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STRONG PSEUDO-CONNES AMENABILITY OF CERTAIN BANACH ALGEBRAS

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In this paper, we introduce a new notion of strong pseudo-Connes amenability for dual Banach algebras. We study the relation between this new notion with classical notions of Connes-amenability. Also we show that for every non-empty set I, $\mathbb{M}_I(\mathbb{C})$ is strong pseudo-Connes amenable if and only if I is finite. We provide some examples of dual Banach algebras and we investigate their strong pseudo-Connes amenability.

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1. Introduction and preliminaries

The concept of amenability for Banach algebras was first introduced by B. E. Johnson [13]. A Banach algebra $\mathcal{A}$ is amenable if and only if there exists a bounded net (m_α) in $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that $a\cdot m_\alpha - m_\alpha \cdot a \to 0$ and $\pi_{\mathcal{A}}(m_\alpha)a \to a$ for every $a \in \mathcal{A}$. By removing the boundedness condition in the definition of amenability, Ghahramani and Zhang introduced the notion of pseudo-amenability [5]. That is a Banach algebra $\mathcal{A}$ is pseudo-amenable, if there exists a net (m_α) in $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that $a\cdot m_\alpha - m_\alpha \cdot a \to 0$ and $\pi_{\mathcal{A}}(m_\alpha)a \to a$ for every $a \in \mathcal{A}$.

Recently Sahami et. al. introduced a notion of amenability, named strong pseudo-amenability [14]. A Banach algebra $\mathcal{A}$ is called strong pseudo-amenable, if there exists a net (m_α) in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that

$$a\cdot m_\alpha - m_\alpha \cdot a \to 0, \quad a\pi^{**}_{\mathcal{A}}(m_\alpha) = \pi^{**}_{\mathcal{A}}(m_\alpha)a \to a \quad (a \in \mathcal{A}).$$

Mahmoodi introduced and studied the notion of pseudo-Connes amenability for dual Banach algebras, see [9]. A dual Banach algebra $\mathcal{A}$ is called pseudo-Connes amenable if there exists a net (m_α) in $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that for every $a \in \mathcal{A}$, $a\cdot m_\alpha - m_\alpha \cdot a \overset{wk^*}{\to} 0$ in $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ and $a\pi_{\sigma wc}(m_\alpha) \overset{wk^*}{\to} a$ in $\mathcal{A}$, see [9, Definition 4.1]. He showed that $\ell^1(\mathbb{N}, \max)$ is pseudo-Connes amenable but it is not Connes amenable [9, Example 6.1].

Motivated by these considerations, we introduce a new notion of strong pseudo-Connes amenability for dual Banach algebra which is weaker than Connes amenability but it is stronger than pseudo-Connes amenability. Here is the definition of our new notion:

Definition 1.1. *A dual Banach algebra $\mathcal{A}$ is called strong pseudo-Connes amenable, if there exists a net (m_α) in $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that*

(i) $a\cdot m_\alpha - m_\alpha \cdot a \xrightarrow{wk^*} 0$ *in* $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ *and*

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(ii) $a\pi_{\sigma wc}(m_\alpha) = \pi_{\sigma wc}(m_\alpha)a \xrightarrow{wk^*} a$ *in* $\mathcal{A}$,

for every $a \in \mathcal{A}$.

In Section 2, we investigate the relation between this new notion with the various notions of Connes-amenability. The Banach algebra of $I \times I$-matrices over $\mathbb{C}$, with finite ℓ^1-norm and matrix multiplication, is denoted by $\mathbb{M}_I(\mathbb{C})$. Using [4, Proposition 3.6] and the fact that $\mathbb{M}_I(\mathbb{C})$ is biflat (see [10, Proposition 2.7]). Applying [4, Proposition 3.5] implies that $\mathbb{M}_I(\mathbb{C})$ is pseudo-amenable. So $\mathbb{M}_I(\mathbb{C})$ is pseudo-Connes amenable for every index set I. But in Section 3 we will show that $\mathbb{M}_I(\mathbb{C})$ is strong pseudo-Connes amenable if and only if I is finite. Also the set of all $I \times I$-upper triangular matrices, say $UP(I, \mathcal{A})$ with finite ℓ^1 norm and the matrix operations becomes a Banach algebra. We prove that $UP(I, \mathcal{A})$ is strong pseudo-Connes amenable if and only if I is singleton and $\mathcal{A}$ is strong pseudo Connes-amenable, provided that $\mathcal{A}$ is a dual Banach algebra which has a wk^*-continuous character. In section 4, we provide some examples of certain dual Banach algebras and we study their strong pseudo-Connes amenability. For instance, we show that $\ell^1(\mathbb{N}, \max)$ is strong pseudo-Connes amenable but it is not Connes-amenable.

