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SUFFICIENT EFFICIENCY CONDITIONS FOR A MINIMIZING FRACTIONAL PROGRAM

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Considerăm problema de minimizare a programului (MFP) în care obiectivul este un vector de câturi de funcţionale integrale curbilinii cu restricţii inecuaţii cu derivate parţiale (IDP) şi/sau ecuaţii cu derivate parţiale (EDP). Scopul acestei lucrări este de a introduce şi studia condiţii suficiente de eficienţă a unei soluţii realizabile a problemei (MFP). Rezultatele prezentate în §2 sunt originale, ele finalizând rezultate recente, al căror studiu este iniţiat în [7] *şi* [8].

We consider the minimizing fractional program (MFP), where the objective is a vector of functionals quotients of paths integrals and the constraints are partial differential inequations (PDI) and partial differential equations (PDE). The aim of this work is to introduce and study sufficient conditions for the efficiency of a feasible solution of the problem (MFP). The results discussed in §2 are new and finalize a recent research initiated in [7] *and* [8].

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1. Minimizing fractional programs

Before presenting our results, we need the following background which is necessary for introducing notations and for the completeness of the exposition. For more details, we address the reader to [7], [8].

Let (T, h) and (M, g) be Riemannian manifolds of dimensions p and n, respectively. Denote by $t = (t^\alpha)$, $\alpha = \overline{1,p}$, and $x = (x^i)$, $i = \overline{1,n}$, the local coordinates on T and M, respectively. Consider $J^1(T, M)$ be the first order jet bundle associated to T and M.

Using the product order relation on $\mathbb{R}^p$, [5], the hyperparallelepiped Ω_{t_0,t_1}, in $\mathbb{R}^p$, with the diagonal opposite points $t_0 = (t_0^1, \dots, t_0^p)$ and $t_1 = (t_1^1, \dots, t_1^p)$, can be written as being the interval $[t_0, t_1]$. Suppose γ_{t_0,t_1} is a piecewise C^1-class curve joining the points t_0 and t_1.

The closed Lagrange 1-forms densities of C^∞-class

$$f_\alpha = (f_\alpha^\ell)\colon J^1(T, M) \to \mathbb{R}^r, \quad k_\alpha = (k_\alpha^\ell)\colon J^1(T, M) \to \mathbb{R}^r, \quad \ell = \overline{1,r}, \quad \alpha = \overline{1,p}$$

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determine the following path independent curvilinear functionals (actions)

$$F^\ell(x(\cdot)) = \int_{\gamma_{t_0,t_1}} f^\ell_\alpha(t, x(t), x_\gamma(t))\, dt^\alpha, \quad K^\ell(x(\cdot)) = \int_{\gamma_{t_0,t_1}} k^\ell_\alpha(t, x(t), x_\gamma(t))\, dt^\alpha,$$

where $x_\gamma(t) = \dfrac{\partial x}{\partial t^\gamma}(t)$, $\gamma = \overline{1,p}$, are partial velocities.

The closeness conditions (complete integrability conditions) are

$$D_\beta f^\ell_\alpha = D_\alpha f^\ell_\beta, \quad D_\beta k^\ell_\alpha = D_\alpha k^\ell_\beta, \quad \alpha, \beta = \overline{1,p},\ \alpha \neq \beta,\ \ell = \overline{1,r},$$

where D_β is the total derivative.

Suppose $K^\ell(x(\cdot)) > 0$, for all $\ell = \overline{1,r}$, and accept that the Lagrange matrix densities

$$g = (g^b_a)\colon J^1(T, M) \to \mathbb{R}^{ms}, \quad a = \overline{1,s}, \quad b = \overline{1,m}, \quad m < n,$$
$$h = (h^b_a)\colon J^1(T, M) \to \mathbb{R}^{qs}, \quad a = \overline{1,s}, \quad b = \overline{1,q},\ q < n,$$

of C^∞-class define the partial differential inequations (PDI) (of evolution)

$$g(t, x(t), x_\gamma(t)) \leqq 0, \quad t \in \Omega_{t_0,t_1},$$

and the partial differential equations (PDE) (of evolution)

$$h(t, x(t), x_\gamma(t)) = 0, \quad t \in \Omega_{t_0,t_1}.$$

On the set $C^\infty(\Omega_{t_0,t_1}, M)$ of all functions $x\colon \Omega_{t_0,t_1} \to M$ of C^∞-class, we set the norm

