A MESHLESS LOCAL DISCRETE GALERKIN (MLDG) METHOD FOR SOLVING NONLINEAR WEAKLY SINGULAR INTEGRAL EQUATIONS

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This article describes a numerical method for solving nonlinear Fredholm integral equations of the second kind with weakly singular kernels. The scheme estimates the solution by the discrete Galerkin method based on the use of moving least squares (MLS) as a local approximation. In order to approximate the singular integrals appeared in the method, we introduced an accurate special quadrature formula. The proposed scheme is constructed on a set of scattered data and does not require any background mesh, so it is meshless. The method can be easily implemented and its algorithm is simple and effective to solve weakly singular integral equations. The error analysis of the method is provided. Finally, numerical examples are included to show the validity and efficiency of the new technique and confirm the theoretical error estimates.

Keywords: Nonlinear integral equation, Weakly singular kernel, Discrete Galerkin method, Moving least squares (MLS), Error analysis.

MSC2010: 65D10, 45G05, 65G99.

1. Introduction

The most usual basic meshfree methods are known in the literature as radial basis functions (RBFs) and the moving least squares (MLS) methods. The MLS scheme as a general case of Shepard's method has been introduced by Lancaster and Salkauskas [16]. This method is an effective technique for the approximation of an unknown function that involves a locally weighted least squares polynomial fitting. The MLS approach is recognized as a meshless method because it is based on a set of scattered points and consequently does not need any domain elements for approximation. This meshless technique has significant importance applications in different problems of the numerical mathematics such as integral equations. The MLS-based meshless methods have been used for solving linear integral equations with logarithmic kernels [3], boundary integral equations [18, 17], Fredholm integro-differential equations [10] and two-dimensional integral equations on non-rectangular domains [4].

Consider the following nonlinear weakly singular Fredholm integral equation of the second kind

$$u(x) - \lambda \int_a^b H(x,y)\varphi(x,y,u(y))\mathrm{d}y = f(x),\ a \le x \le b,\ \lambda \neq 0, \tag{1}$$

where the known function $H(x,y)$ has an infinite singularity in $x, y \in [a,b]$, $x = y$ and the most important examples are $\ln|x-y|$, $|x-y|^\alpha$ for some $-1 < \alpha < 0$ and variants of them, φ is a given several times continuously differentiable function, f is a known function, u is the unknown function which must be determined and λ is a constant. These types of integral

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equations often arise in practical applications such as investigation of electrostatic and low frequency electromagnetic problems [20], methods for computing the conformal mapping of a given domain [23], solution of electromagnetic scattering problems [2], determination of propagation of acoustical and elastically waves [7] and the reformulation of some exterior boundary value problems [24].

The projection and discrete projection methods are the commonly used approaches for the numerical solutions of nonlinear weakly singular Fredholm integral equations. The discrete Petrov-Galerkin methods [9], piecewise polynomial collocation and Galerkin methods [22], Sinc-collocation methods [21], hybrid collocation methods [8], high-order collocation methods [11], iterated fast multiscale Galerkin methods [19], Galerkin-wavelet methods [1, 12] and Bubnov-Galerkin methods [14] have been used to solve these types of integral equations. Author of [24] has investigated Adomian decomposition methods for solving weakly singular Fredholm integral equations of the second kind.

The main purpose of this article is to present a numerical method for solving the weakly singular integral equation (1) by the discrete Galerkin method with the shape functions of MLS approximation constructed on a set of disordered data as basis. Since the singular integrals appeared in the proposed method cannot be estimated by classical integration schemes, we apply a quadrature formula established based on the non-uniform composite Gauss-Legendre integration rule. The new technique does not require any domain elements, so it is a meshless method. The presented scheme developed in the current paper is simple, computationally attractive and more flexible for most classes of integral equations such as weakly singular Volterra integral equations and weakly singular integro-differential equations. We also obtain the error bound and the convergence rate for the proposed method.

The outline of the paper is as follows. In Section 2, we review some basic formulations and properties of the MLS approximation. In Sections 3, we present a computational method for solving the integral equation (1) by combining MLS and the discrete Galerkin methods. In Section 4, we provide the error analysis for the proposed method. Numerical examples are given in Section 5. Finally, we conclude the article in Section 6.