Here first we recall some notations and definitions that we shall need in this paper. The class of dual Banach algebras were introduced by Runde [11]. Let $\mathcal{A}$ be a Banach algebra and let E be a Banach $\mathcal{A}$-bimodule. An $\mathcal{A}$-bimodule E is called dual if there is a closed submodule E_* of E^* such that $E = (E_*)^*$. The Banach algebra $\mathcal{A}$ is called dual if it is dual as a Banach $\mathcal{A}$-bimodule. For a given dual Banach algebra $\mathcal{A}$ and a Banach $\mathcal{A}$-bimodule E, we denote by $\sigma wc(E)$, the set of all elements $x \in E$ such that the module maps $\mathcal{A} \to E$; $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are wk^*-wk-continuous, which is a closed submodule of E. Since $\sigma wc(\mathcal{A}_*) = \mathcal{A}_*$, the adjoint of $\pi_{\mathcal{A}}$ maps $\mathcal{A}_*$ into $\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*$. Therefore, $\pi_{\mathcal{A}}^{**}$ drops to an $\mathcal{A}$-bimodule morphism $\pi_{\sigma wc} : (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^* \to \mathcal{A}$. A suitable concept of amenability for dual Banach algebras is the Connes amenability. This notion under different name, for the first time was introduced by Johnson, Kadison, and Ringrose for von Neumann algebras, see [6]. The concept of Connes amenability for the larger class of dual Banach algebras was later extended by Runde. For more details and the history of Connes-amenability, see [11]. A dual Banach algebra $\mathcal{A}$ is called Connes-amenable if and only if $\mathcal{A}$ has a σwc-virtual diagonal, that is, there exists an element $M \in (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that $a \cdot M = M \cdot a$ and $a\pi_{\sigma wc}(M) = a$, for every $a \in \mathcal{A}$, see [12].

2. **Strong pseudo-Connes amenability**

In this section, we study the relation between the notion of strong pseudo-Connes amenability with pseudo-Connes amenability and Connes-amenability.

Remark 2.1. *Let* $\mathcal{A}$ *be a dual Banach algebra and let* X *be a Banach* $\mathcal{A}$*-bimodule. Since* $\sigma wc(X^*)$ *is a closed* $\mathcal{A}$*-submodule of* X^* *and* $\sigma wc(X^*)^* = \dfrac{X^{**}}{\sigma wc(X^*)^\perp}$*, we have a quotient map* $q : X^{**} \longrightarrow \sigma wc(X^*)^*$*, where* $q(x^{**}) = x^{**}|_{\sigma wc(X^*)}$ *for every* $x^{**} \in X^{**}$.

Proposition 2.1. *Let* $\mathcal{A}$ *be a strong pseudo-Connes amenable dual Banach algebra. Then* $\mathcal{A}$ *is pseudo-Connes amenable.*

Proof. Since $\mathcal{A}$ is strong pseudo-Connes amenable, there exists a net $(m_\alpha)_{\alpha\in I}$ in $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that

$$a \cdot m_\alpha - m_\alpha \cdot a \xrightarrow{wk^*} 0 \quad a\pi_{\sigma wc}(m_\alpha) = \pi_{\sigma wc}(m_\alpha)a \xrightarrow{wk^*} a \quad (a \in \mathcal{A}).$$

We denote inclusion map with $i : \mathcal{A}\hat{\otimes}\mathcal{A} \hookrightarrow (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ and also we denote $q : (\mathcal{A}\hat{\otimes}\mathcal{A})^{**} \longrightarrow (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ for the quotient map, as in Remark 2.1. Compose i with q and by Goldstein's theorem, we obtain a continuous $\mathcal{A}$-bimodule map $\eta : \mathcal{A}\hat{\otimes}\mathcal{A} \longrightarrow (\sigma\omega c(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ which has a wk^*-dense range. So there exists a net $(u^\alpha_\beta)_{\beta\in\Theta}$ in $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that $wk^*\text{-}\lim_\beta u^\alpha_\beta = m_\alpha$ in $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$. Thus

$$wk^*\text{-}\lim_\alpha wk^*\text{-}\lim_\beta(a\cdot u^\alpha_\beta - u^\alpha_\beta\cdot a) = wk^*\text{-}\lim_\alpha(a\cdot m_\alpha - m_\alpha\cdot a) = 0 \quad \text{in } (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*,$$

for every $a\in\mathcal{A}$. Since $\pi_{\sigma wc}$ is wk^*-continuous and the multiplication in $\mathcal{A}$ is separately wk^*-continuous [13, Exercise 4.4.1], we have

$$wk^*\text{-}\lim_\alpha wk^*\text{-}\lim_\beta a\pi_{\sigma wc}(u^\alpha_\beta) = wk^*\text{-}\lim_\alpha a\pi_{\sigma wc}(m_\alpha) = a \quad \text{in } \mathcal{A}. \tag{1}$$

Let $E = I\times\Theta^I$ be a directed set with product ordering which is defined by

$$(\alpha,\beta)\le_E(\alpha',\beta') \Leftrightarrow \alpha\le_I\alpha', \beta\le_{\Theta^I}\beta' \qquad (\alpha,\alpha'\in I, \quad \beta,\beta'\in\Theta^I),$$

where Θ^I is the set of all functions from I into Θ and $\beta\le_{\Theta^I}\beta'$ means that $\beta(d)\le_\Theta\beta'(d)$ for every $d\in I$. Suppose that $\gamma=(\alpha,\beta_\alpha)$ and $n_\gamma = u^\alpha_\beta$. By iterated limit theorem [7, Page 69], one can see that $wk^*\text{-}\lim_\gamma a\cdot n_\gamma - n_\gamma\cdot a = 0$ in $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ and $wk^*\text{-}\lim_\gamma a\pi_{\sigma wc}(n_\gamma) = a$ in $\mathcal{A}$. So by [9, Definition 4.3] $\mathcal{A}$ is pseudo-Connes amenable. □