$$\|x\| = \|x\|_\infty + \sum_{\alpha=1}^{p} \|x_\alpha\|_\infty.$$

Denote by

$$\mathcal{F}(\Omega_{t_0,t_1}) = \Big\{ x \in C^\infty(\Omega_{t_0,t_1}, M) \,|\, x(t_0) = x_0,\ x(t_1) = x_1,\ \text{or}\ x(t)|_{\partial\Omega_{t_0,t_1}} = \text{given},$$
$$g(t, x(t), x_\gamma(t)) \leqq 0,\ h(t, x(t), x_\gamma(t)) = 0,\ t \in \Omega_{t_0,t_1} \Big\}$$

the *set of all feasible solutions* of the problem (MFP).

The aim of this work is to introduce and study sufficient efficiency conditions for the variational problem

$$\text{(MFP)} \quad \begin{cases} \min\limits_{x(\cdot)} \quad \left(\dfrac{F^1(x(\cdot))}{K^1(x(\cdot))}, \dots, \dfrac{F^r(x(\cdot))}{K^r(x(\cdot))} \right), \\ \qquad \text{subject to} \quad x(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1}). \end{cases}$$

The authors of this paper and Ştefan Mititelu have introduced and studied such variational problems. More exactly, they have given necessary conditions for the efficiency of a feasible solution of the problem (MFP) and studied some types of dualities [7], [8].

Definition 1.1. A feasible solution $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is called *efficient* for the program (MFP) if and only if for any feasible solution $x(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$, the inequality $\dfrac{F(x(\cdot))}{K(x(\cdot))} \leqq \dfrac{F(x^\circ(\cdot))}{K(x^\circ(\cdot))}$ implies the equality $\dfrac{F(x(\cdot))}{K(x(\cdot))} = \dfrac{F(x^\circ(\cdot))}{K(x^\circ(\cdot))}$.

Definition 1.2. Let x° be an optimal solution of the problem (MFP). Suppose there are in $\mathbb{R}^r$ the vectors $\Lambda^{1\circ}$ and $\Lambda^{2\circ}$ having all components nonnegative but al least one positive and the smooth matrix functions μ_α° and ν_α° such that

$$< \Lambda^{1\circ}, \frac{\partial f_\alpha}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t)) > - < \Lambda^{2\circ}, \frac{\partial k_\alpha}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t)) >$$
$$+ < \mu_\alpha^\circ(t), \frac{\partial g}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t)) > + < \nu_\alpha^\circ(t), \frac{\partial h}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t)) >$$
$$-D_\gamma\Bigg(< \Lambda^{1\circ}, \frac{\partial f_\alpha}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) > - < \Lambda^{2\circ}, \frac{\partial k_\alpha}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) >$$
$$+ < \mu_\alpha^\circ(t, x^\circ(t), x_\gamma^\circ(t)), \frac{\partial g}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) >$$
$$+ < \nu_\alpha^\circ(t), \frac{\partial h}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) > \Bigg) = 0,$$
$$t \in \Omega_{t_0,t_1},\ \alpha = \overline{1,p} \quad \text{(Euler – Lagrange PDEs)}.$$

Then $x^\circ(\cdot)$ is called *normal optimal solution* of problem (MFP).

Let ρ be a real number and $b\colon C^\infty(\Omega_{t_0,t_1}, M) \times C^\infty(\Omega_{t_0,t_1}, M) \to [0, \infty)$ a functional. To any closed 1-form $a = (a_\alpha)$ we associate the path independent curvilinear functional

$$A(x(\cdot)) = \int_{\gamma_{t_0,t_1}} a_\alpha(t, x(t), x_\gamma(t))\, dt^\alpha.$$

The following definition of the quasiinvexity [5], [7], [8], helps us to state the results included in our main section.