2. The MLS approximation

The aim of the moving least squares (MLS) method is to approximate the function $u(\mathbf{x})$ in the closed domain $D \subset \mathbb{R}^d$ for every point $\mathbf{x} \in D$ in a weighted least square sense. We consider the data values of the function $u(\mathbf{x})$ at certain data sites $X = \{\mathbf{x}_1, ..., \mathbf{x}_N\}$. For $\mathbf{x} \in D$, the value $s_{u,X}(\mathbf{x})$ of the MLS approximation is taken by the solution of [16]

$$\min \left\{ \sum_{i=1}^{N} [u(\mathbf{x}_i) - p(\mathbf{x}_i)]^2 w(\mathbf{x}, \mathbf{x}_i) : p \in \Pi_q(\mathbb{R}^d) \right\}, \tag{2}$$

where $w : D \times D \to [0, \infty)$ is a continuous weight function and $\Pi_q(\mathbb{R}^d)$ is the linear space of polynomials of total degree less than or equal to q in d variables with the basis $\{p_1, ..., p_Q\}$ [25]. We are mainly interested in local continuous weight function w which gets smaller as its arguments move away from each other. Therefore, we can assume that

$$w(\mathbf{x}, \mathbf{y}) = \phi(\frac{\|\mathbf{x} - \mathbf{y}\|_2}{\delta}), \ \delta > 0, \tag{3}$$

where the univariate function ϕ is nonnegative which $\phi(r) = 0$ for $r \geq 1$ [25].
In the following theorem, we will find a direct approach to obtain the solution of problem (2), but prior to that, the definition of unisolvent set is required.

Definition 2.1. [13] *A set of points $X = \{\mathbf{x}_1, ..., \mathbf{x}_N\} \subset \mathbb{R}^d$ is called as q-unisolvent if the only polynomial of total degree at most q, interpolating zero data on X is the zero polynomial.*

Theorem 2.1. [16, 25] *Suppose that the set $X = \{\mathbf{x}_1, ..., \mathbf{x}_N\}$ is q-unisolvent, then the problem (2) is uniquely solvable and the solution $s_{u,X}(\mathbf{x})$ for every $\mathbf{x} \in D$ can be represented as*

$$s_{u,X}(\mathbf{x}) = \sum_{i=1}^{N} \psi_i(\mathbf{x}) u(\mathbf{x}_i), \tag{4}$$

where the basis functions $\psi_i(\mathbf{x})$ are called as shape functions corresponding to X and determined by

$$\psi_i(\mathbf{x}) = w(\mathbf{x}, \mathbf{x}_i) \sum_{k=0}^{Q} z_k p_k(\mathbf{x}_i), \tag{5}$$

in which the z_k are the unique solution of

$$\sum_{k=0}^{Q} z_k \sum_{i=1}^{N} w(\mathbf{x}, \mathbf{x}_i) p_k(\mathbf{x}_i) p_l(\mathbf{x}_i) = p_l(\mathbf{x}), \ 0 \le l \le Q. \tag{6}$$

The Gaussian weight function is applied in the present work which is defined as follows:

$$w(\mathbf{x}, \mathbf{x}_j) = \begin{cases} \frac{\exp[-(d_j/\alpha)^2] - \exp[-(\delta/\alpha)^2]}{1 - \exp[-(\delta/\alpha)^2]}, & 0 \le d_j \le \delta, \\ 0, & d_j > \delta, \end{cases} \tag{7}$$

where $d_j = \| \mathbf{x} - \mathbf{x}_j \|_2$ (the Euclidean distance between $\mathbf{x}$ and $\mathbf{x}_j$), α is a constant controlling the shape of the weight function w and δ is the size of the support domain.
Now, we proceed by discussing on the convergence of the MLS method in terms of the fill distance parameter $h_{X,D}$.
We restrict ourselves to the domains satisfying an interior cone condition with angle $\theta \in (0, \pi/2)$ and radius $r > 0$ (for more details, see [13, 25]). In the following, we present some definitions which are important to measure the quality of data points and to estimate the convergence rates of the MLS method.