Lemma 2.1. *Let $\mathcal{A}$ be a commutative pseudo-Connes amenable dual Banach algebra. Then $\mathcal{A}$ is strong pseudo-Connes amenable.*

Proof. Let $\mathcal{A}$ be a pseudo-Connes amenable dual Banach algebra. Then there exists a net (m_α) in $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that for every $a\in\mathcal{A}$, $a\cdot m_\alpha - m_\alpha\cdot a \stackrel{wk^*}{\to} 0$ in $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ and $a\pi_{\sigma wc}(m_\alpha)\stackrel{wk^*}{\to} a$ in $\mathcal{A}$. Since $\mathcal{A}$ is commutative, $\pi_{\sigma wc}(m_\alpha)a = a\pi_{\sigma wc}(m_\alpha)$. Hence $\mathcal{A}$ is strong pseudo-Connes amenable. □

Proposition 2.2. *Let $\mathcal{A}$ be a dual Banach algebra. If $\mathcal{A}$ is Connes-amenable, then $\mathcal{A}$ is strong pseudo-Connes amenable.*

Proof. Let $\mathcal{A}$ be a Connes-amenable Banach algebra. Then by [12, Theorem 4.8], there is an element $M\in(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that

$$a\cdot M = M\cdot a \quad \text{and} \quad \pi_{\sigma wc}(M)a = a \quad (a\in\mathcal{A}).$$

Since $\pi_{\sigma wc}$ is an $\mathcal{A}$-bimodule homomorphism, $a\pi_{\sigma wc}(M) = \pi_{\sigma wc}(M)a$. So $\mathcal{A}$ is strong pseudo-Connes amenable. □

Remark 2.2. *Let $\mathcal{A}$ be a strong pseudo-Connes amenable dual Banach algebra. In the Definition 1.1 if (m_α) is considered as a bounded net, then by Banach-Alaoglu theorem, there is a wk^*-limit point for the net (m_α). Let $M = wk^*\text{-}\lim_\alpha m_\alpha$. One can see that M is a σwc-virtual diagonal for $\mathcal{A}$. So $\mathcal{A}$ is Connes-amenable, see* [12, Theorem 4.8].

Proposition 2.3. *Let $\mathcal{A}$ be a dual Banach algebra. If $\mathcal{A}$ is strong pseudo-amenable, then $\mathcal{A}$ is strong pseudo Connes-amenable.*

Proof. Since $\mathcal{A}$ is strong pseudo-amenable, there exists a net (m_α) in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that

$$a\cdot m_\alpha - m_\alpha\cdot a\to 0, \quad a\pi^{**}_{\mathcal{A}}(m_\alpha) = \pi^{**}_{\mathcal{A}}(m_\alpha)a\to a, \tag{2}$$

for every $a\in\mathcal{A}$. Let $\tilde{m}_\alpha = q(m_\alpha)$, where $q:(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}\to(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ is the quotient map as in Remark 2.1. So $a\cdot\tilde{m}_\alpha - \tilde{m}_\alpha\cdot a\stackrel{wk^*}{\to} 0$ for every $a\in\mathcal{A}$. Consider

$$\begin{aligned}\langle f, \pi_{\sigma wc}q(m_\alpha)\rangle &= \langle \pi^*|_{\mathcal{A}_*}(f), q(m_\alpha)\rangle = \langle\pi^*|_{\mathcal{A}_*}(f), m_\alpha\rangle\\ &= \langle\pi^*_{\mathcal{A}}(f), m_\alpha\rangle = \langle f, \pi^{**}_{\mathcal{A}}(m_\alpha)\rangle,\end{aligned}$$

for every $f\in\mathcal{A}_*$. It follows that

$$\pi_{\sigma wc}q = \pi^{**}_{\mathcal{A}}. \tag{3}$$

By (3) and (2), we have

$$wk^*\text{-}\lim_\alpha a\pi_{\sigma wc}(\tilde{m}_\alpha) = wk^*\text{-}\lim_\alpha a\pi_{\sigma wc}q(m_\alpha) = wk^*\text{-}\lim_\alpha a\pi_{\mathcal{A}}^{**}(m_\alpha) = a,$$

and also

$$wk^*\text{-}\lim_\alpha \pi_{\sigma wc}(\tilde{m}_\alpha)a = wk^*\text{-}\lim_\alpha \pi_{\sigma wc}q(m_\alpha)a = wk^*\text{-}\lim_\alpha \pi_{\mathcal{A}}^{**}(m_\alpha)a = a.$$

Hence $\mathcal{A}$ is strong pseudo Connes-amenable. □

A dual Banach algebra $\mathcal{A}$ is called Connes biprojective, if there exists a bounded $\mathcal{A}$-bimodule morphism $\rho : \mathcal{A} \longrightarrow (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that $\pi_{\sigma wc} \circ \rho = id_{\mathcal{A}}$. Shirinkalam and Pourabbas in [16] showed that a dual Banach algebra $\mathcal{A}$ is Connes amenable if and only if $\mathcal{A}$ is Connes biprojective and has an identity.