Definition 1.3. The functional A is called [*strictly*] (ρ, b)-*quasiinvex* at the point $x^\circ(\cdot)$ if there is a vector function $\eta\colon J^1(\Omega_{t_0,t_1}, M) \times J^1(\Omega_{t_0,t_1}, M) \to \mathbb{R}^n$, such that

$$\eta(t, x^\circ(t), x_\gamma^\circ(t), x^\circ(t), x_\gamma^\circ(t)) = 0,$$

and the functional $\theta\colon C^\infty(\Omega_{t_0,t_1}, M) \times C^\infty(\Omega_{t_0,t_1}, M) \to \mathbb{R}^n$, such that for any $x(\cdot)$ $[x(\cdot) \neq x^\circ(\cdot)]$, the following implication holds

$$(A(x(\cdot)) \leqq A(x^\circ(\cdot))) \Rightarrow \Bigg(b(x((\cdot), x^\circ(\cdot)) \int_{\gamma_{t_0,t_1}} \Bigg[< \eta(t, x(t), x_\gamma(t), x^\circ(t), x_\gamma^\circ(t)),$$
$$\frac{\partial a_\alpha}{\partial x}(t, x^\circ(t), x_\gamma^\circ(t)) > + < D_\gamma \eta(t, x(t), x_\gamma(t), x^\circ(t), x_\gamma^\circ(t)),$$
$$\frac{\partial a_\alpha}{\partial x_\gamma}(t, x^\circ(t), x_\gamma^\circ(t)) > \Bigg] dt^\alpha [<] \leqq -\rho b(x(\cdot), x^\circ(\cdot)) \|\theta(x(\cdot), x^\circ(\cdot))\|^2 \Bigg).$$

In order to find some sufficient efficiency conditions for the problem (MFP), we need the following [7]

Theorem 1.1 (Necessary efficiency conditions)**.** *Let the function* $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ *be a normal efficient solution of problem* (MFP). *Then there exist* $\Lambda^{1\circ}$, $\Lambda^{2\circ} \in \mathbb{R}^r$

and the smooth functions $\mu^\circ\colon \Omega_{t_0,t_1} \to \mathbb{R}^{msp}$, $\nu^\circ\colon \Omega_{t_0,t_1} \to \mathbb{R}^{qsp}$, *such that we have*

$$
(\mathrm{MFP})_\circ \begin{cases}
< \Lambda^{1\circ}, \dfrac{\partial f_\alpha}{\partial x}(t, x^\circ(t), x^\circ_\gamma(t)) > - < \Lambda^{2\circ}, \dfrac{\partial k_\alpha}{\partial x}(t, x^\circ(t), x^\circ_\gamma(t)) > \\
+ < \mu^\circ_\alpha(t), \dfrac{\partial g}{\partial x}(t, x^\circ(t), x^\circ_\gamma(t)) > + < \nu^\circ_\alpha(t), \dfrac{\partial h}{\partial x}(t, x^\circ(t), x^\circ_\gamma(t)) > \\
-D_\gamma\Big(< \Lambda^{1\circ}, \dfrac{\partial f_\alpha}{\partial x_\gamma}(t, x^\circ(t), x^\circ_\gamma(t)) > - < \Lambda^{2\circ}, \dfrac{\partial k_\alpha}{\partial x_\gamma}(t, x^\circ(t), x^\circ_\gamma(t)) > \\
+ < \mu^\circ_\alpha(t, x^\circ(t), x^\circ_\gamma(t)), \dfrac{\partial g}{\partial x_\gamma}(t, x^\circ(t), x^\circ_\gamma(t)) > \\
+ < \nu^\circ_\alpha(t), \dfrac{\partial h}{\partial x_\gamma}(t, x^\circ(t), x^\circ_\gamma(t)) > \Big) = 0, \\
\qquad t \in \Omega_{t_0,t_1},\ \alpha = \overline{1,p} \quad (\text{Euler} - \text{Lagrange PDEs}) \\
< \mu^\circ_\alpha(t), g(t, x^\circ(t), x^\circ_\gamma(t)) >= 0, \quad t \in \Omega_{t_0,t_1},\ \alpha = \overline{1,p}, \\
\mu^\circ_\alpha(t) \geqq 0, \quad t \in \Omega_{t_0,t_1}, \quad \alpha = \overline{1,p}, \\
\Lambda^{1\circ} \geq 0, \qquad < e, \Lambda^{1\circ} >= 1, \quad e = (1, \ldots, 1) \in \mathbb{R}^r.
\end{cases}
$$

2. **Efficiency sufficient conditions**

We shall establish efficiency sufficient conditions for the problem (MFP).