Definition 2.2. [13] *The fill distance of a set of points $X = \{\mathbf{x}_1, ..., \mathbf{x}_N\} \subseteq D$ for a bounded domain D is defined by*

$$h_{X,D} = \sup_{x \in D} \min_{0 \le j \le N} \|\mathbf{x} - \mathbf{x}_j\|_2.$$

Definition 2.3. [13] *The separation distance of $X = \{\mathbf{x}_1, ..., \mathbf{x}_N\}$ is defined by*

$$q_X = \frac{1}{2} \min_{i \ne j} \|\mathbf{x}_i - \mathbf{x}_j\|_2.$$

The set X is said to be quasi-uniform with respect to a constant $c > 0$ if

$$q_X \le h_{X,D} \le c q_X.$$

We can ready to represent the following theorem about the error bound for approximating a function using the MLS scheme for every $\mathbf{x} \in D$.

Theorem 2.2. [25] *Let D^* be the closure of $\cup_{\mathbf{x} \in D} B(\mathbf{x}, 2r)$ and $h_0 = r/\tau$ where $\tau = \frac{16(1+\sin\theta)^2 q^2}{3\sin^2\theta}$, then there exists a constant $C > 0$ that can be computed explicitly such that for all $u \in C^{q+1}(D^*)$ and all quasi-uniform $X \subset D$ with $h_{X,D} \le h_0$, the approximation error is bounded as follows:*

$$\|u - s_{u,X}\|_{L^\infty(D)} \le C h_{X,D}^{q+1} |u|_{C^{q+1}(D^*)}. \tag{8}$$

The semi-norm on the right-hand side is defined by [25]

$$|u|_{C^{q+1}(D^*)} = \max_{|\alpha| = q+1} \|D^\alpha u\|_{L^\infty(D^*)}.$$

3. Solution of weakly singular integral equations

In this section, we utilize the moving least squares (MLS) method to the numerical solution of the following weakly singular Fredholm integral equation of the second kind:

$$u(x) - \lambda \int_a^b H(x,y)\varphi(x,y,u(y))\mathrm{d}y = f(x),\ a \le x \le b,\ \lambda \in \mathbb{R}, \tag{9}$$

where the weakly singular function $H(x,y)$ and the right hand side function $f \in L^2[a,b]$ are given, u is an unknown function to be determined and the given function φ is sufficiently smooth and nonlinear corresponding to the variable u. We assume that $H(x,y)$ is continuous for all $x,y \in [a,b]$, $x \neq y$, and there exist positive constants ξ and $\alpha \in (0,\ell]$ such that for all $x,y \in [a,b]$, $x \neq y$, we have

$$|H(x,y)| \le \xi |x-y|^{\alpha-\ell}. \tag{10}$$

In other words, the discontinuous functions $H(x,y)$ have an infinite singularity [5] and the most important examples are $\ln|x-y|$, $|x-y|^\alpha$ for some $-1 < \alpha < 0$ and variants of them. To apply the MLS method, we require N nodal points in the interval $[a,b]$. Since the interval $[a,b]$ should be transferred to the interval $[0,1]$ by a simple change of variables, without loss of generality, we can assume that $a = 0$ and $b = 1$. The distribution of these nodes could be selected regularly or randomly as $0 < x_1 < x_2 < ... < x_N < 1$. Therefore the unknown function $u(x)$ can be approximated by the MLS method as

$$u(x) \approx u_N(x) = \sum_{i=1}^{N} c_i \psi_i(x) = C^t \Psi(\mathbf{x}),\ x \in [0,1], \tag{11}$$

where

$$\Psi(\mathbf{x}) = [\psi_1(\mathbf{x}), ..., \psi_N(\mathbf{x})]^t,\ C = [c_1, ..., c_N]^t.$$

Replacing the expansions (11) with $u(x)$ in the integral equation (9) yields

$$C^t \Psi(x) - \lambda \int_0^1 H(x,y)\varphi(x,y,C^t\Psi(y))\mathrm{d}y = f(x). \tag{12}$$

By taking inner product $\langle ., \Psi^t(x)\rangle$ upon both sides of (12), we have

$$C^t \left\langle \Psi(x), \Psi^t(x)\right\rangle - \lambda \left\langle \int_0^1 H(x,y)\varphi\left(x,y,C^t\Psi(y)\right)\mathrm{d}y, \Psi^t(x)\right\rangle = \left\langle f(x), \Psi^t(x)\right\rangle. \tag{13}$$