Proposition 2.4. *Let $\mathcal{A}$ be a dual Banach algebra with a central approximate identity. If $\mathcal{A}$ is Connes biprojective, then $\mathcal{A}$ is strong pseudo-Connes amenable.*

Proof. Let (e_α) be a central approximate identity for $\mathcal{A}$ and let $\rho : \mathcal{A} \longrightarrow (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ be a bounded $\mathcal{A}$-bimodule morphism such that $\pi_{\sigma wc} \circ \rho = id_{\mathcal{A}}$. Put $m_\alpha = \rho(e_\alpha)$. Since ρ is a bounded $\mathcal{A}$-bimodule morphism, one can see that $a \cdot m_\alpha = m_\alpha \cdot a$ and $a\pi_{\sigma wc}(m_\alpha) = \pi_{\sigma wc}(m_\alpha)a \longrightarrow a$ for every $a \in \mathcal{A}$. So $\mathcal{A}$ is strong pseudo-Connes amenable. □

Proposition 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be dual Banach algebras. Suppose that $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ is a continuous epimorphism which is also wk^*-continuous. If $\mathcal{A}$ is strong pseudo-Connes amenable, then $\mathcal{B}$ is strong pseudo-Connes amenable.*

Proof. Since $\mathcal{A}$ is strong pseudo-Connes amenable, there exists a net (m_α) in $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that

$$a \cdot m_\alpha - m_\alpha \cdot a \xrightarrow{wk^*} 0, \quad a\pi_{\sigma wc}^{\mathcal{A}}(m_\alpha) = \pi_{\sigma wc}^{\mathcal{A}}(m_\alpha)a \xrightarrow{wk^*} a, \tag{4}$$

for every $a \in \mathcal{A}$. Define $\theta\otimes\theta : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{B}\hat{\otimes}\mathcal{B}$ by $\theta\otimes\theta(x\otimes y) = \theta(x)\otimes\theta(y)$, for every $x, y \in \mathcal{A}$. One can see that $\theta\otimes\theta$ is a bounded linear map. Also for each $a \in \mathcal{A}$ and $u \in \mathcal{A}\hat{\otimes}\mathcal{A}$, we have

$$\theta(a)\cdot(\theta\otimes\theta)(u) = (\theta\otimes\theta)(a\cdot u), \qquad (\theta\otimes\theta)(u)\cdot\theta(a) = (\theta\otimes\theta)(u\cdot a).$$

By [9, Lemma 4.4], we have

$$a\cdot(\theta\otimes\theta)^*(f) = (\theta\otimes\theta)^*(\theta(a)\cdot f), \qquad (\theta\otimes\theta)^*(f)\cdot a = (\theta\otimes\theta)^*(f\cdot\theta(a)), \tag{5}$$

for every $a \in \mathcal{A}$ and $f \in (\mathcal{B}\hat{\otimes}\mathcal{B})^*$. So

$$(\theta\otimes\theta)^*(\sigma wc(\mathcal{B}\hat{\otimes}\mathcal{B})^*) \subseteq \sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*. \tag{6}$$

Define the map

$$\Psi := ((\theta\otimes\theta)^*|_{\sigma wc(\mathcal{B}\hat{\otimes}\mathcal{B})^*})^* : (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^* \to (\sigma wc(\mathcal{B}\hat{\otimes}\mathcal{B})^*)^*.$$

Set $n_\alpha = \Psi(m_\alpha)$. Then (5) implies that for every $a \in \mathcal{A}$ and $T \in \sigma wc(\mathcal{B}\hat{\otimes}\mathcal{B})^*$

$$\begin{aligned}\langle T, \Psi(a\cdot m_\alpha)\rangle &= \langle(\theta\otimes\theta)^*(T), a\cdot m_\alpha\rangle = \langle(\theta\otimes\theta)^*(T)\cdot a, m_\alpha\rangle\\ &= \langle(\theta\otimes\theta)^*(T\cdot\theta(a)), m_\alpha\rangle = \langle(\theta\otimes\theta)^*|_{\sigma wc(\mathcal{B}\hat{\otimes}\mathcal{B})^*}(T\cdot\theta(a)), m_\alpha\rangle\\ &= \langle T\cdot\theta(a), \Psi(m_\alpha)\rangle = \langle T, \theta(a)\cdot\Psi(m_\alpha)\rangle,\end{aligned}$$

The right action is similar, then we have $\Psi(m_\alpha\cdot a) = \Psi(m_\alpha)\cdot\theta(a)$ for every $a \in \mathcal{A}$. Since Ψ is wk^*-continuous,

$$\lim_\alpha\langle T, \theta(a)\cdot n_\alpha - n_\alpha\cdot\theta(a)\rangle = \lim_\alpha\langle T, \Psi(a\cdot m_\alpha - m_\alpha\cdot a)\rangle = 0 \quad (T \in \sigma wc(\mathcal{B}\hat{\otimes}\mathcal{B})^*).$$

