphi)^{-1}(0)$.*

Proof. Based on Propositions 3.1-3.3, we have the following results in the hand: (i) $\forall z^\dagger \in (f+\varphi)^{-1}$, $\lim_{n\to\infty} \|x_n - z^\dagger\|$ exists; (ii) $\omega_w(x_n) \subset (f+\varphi)^{-1}$. Therefore, utilizing Lemma 2.1, we can conclude that the sequence $\{x_n\}$ generated by Algorithm 3.1 converges weakly to some point in $(f+\varphi)^{-1}$. This completes the proof. □

4. **Concluding remarks**

In this paper, we devote to construct an iterative algorithm for solving the variational inclusion problem (1) in Hilbert spaces. A popular algorithm for finding a solution of (1) is to use the well-known forward-backward algorithm in which the investigated operator f should be (inverse) strongly monotone in order to ensure the convergence of the algorithm. In our paper, we propose a self-adaptive Tseng-type iterative algorithm [Algorithm 3.1] in which the involved operator f is a general monotone operator. Under some additional conditional, we prove that the sequence $\{x_n\}$ generated by Algorithm 3.1 has weak convergence.

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