The discrete Galerkin method results from the numerical integration of all integrals in the nonlinear system (13) associated with the Galerkin method [5]. To approximate the inner products $\langle f, \psi_j\rangle$ and $\langle \psi_j, \psi_i\rangle$, we use the composite q_k-point Gauss-Legendre rule with M uniform subdivisions relative to the coefficients $\{v_\ell\}$ and weights $\{w_\ell\}$ in interval $[-1,1]$ as

$$\langle f, \psi_j\rangle = \int_0^1 f(x)\psi_j(x)\mathrm{d}x = \frac{\Delta x}{2}\sum_{\ell=1}^{q_k} w_\ell \sum_{q=1}^{M} f(\tau_\ell^q)\psi_j(\tau_\ell^q) + \mathcal{O}(\frac{1}{M^{2q_k}}), \tag{14}$$

and

$$\langle \psi_j, \psi_i\rangle = \int_0^1 \psi_i(x)\psi_j(x)\mathrm{d}x = \frac{\Delta x}{2}\sum_{\ell=1}^{q_k} w_\ell \sum_{q=1}^{M} \psi_i(\tau_\ell^q)\psi_j(\tau_\ell^q) + \mathcal{O}(\frac{1}{M^{2q_k}}), \tag{15}$$

where $\Delta x = \frac{1}{M}$ and $\tau_\ell^q = \frac{\Delta x}{2}v_\ell + (q - \frac{1}{2})\Delta x$.
Since $H(x,y)$ is a weakly singular function along the diagonal $x = y$, the double integrals in the nonlinear system (13) cannot be computed by the classical numerical integration rule. Thus a particular quadrature formula is required. For approximating these integrals, we use the composite q_k-point Gauss-Legendre rule with M non-uniform subdivisions [12]. But this quadrature rule can not be applied for approximating them, because these formulae are

utilized when the singularities occur on the vertical axis. The following change of variables for the integrals are most useful [12, 15]:

$$t = y - x \text{ and } s = y + x.$$

In fact, with the change of variables, the unit square $[0,1] \times [0,1]$ is transformed to the diamond $\{(t,s) : |t| + |s-1| \leq 1\}$, and so

$$\int_0^1 \int_0^1 H(x,y)\varphi\left(x, y, \sum_{i=1}^{N} c_i\psi_i(y)\right)\psi_j(x)\mathrm{d}y\mathrm{d}x = \int_{-1}^{1} h(t)\mathrm{d}t, \tag{16}$$

where

$$h(t) = \int_{\alpha(t)}^{\beta(t)} K\left(t, s, \sum_{i=1}^{N} c_i\psi_i(\frac{s+t}{2})\right)\psi_j\left(\frac{s-t}{2}\right)\mathrm{d}s, \tag{17}$$

with

$$K\left(t, s, \sum_{i=1}^{N} c_i\psi_i(\frac{s+t}{2})\right) = H\left(\frac{s-t}{2}, \frac{s+t}{2}\right)\varphi\left(\frac{s-t}{2}, \frac{s+t}{2}, \sum_{i=1}^{N} c_i\psi_i(\frac{s+t}{2})\right),$$

$\alpha(t) = \max\{-t, t\}$ and $\beta(t) = \min\{2-t, 2+t\}$.
The integrand of $h(t)$ has a weakly singularity only at $t = 0$ and is sufficiently smooth for every s. Consequently $h(t)$ satisfies the condition

$$|h^{(2q_k)}(t)| \leq Ct^{-\epsilon-2q_k}, \ \textit{for all } t \in (0,1), \tag{18}$$

for any positive integer k and for any small positive number ϵ [12], so by the composite q_k-point Gauss-Legendre rule with M non-uniform subdivisions [12], we have

$$\int_{-1}^{1} h(t)\mathrm{d}t = \int_0^1 [h(t) + h(-t)]\,\mathrm{d}t = \sum_{\ell=1}^{q_k} w_\ell \sum_{q=1}^{M} \frac{\Delta t_q}{2}\left[h(\theta_\ell^q) + h(-\theta_\ell^q)\right] + \mathcal{O}(\frac{1}{M^{2q_k}}). \tag{19}$$