Hence $\theta(a)\cdot n_\alpha - n_\alpha\cdot\theta(a) \xrightarrow{wk^*} 0$. By using the argument of as in the proof of [9, Proposition 4.5 (ii)] we have $\pi_{\sigma wc}^{\mathcal{B}}\circ\theta\otimes\theta(u) = \theta\circ\pi_{\sigma wc}^{\mathcal{A}}$, for every $u \in \mathcal{A}\hat{\otimes}\mathcal{A}$. Since $i : \mathcal{A}\hat{\otimes}\mathcal{A} \to (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$

is a wk^*-dense range map, for each α, there exists (u_β^α) in $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that $wk^*\text{-}\lim_\alpha u_\beta^\alpha = m_\alpha$ in $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$. The wk^*-continuity of the maps $\pi_{\sigma wc}^{\mathcal{B}}$, Ψ, θ and $\pi_{\sigma wc}^{\mathcal{A}}$ imply that

$$\begin{aligned}\pi_{\sigma wc}^{\mathcal{B}} \circ \Psi(m_\alpha) &= wk^*\text{-}\lim_\alpha \pi_{\sigma wc}^{\mathcal{B}} \circ \Psi(u_\beta^\alpha) = wk^*\text{-}\lim_\alpha \pi_{\sigma wc}^{\mathcal{B}} \circ \theta \otimes \theta(u_\beta^\alpha)\\ &= wk^*\text{-}\lim_\alpha \theta \circ \pi_{\sigma wc}^{\mathcal{A}}(u_\beta^\alpha) = \theta \circ \pi_{\sigma wc}^{\mathcal{A}}(m_\alpha).\end{aligned}$$

Since θ is wk^*-continuous, (4) follows that

$$\theta(a)\pi_{\sigma wc}^{\mathcal{B}}(\Psi(m_\alpha)) = \pi_{\sigma wc}^{\mathcal{B}}(\Psi(m_\alpha))\theta(a) \xrightarrow{wk^*} \theta(a).$$

So $\mathcal{B}$ is strong pseudo-Connes amenable. □

Corollary 2.1. *Let $\mathcal{A}$ be a dual Banach algebra and let I be a wk^*-closed ideal of $\mathcal{A}$. If $\mathcal{A}$ is strong pseudo-Connes amenable, then $\mathcal{A}/I$ is strong pseudo-Connes amenable.*

Proof. Since I is wk^*-closed, the quotient algebra $\mathcal{A}/I$ is dual with respect the predual ${}^\perp I$. So we have the inclusion map $i : {}^\perp I \hookrightarrow \mathcal{A}_*$. It follows that the quotient map $q : \mathcal{A} \to \mathcal{A}/I$ is wk^*-continuous. By Proposition 2.5 the dual Banach algebra $\mathcal{A}/I$ is strong pseudo-Connes amenable. □

Lemma 2.2. *Let $\mathcal{A}$ be a dual Banach algebra and $\varphi \in \Delta_{wk^*}(\mathcal{A})$. If $\mathcal{A}$ is strong pseudo-Connes amenable, then there is a net (n_α) in $\mathcal{A}$ such that*

$$an_\alpha - \varphi(a)n_\alpha \xrightarrow{wk^*} 0, \qquad \varphi(n_\alpha) \to 1 \quad (a \in \mathcal{A}).$$

Proof. Since $\mathcal{A}$ is strong pseudo-Connes amenable, there is a net (m_α) in $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that for every $a \in \mathcal{A}$

$$a \cdot m_\alpha - m_\alpha \cdot a \xrightarrow{wk^*} 0, \quad a\pi_{\sigma wc}(m_\alpha) = \pi_{\sigma wc}(m_\alpha)a \xrightarrow{wk^*} a.$$

Define $\theta : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$ by $\theta(a \otimes b) = \varphi(b)a$ for every $a, b \in \mathcal{A}$. It is easy to see that

$$a \cdot \theta^*(f) = \varphi(a)\theta^*(f), \quad \theta^*(f) \cdot a = \theta^*(f \cdot a) \quad (a \in \mathcal{A}, f \in \mathcal{A}^*), \tag{7}$$

and also

$$\langle \theta(u), \varphi \rangle = \langle \pi(u), \varphi \rangle \qquad (u \in \mathcal{A}\hat{\otimes}\mathcal{A}). \tag{8}$$

Since φ is wk^*-continuous, $\varphi \in \mathcal{A}_*$. So (7) implies that $\theta^*(\mathcal{A}_*) \subseteq \sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*$. Define $\tau := (\theta^*|_{\mathcal{A}_*})^* : (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^* \to \mathcal{A}$. Set $n_\alpha = \tau(m_\alpha)$ for every α. Thus by (7), we have

$$\begin{aligned}\langle an_\alpha, f\rangle &= \langle a\tau(m_\alpha), f\rangle = \langle \tau(m_\alpha), f \cdot a\rangle\\ &= \langle m_\alpha, \theta^*(f \cdot a)\rangle = \langle m_\alpha, \theta^*(f) \cdot a\rangle\\ &= \langle a \cdot m_\alpha, \theta^*(f)\rangle,\end{aligned} \tag{9}$$

for every $a \in \mathcal{A}$ and $f \in \mathcal{A}_*$. Also

$$\begin{aligned}\langle \varphi(a)n_\alpha, f\rangle &= \langle \tau(m_\alpha), \varphi(a)f\rangle = \langle m_\alpha, \theta^*(\varphi(a)f)\rangle = \langle m_\alpha, \varphi(a)\theta^*(f)\rangle\\ &= \langle m_\alpha, a \cdot \theta^*(f)\rangle = \langle m_\alpha \cdot a, \theta^*(f)\rangle.\end{aligned} \tag{10}$$