Theorem 2.1. *Let us consider the vectors* $\Lambda^{1\circ}$, $\Lambda^{2\circ}$ *from* $\mathbb{R}^r$ *and the functions* $x^\circ(\cdot)$, $\mu^\circ(\cdot)$, $\nu^\circ(\cdot)$ *which satisfy the conditions* $(\mathrm{MFP})_\circ$. *Suppose that the following properties hold:*

a) *the functional*

$$
< \Lambda^{1\circ}, F(x(\cdot)) > - < \Lambda^{2\circ}, K(x(\cdot)) > = \int_{\gamma_{t_0,t_1}} \big[< \Lambda^{1\circ}, f_\alpha(t, x(t), x_\gamma(t)) > - < \Lambda^{2\circ}, k_\alpha(t, x(t), x_\gamma(t)) > \big] dt^\alpha
$$

is (ρ_1, b)*-quasiinvex at the point* $x^\circ(\cdot)$ *with respect to* η *and* θ*;*

b) *the functional* $\displaystyle\int_{\gamma_{t_0,t_1}} < \mu^\circ_\alpha(t), g(t, x(t), x_\gamma(t)) > dt^\alpha$ *is* (ρ_2, b) *-quasiinvex at the point* $x^\circ(\cdot)$ *with respect to* η *and* θ*;*

c) *the functional* $\displaystyle\int_{\gamma_{t_0,t_1}} < \nu^\circ_\alpha(t), h(t, x(t), x_\gamma(t)) > dt^\alpha$ *is* (ρ_3, b)*-quasiinvex at the point* $x^\circ(\cdot)$ *with respect to* η *and* θ*;*

d) *one of the integrals of* a)-c) *is strictly* (ρ_1, b), (ρ_2, b) *or* (ρ_3, b)*-quasiinvex at the point* $x^\circ(\cdot)$ *with respect to* η *and* θ*;*

e) $\rho_1 + \rho_2 + \rho_3 \geq 0$*;*

f) $\Lambda^{1\circ}_\ell F^\ell(x^\circ(\cdot)) - \Lambda^{2\circ}_\ell K^\ell(x^\circ(\cdot)) = 0$, *for each* $\ell = \overline{1,r}$.

Then the point $x^\circ(\cdot)$ *is an efficient solution of problem* (MFP).

Proof. Let us suppose that the point $x^\circ(\cdot)$ is not an efficient solution for problem (MFP). Then, there is a feasible solution $x(\cdot)$ for problem (MFP), such that

$$\frac{F^\ell(x(\cdot))}{K^\ell(x(\cdot))} \leq \frac{F^\ell(x^\circ(\cdot))}{K^\ell(x^\circ(\cdot))}, \quad \ell = \overline{1,r},$$

the case $x(\cdot) = x^\circ(\cdot)$ being excluded. That is

$$\Lambda_\ell^{1\circ} F^\ell(x(\cdot)) - \Lambda_\ell^{2\circ} K^\ell(x(\cdot)) \leq \Lambda_\ell^{1\circ} F^\ell(x^\circ(\cdot)) - \Lambda_\ell^{2\circ} K^\ell(x^\circ(\cdot)), \quad \ell = \overline{1,r}.$$

Making the sum after $\ell = \overline{1,r}$, we get

$$< \Lambda^{1\circ}, F(x(\cdot)) > - < \Lambda^{2\circ}, K(x(\cdot)) > \leq < \Lambda^{1\circ}, F(x^\circ(\cdot)) > - < \Lambda^{2\circ}, K(x^\circ(\cdot)) > .$$

According to condition a), it follows

$$\begin{aligned} &b(x(\cdot), x^\circ(\cdot)) \int_{\gamma_{t_0,t_1}} \Big[< \eta(t, x(t), x_\gamma(t), x^\circ(t), x^\circ_\gamma(t)), < \Lambda^{1\circ}, \frac{\partial f_\alpha}{\partial x}(t, x^\circ(t), x^\circ_\gamma(t)) > \\ &\quad - < \Lambda^{2\circ}, \frac{\partial k_\alpha}{\partial x}(t, x^\circ(t), x^\circ_\gamma(t)) >> + < D_\gamma \eta(t, x(t), x_\gamma(t), x^\circ(t), x^\circ_\gamma(t)), \\ &\quad < \Lambda^{1\circ}, \frac{\partial f_\alpha}{\partial x_\gamma}(t, x(t), x_\gamma(t)) > - < \Lambda^{2\circ}, \frac{\partial k_\alpha}{\partial x_\gamma}(t, x^\circ(t), x^\circ_\gamma(t)) >> \Big] dt^\alpha \\ &\leqq - \rho_1 b(x(\cdot), x^\circ(\cdot)) \|\theta(x(\cdot), x^\circ(\cdot))\|^2. \end{aligned} \tag{1}$$