Since $\lim_\alpha \langle a \cdot m_\alpha - m_\alpha \cdot a, \theta^*(f)\rangle = 0$, by (9) and (10) we have $an_\alpha - \varphi(a)n_\alpha \xrightarrow{wk^*} 0$. Using the Goldstein's Theorem for every $u \in (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ there exists a net (x_α) in $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that $wk^*\text{-}\lim_\alpha x_\alpha = u$ in $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$. One can see that $\pi_{\sigma wc}(u) = wk^*\text{-}\lim_\alpha \pi_{\sigma wc}(x_\alpha) = wk^*\text{-}\lim_\alpha \pi(x_\alpha)$ and $\tau(u) = wk^*\text{-}\lim_\alpha \tau(x_\alpha) = wk^*\text{-}\lim_\alpha \theta(x_\alpha)$. So by (8) we have $\langle \varphi, \pi_{\sigma wc}(u)\rangle = \lim_\alpha \langle \varphi, \pi(x_\alpha)\rangle = \lim_\alpha \langle \varphi, \theta(x_\alpha)\rangle = \langle \varphi, \tau(u)\rangle$. Then

$$\langle \varphi, \pi_{\sigma wc}(m_\alpha)\rangle = \langle \varphi, \tau(m_\alpha)\rangle, \tag{11}$$

for all α. Since φ is wk^*-continuous, $\varphi \in \mathcal{A}_*$. Therefore $\varphi(\pi_{\sigma wc}(m_\alpha))\varphi(a) \to \varphi(a)$. Thus $\varphi(\pi_{\sigma wc}(m_\alpha)) \to 1$ in $\mathbb{C}$. By (11), $\varphi(n_\alpha) \to 1$. □

Remark 2.3. *Note that in the proof of Lemma 2.2 with the same conditions, if we define* $\theta : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$ *by* $\theta(a \otimes b) = \varphi(a)b$ *for every* $a, b \in \mathcal{A}$*, then there exists a net* (n_α) *in* $\mathcal{A}$ *such that* $n_\alpha a - \varphi(a)n_\alpha \stackrel{wk^*}{\to} 0, \qquad \varphi(n_\alpha) \to 1 \quad (a \in \mathcal{A})$.

3. Some applications for matrix algebras

Following [3], let $\mathcal{A}$ be a Banach algebra. Suppose that I and J are arbitrary non-empty sets. Let P be a $J \times I$ matrix over $\mathcal{A}$ such that $\|P\|_\infty = \sup\{\|P_{j,i}\| : j \in J, i \in I\} \leq 1$. The set of all $I \times J$ matrices over $\mathcal{A}$ with finite ℓ^1-norm and product $XY = XPY$ is a Banach algebra, which is denoted by $LM(\mathcal{A}, P)$ and it is called the ℓ^1-Munn $I \times J$ matrix algebra over $\mathcal{A}$ with sandwich matrix P.
We recall that $\mathbb{M}_I(\mathbb{C})$ the Banach algebra of $I \times I$-matrices over $\mathbb{C}$, with finite ℓ^1-norm and matrix operations, is a dual ℓ^1-Munn algebra [17]. Note that for a Banach algebra $\mathcal{A}$, the map $\theta : \mathbb{M}_I(\mathcal{A}) \to \mathcal{A} \otimes_p \mathbb{M}_I(\mathbb{C})$ defined by $\theta(a_{ij}) = \sum_{i,j\in I} a_{ij} \otimes E_{ij}$ is an isometric isomorphism, where E_{ij} are the matrix units.

Theorem 3.1. *Let* I *be a non-empty set. Then* $\mathbb{M}_I(\mathbb{C})$ *is strong pseudo-Connes amenable if and only if* I *is finite.*

Proof. Let $\mathcal{A} = \mathbb{M}_I(\mathbb{C})$ be strong pseudo-Connes amenable. Then there exists a net (m_α) in $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that

$$a \cdot m_\alpha - m_\alpha \cdot a \xrightarrow{wk^*} 0, \quad a\pi_{\sigma wc}(m_\alpha) = \pi_{\sigma wc}(m_\alpha)a \xrightarrow{wk^*} a, \tag{12}$$

for every $a \in \mathcal{A}$. Set $e_\alpha = \pi_{\sigma wc}(m_\alpha)$. Clearly (e_α) is a net in $\mathcal{A} = \mathbb{M}_I(\mathbb{C}) \cong \mathbb{C} \otimes_p \mathbb{M}_I(\mathbb{C})$ such that $ae_\alpha = e_\alpha a$ and $e_\alpha a \xrightarrow{wk^*} a$ for each $a \in \mathcal{A}$. Then $e_\alpha = \sum_{i,j\in I} a_{ij}^{(\alpha)} \otimes E_{i,j}$ where (a_{ij}) is a net in $\mathbb{C}$ such that $\sum_{i,j\in I} |a_{i,j}| < \infty$. Define $F_{ij} = 1 \otimes E_{ij}$. Since e_α commutes with each element of $\mathcal{A}$, for each k, l and α in I we have $F_{kl}e_\alpha = e_\alpha F_{kl}$. It follows that

$$\sum_{j\in I} a_{lj}^{(\alpha)} \otimes E_{kj} = \sum_{i\in I} a_{ik}^{(\alpha)} \otimes E_{il}, \quad (\alpha \in I). \tag{13}$$

Suppose that k and l are two different elements of I. Applying (13) gives that $a_{lk}^{(\alpha)} = 0$. Also it follows that $a_{ll}^{(\alpha)} = a_{kk}^{(\alpha)}$. We claim that I is finite. We assume in contradiction that I is infinite. Since $\sum_{ij\in I} |a_{ij}^{(\alpha)}| < \infty$ and $a_{ll}^{(\alpha)} = a_{kk}^{(\alpha)}$, we have $a_{ij}^{(\alpha)} = 0$, for each i, j and α in I. Thus $e_\alpha = 0$. But $e_\alpha a \xrightarrow{wk^*} a$, follows that $a = 0$ for all $a \in \mathcal{A}$ which is a contradiction.

Conversely, if I is finite, then $\mathcal{A}$ is Connes amenable [8, Theorem 3.7]. So by Proposition 2.2, $\mathcal{A}$ is strong pseudo-Connes amenable. □

Remark 3.1. *In the previous Theorem, we give a pseudo-Connes amenable Banach algebra which is not strong pseudo-Connes amenable. Indeed Ramsden in* [10, Proposition 2.7] *showed that* $\mathbb{M}_I(\mathbb{C})$ *is biflat. Since* $\mathbb{M}_I(\mathbb{C})$ *has an approximate identity(see* [4, Proposition 3.6]*), by* [4, Proposition 3.5] $\mathbb{M}_I(\mathbb{C})$ *is pseudo-amenable. So* $M_I(\mathbb{C})$ *is pseudo Connes-amenable for each index set* I*. But if* I *is infinite, by previous Theorem* $\mathbb{M}_I(\mathbb{C})$ *is not strong pseudo-Connes amenable.*

Let $\mathcal{A}$ be a dual Banach algebra and let I be a totally ordered set. Then the set of all $I \times I$-upper triangular matrices with the usual matrix operations and the norm $\| [a_{i,j}]_{i,j\in I} \| = \sum_{i,j\in I} \| a_{i,j} \| < \infty$, becomes a Banach algebra and it is denoted by

$$UP(I, \mathcal{A}) = \{[\ a_{i,j}\ ]_{i,j\in I}\,; a_{i,j} \in A \text{ and } a_{i,j} = 0 \text{ for every } i > j\}.$$

Authors showed that $UP(I, \mathcal{A})$ is a dual Banach algebra, see [15, Theorem 3.1].

Theorem 3.2. *Let $\mathcal{A}$ be a dual Banach algebra with $\varphi \in \Delta_{wk^*}(\mathcal{A})$ and let I be a totally ordered set with smallest element. Then $UP(I,\mathcal{A})$ is strong pseudo-Connes amenable if and only if $\mathcal{A}$ is strong pseudo-Connes amenable and $|I| = 1$.*

Proof. Let $UP(I,\mathcal{A})$ be strong pseudo-Connes amenable. Assume that i_0 be a smallest element and $\varphi \in \Delta_{wk^*}(\mathcal{A})$. We define a map $\psi : UP(I,\mathcal{A}) \longrightarrow \mathbb{C}$ by $[a_{i,j}]_{i,j\in I} \longmapsto \varphi(a_{i_0,i_0})$ for every $[a_{i,j}]_{i,j\in I} \in UP(I,\mathcal{A})$. We claim that $\psi \in \Delta_{wk^*}(UP(I,\mathcal{A}))$. Suppose that (X_α) is a net in $UP(I,\mathcal{A})$, where $X_\alpha = \left[a^\alpha_{i,j}\right]_{i,j\in I}$ for every α and $X = [a_{i,j}]_{i,j\in I}$ be an element in $UP(I,\mathcal{A})$ such that $X_\alpha \overset{wk^*}{\to} X$. For an arbitrary $f \in \mathcal{A}_*$, let F be a matrix with f in (i_0,i_0)-th position and 0 elsewhere. So $F \in UP(I,\mathcal{A})_*$ and $\langle F, X_\alpha\rangle \to \langle F, X\rangle$. Then $\langle f, a^\alpha_{i_0,i_0}\rangle \to \langle f, a_{i_0,i_0}\rangle$. Since f is an arbitrary element, $a^\alpha_{i_0,i_0} \overset{wk^*}{\to} a_{i_0,i_0}$ in $\mathcal{A}$. Since φ is wk^*-continuous, $\varphi(a^\alpha_{i_0,i_0}) \to \varphi(a_{i_0,i_0})$. By Lemma 2.2 and Remark 2.3, there exists a net (n_α) in $UP(I,\mathcal{A})$ such that $n_\alpha a - \psi(a)n_\alpha \overset{wk^*}{\to} 0, \quad \psi(n_\alpha) \to 1 \quad (a \in UP(I,\mathcal{A}))$. Following the arguments in the proof [14, Theorem 3.1], suppose that $J = \left\{ [a_{i,j}]_{i,j\in I} \in UP(I,\mathcal{A}) \mid a_{i,j} = 0 \quad \forall i \neq i_0 \right\}$. One can see that J is a wk^*-closed ideal in $UP(I,\mathcal{A})$ and $\psi|_J \neq 0$. Consider $j_0 \in J$ such that $\psi(j_0) = 1$. Let $m_\alpha = j_0 n_\alpha$. Since the multiplication in $UP(I,\mathcal{A})$ is separately wk^*-continuous [13, Exercise 4.4.1], we may assume that (m_α) is a net in J such that

$$m_\alpha a - \psi(a)m_\alpha \overset{wk^*}{\to} 0, \qquad \psi(m_\alpha) \to 1 \quad (a \in J). \tag{14}$$