Applying property b), the inequality

$$\int_{\gamma_{t_0,t_1}} < \mu^\circ_\alpha(t), g(t, x(t), x_\gamma(t)) > dt^\alpha \leqq \int_{\gamma_{t_0,t_1}} < \mu^\circ_\alpha(t), g(t, x^\circ(t), x^\circ_\gamma(t)) > dt^\alpha$$

leads us to

$$\begin{aligned} &b(x(\cdot), x^\circ(\cdot)) \int_{\gamma_{t_0,t_1}} \Big(< \eta(t, x(t), x_\gamma(t), x^\circ(t), x^\circ_\gamma(t)), < \mu^\circ_\alpha(t), \frac{\partial g}{\partial x}(t, x^\circ(t), x^\circ_\gamma(t)) >> \\ &\quad + < D_\gamma \eta(t, x(t), x_\gamma(t), x^\circ(t), x^\circ_\gamma(t)), < \mu^\circ_\alpha(t), \frac{\partial g}{\partial x_\gamma}(t, x^\circ(t), x^\circ_\gamma(t)) >> \Big) dt^\alpha \\ &\quad \leqq - \rho_2 b(x(\cdot), x^\circ(\cdot)) \|\theta(x(\cdot), x^\circ(\cdot))\|^2. \end{aligned} \tag{2}$$

Taking into account condition c), the equality

$$\int_{\gamma_{t_0,t_1}} < \nu^\circ_\alpha(t), h(t, x(t), x_\gamma(t)) > dt^\alpha = \int_{\gamma_{t_0,t_1}} < \nu^\circ_\alpha(t), h(t, x^\circ(t), x^\circ_\gamma(t)) > dt^\alpha$$

implies

$$\begin{aligned} &b(x(\cdot), x^\circ(\cdot)) \int_{\gamma_{t_0,t_1}} \Big(< \eta(t, x(t), x_\gamma(t), x^\circ(t), x^\circ_\gamma(t)), < \nu^\circ_\alpha(t), \frac{\partial h}{\partial x}(t, x^\circ(t), x^\circ_\gamma(t)) >> \\ &\quad + < D_\gamma \eta(t, x(t), x_\gamma(t), x^\circ(t), x^\circ_\gamma(t)), < \nu^\circ_\alpha(t), \frac{\partial h}{\partial x_\gamma}(t, x^\circ(t), x^\circ_\gamma(t)) >> \Big) dt^\alpha \\ &\quad \leqq - \rho_3 b(x(\cdot), x^\circ(\cdot)) \|\theta(x(\cdot), x^\circ(\cdot))\|^2. \end{aligned} \tag{3}$$

Summing side by side relations (1), (2), (3) and using condition d), it follows

$$b(x(\cdot),x^\circ(\cdot))\int_{\gamma_{t_0,t_1}} < \eta(t,x(t),x_\gamma(t),x^\circ(t),x^\circ_\gamma(t)), < \Lambda^{1\circ}, \frac{\partial f_\alpha}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t)) >$$
$$- < \Lambda^{2\circ}, \frac{\partial k_\alpha}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t)) > + < \mu^\circ_\alpha(t), \frac{\partial g}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t)) >$$
$$+ < \nu^\circ_\alpha(t), \frac{\partial h}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t)) >> dt^\alpha + b(x(\cdot),x^\circ(\cdot))\int_{\gamma_{t_0,t_1}} < D_\gamma \eta(t,x(t),$$
$$x_\gamma(t),x^\circ(t),x^\circ_\gamma(t)), < \Lambda^{1\circ}, \frac{\partial f_\alpha}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t)) > - < \Lambda^{2\circ}, \frac{\partial k_\alpha}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t)) >$$
$$+ < \mu^\circ_\alpha(t), \frac{\partial g}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t)) > + < \nu^\circ_\alpha(t), \frac{\partial h}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t)) >> dt^\alpha$$
$$< -(\rho_1+\rho_2+\rho_3)\, b(x(\cdot),x^\circ(\cdot))\|\theta(x(\cdot),x^\circ(\cdot))\|^2.$$