Suppose that $|I| > 1$ and m_α has a form $\begin{pmatrix} a^\alpha_{i_0,i_0} & a^\alpha_{i_0,i} & \cdots \\ 0 & 0 & \cdots \\ \vdots & \cdots & \vdots \end{pmatrix}$, for some nets $(a^\alpha_{i_0,i_0})$ in $\mathcal{A}$. So $\varphi(a^\alpha_{i_0,i_0}) = \psi(m_\alpha) \to 1$. Consider $x \in \mathcal{A}$ such that $\varphi(x) = 1$. Let $a = \begin{pmatrix} 0 & x & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \cdots & \vdots \end{pmatrix} \in J$. Since $\psi(a) = 0$, (14) implies that $m_\alpha a \overset{wk^*}{\to} 0$. A simple calculation follows that $a^\alpha_{i_0,i_0} x \overset{wk^*}{\to} 0$. Since φ is wk^*-continuous, $\varphi(a^\alpha_{i_0,i_0}) = \varphi(a^\alpha_{i_0,i_0})\varphi(x) = \varphi(a^\alpha_{i_0,i_0} x) \to 0$, which is a contradiction. So $|I| = 1$ and $\mathcal{A} = UP(I,\mathcal{A})$ is strong pseudo-Connes amenable.

Converse is clear. □

4. **Examples**

Example 4.1. *Consider the Banach algebra ℓ^1 of all sequences $a = (a_n)$ of complex numbers with $\|a\| = \sum_{n=1}^{\infty} |a_n| < \infty$, and the following product*

$$(a * b)(n) = \begin{cases} a(1)b(1) & \text{if} \quad n = 1 \\ a(1)b(n) + b(1)a(n) + a(n)b(n) & \text{if} \quad n > 1 \end{cases}$$

for every $a, b \in \ell^1$. It is easy to see that $\Delta(\ell^1) = \{\varphi_1\} \cup \{\varphi_1 + \varphi_n : n \geq 2\}$, where $\varphi_n(a) = a(n)$ for every $a \in \ell^1$. We claim that $(\ell^1, *)$ is a dual Banach algebra with respect to c_0. We show that c_0 is an ℓ^1-module with dual actions. In fact we have

$$a \cdot \lambda(n) = \begin{cases} \sum_{k=1}^{\infty} a(k)\lambda(k) & \text{if} \quad n = 1 \\ (a(1) + a(n))\lambda(n) & \text{if} \quad n > 1, \end{cases}$$

for every $a \in \ell^1$ and $\lambda \in c_0$. Since λ vanishes at infinity and $\sup_n |a(n)| < \infty$, one can see that $a \cdot \lambda$ vanishes at infinity. Similarly the right action is also vanishes at infinity. So c_0 is a closed ℓ^1-submodule of ℓ^∞. We claim that ℓ^1 is not strong pseudo-Connes amenable.

Suppose conversely that ℓ^1 *is strong pseudo-Connes amenable. Since* φ_1 *is* wk^**-continuous, by Lemma 2.2 there is a bounded net* (m_α) *in* ℓ^1 *such that*

$$a * m_\alpha - \varphi_1(a)m_\alpha \xrightarrow{wk^*} 0 \quad \text{and} \quad \varphi_1(m_\alpha) \longrightarrow 1 \qquad (a \in \ell^1). \tag{15}$$

Choose $a = \delta_n$ *in* ℓ^1*, where* $n \geq 2$*. So* $\varphi_1(\delta_n) = 0$*. (15) implies that* $\delta_n * m_\alpha \xrightarrow{wk^*} 0$ *in* ℓ^1*. One can see that* $\delta_n * m_\alpha = (m_\alpha(1) + m_\alpha(n))\delta_n$*. Consider* δ_n *as an element in* c_0*, where* $n \geq 2$*. So* $\lim_\alpha \langle \delta_n, \delta_n * m_\alpha \rangle = \lim_\alpha m_\alpha(1) + m_\alpha(n) = 0$*. Since* $\lim_\alpha m_\alpha(1) = 1$ *and* $\lim_\alpha m_\alpha(n) = -1$ *for every* $n \geq 2$*, we have* $\sup_\alpha \|m_\alpha\| = \infty$*, which contradicts with the boundedness of the net* (m_α)*.*

Example 4.2. *We give a strong psuedo-Connes amenable Banach algebra which is not Connes amenable.*

Let $S = \mathbb{N}$ *be the set of natural numbers with the binary operation* $(m, n) \longmapsto \max\{m, n\}$*, for every* m *and* n *in* $\mathbb{N}$*. Then* S *is a weakly cancellative semigroup, that is, for every* $s, t \in S$ *the set* $\{x \in S : sx = t\}$ *is finite. So* $\ell^1(S)$ *is a dual Banach algebra with predual* $c_0(S)$*, see* [1, Theorem 4.6]*. By* [9, Example 6.1]*,* $\ell^1(S)$ *is pseudo-Connes amenable. Since* $\ell^1(S)$ *is commutative, by Lemma 2.1 it is strong pseudo-Connes amenable. But* $\ell^1(S)$ *is not Connes amenable, see* [2, Theorem 5.13].

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