This inequality implies that $b(x(\cdot),x^\circ(\cdot)) > 0$, therefore we obtain

$$\int_{\gamma_{t_0,t_1}} < \eta(t,x(t),x_\gamma(t),x^\circ(t),x^\circ_\gamma(t)), < \Lambda^{1\circ}, \frac{\partial f_\alpha}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t)) >$$
$$- < \Lambda^{2\circ}, \frac{\partial k_\alpha}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t)) > + < \mu^\circ_\alpha(t), \frac{\partial g}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t)) >$$
$$+ < \nu^\circ_\alpha(t), \frac{\partial h}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t)) >> dt^\alpha$$
$$+ \int_{\gamma_{t_0,t_1}} < D_\gamma \eta(t,x(t),x_\gamma(t),x^\circ(t),x^\circ_\gamma(t)), < \Lambda^{1\circ}, \frac{\partial f_\alpha}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t)) >$$
$$- < \Lambda^{2\circ}, \frac{\partial k_\alpha}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t)) > + < \mu^\circ_\alpha(t), \frac{\partial g}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t)) >$$
$$+ < \nu^\circ_\alpha(t), \frac{\partial h}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t)) >> dt^\alpha$$
$$< -(\rho_1+\rho_2+\rho_3)\,\|\theta(x(\cdot),x^\circ(\cdot))\|^2.$$

According to [13], §9, we have the following

Lemma 2.1. *A total divergence is equal to a total derivative.*

Integrating by parts the second integral and using Lemma 2.1, the previous inequality leads us to a contradiction, that is

$$0 < -(\rho_1+\rho_2+\rho_3)\,\|\theta(x(\cdot),x^\circ(\cdot))\|^2.$$

Therefore, the point $x^\circ(\cdot)$ is an efficient solution for problem (MFP). □

Replacing the integrals from hypotheses b), c), of Theorem 2.1 by the integral

$$\int_{\gamma_{t_0,t_1}} \big[< \mu^\circ_\alpha(t), g(t,x(t),x_\gamma(t)) > + < \nu^\circ_\alpha(t), h(t,x(t),x_\gamma(t)) > \big] dt^\alpha,$$

the following statement is obtained.

Corollary 2.1. *Let $x^\circ(\cdot)$ be a feasible solution of problem* (MFP), *$\mu^\circ(\cdot)$, $\nu^\circ(\cdot)$ be functions and $\Lambda^{1\circ}$, $\Lambda^{2\circ}$ vectors from $\mathbb{R}^r$ such that the relations* $(\mathrm{MFP})_\circ$ *are satisfied. Suppose that the following conditions are fulfilled:*

a) *the functional*

$$< \Lambda^{1\circ}, F(x(\cdot)) > - < \Lambda^{2\circ}, K(x(\cdot)) > = \int_{\gamma_{t_0,t_1}} \big[< \Lambda^{1\circ}, f_\alpha(t, x(t), x_\gamma(t)) > - < \Lambda^{2\circ}, k_\alpha(t, x(t), x_\gamma(t)) > \big] dt^\alpha$$

is (ρ_1, b)-quasiinvex at the point $x^\circ(\cdot)$ with respect to η and θ;

b) *the functional*

$$\int_{\gamma_{t_0,t_1}} (< \mu^\circ_\alpha(t), g(t, x(t), x_\gamma(t)) > + < \nu^\circ_\alpha(t), h(t, x(t), x_\gamma(t)) >) dt^\alpha$$

is (ρ_2, b)-quasiinvex at the point $x^\circ(\cdot)$ with respect to η and θ;

c) *one of the integrals from* a) *or* b) *is strictly-quasiinvex at the point $x^\circ(\cdot)$;*

d) $\rho_1 + \rho_2 \geq 0$;

e) $\Lambda^{1\circ}_\ell F^\ell(x^\circ(\cdot)) - \Lambda^{2\circ}_\ell K^\ell(x^\circ(\cdot)) = 0$, *for each $\ell = \overline{1, r}$.*

Then, the point $x^\circ(\cdot)$ is an efficient solution of problem (MFP).

For other developments of optimization problems of path independent curvilinear integrals with PDE constraints or with isoperimetric constraints as multiple integrals or path independent curvilinear integrals, see [2] ÷ [6] and [9] ÷ [16]. For a computer aided study of PDE and/or PDI optimization problems using Maple, see [1] and [14].

3. Conclusions

We considered the minimizing fractional program (MFP), where the objective is a vector of functionals quotients of paths integrals and the constraints are partial differential inequations (PDI) and equations (PDE). In this work, we introduced and studied sufficient conditions for the efficiency of a feasible solution of problem (MFP). The present study completes previous results obtained with Ştefan Mititelu and included in papers [7] and [8].

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