To approximate $h(t)$, we utilize the common m_N-point numerical integration scheme over $[-1,1]$ with M uniform subdivisions relative to the coefficients $\{v_r\}$ and weights $\{w_r\}$. Thus, we obtain

$$h(t) = \frac{\Delta s}{2}\sum_{r=1}^{q_k} w_r \sum_{p=1}^{M} K\left(t, \eta_r^p, \sum_{i=1}^{N} c_i\psi_i(\frac{\eta_r^p + t}{2})\right)\psi_j\left(\frac{\eta_r^p - t}{2}\right) + \mathcal{O}(\frac{1}{M^{2q_k}}), \tag{20}$$

where $\Delta s = \frac{\beta(t)-\alpha(t)}{M_t}$ with $M_t = 1 + [M(\beta(t) - \alpha(t))]$ and $\eta_r^p = \frac{\Delta s}{2}v_r + (p - \frac{1}{2})\Delta s$.
Now, using the quadrature formulae (14), (15) and (19) to approximate the integrals in the system (13), the following nonlinear system of algebraic equations for unknowns $\{c_1, \ldots, c_N\}$ is obtained:

$$\frac{\Delta x}{2}\sum_{\ell=1}^{q_k} w_\ell \sum_{q=1}^{M}\left[\hat{c}_j\psi_i(\tau_\ell^q) - f(\tau_\ell^q)\right]\psi_j(\tau_\ell^q) = \lambda\sum_{\ell=1}^{q_k} w_\ell \sum_{q=1}^{M}\frac{\Delta t_q}{2}\left[\hat{h}(\theta_\ell^q) + \hat{h}(-\theta_\ell^q)\right], \tag{21}$$

where

$$\hat{h}(\theta_\ell^q) = \frac{\Delta s}{2}\sum_{r=1}^{q_k} w_r \sum_{p=1}^{M} K\left(\theta_\ell^q, \eta_r^p, \sum_{i=1}^{2^{k-1}M}\hat{c}_i\psi_i(\frac{\eta_r^p + \theta_\ell^q}{2})\right)\psi_j\left(\frac{\eta_r^p - \theta_\ell^q}{2}\right).$$

Thus, the values of $u(x)$ at any point $x \in [0,1]$ can be approximated by

$$\hat{u}_N(x) = \sum_{i=1}^{N}\hat{c}_i\psi_i(x).$$

4. Error analysis

This section includes the error estimate and the rate of convergence of the presented method. We define $\mathcal{P}_N : L^2[a,b] \to V_N$ as a Galerkin projection operator by

$$\mathcal{P}_N u(x) = \sum_{i=1}^{N} c_i \psi_i(x), \ x \in [a,b], \tag{22}$$

where the space $V_N = \text{span}\{\psi_1, ..., \psi_N\} \subset L^2[a,b]$ with the dimension d_N and the coefficients $\{c_1, ..., c_N\}$ determined by solving the linear system

$$\langle u, \psi_j \rangle = \sum_{i=1}^{N} c_i \langle \psi_i, \psi_j \rangle \ \ j = 1, ..., N. \tag{23}$$

The weakly singular operator $T : L^2[a,b] \to L^2[a,b]$ is introduced as

$$Tu(x) = \lambda \int_a^b H(x,y)\varphi(x,y,u(y))\mathrm{d}y + f(x). \tag{24}$$

The weakly singular integral operator (24) is compact on $L^2[a,b]$ [5].
Utilizing the operator (24), we can rewrite the integral equation (9) as

$$u = Tu, \tag{25}$$

and so solving the nonlinear integral equation (9) is equivalent to finding the fixed points of T.
Based on the q_N-point Gauss-Legendre rule with M subdivisions, we introduce the discrete semi-definite inner product as

$$\langle f, g \rangle_{q_N} = \sum_{k=1}^{q_N} \bar{w}_k \sum_{q=1}^{M} f(\omega_N^q) g(\omega_N^q), \ \ f, g \in L^2[a,b]. \tag{26}$$

Therefore, we can introduce the discrete projection operator as follows:

$$\Omega_N u(x) = \sum_{k=1}^{N} c_k \psi_k(x), \ x \in [a,b], \tag{27}$$

where the coefficients $\{c_1, ..., c_N\}$ determined by solving the linear system

$$\langle u, \psi_j \rangle_{q_N} = \sum_{k=1}^{N} c_k \langle \psi_k, \psi_j \rangle_{q_N} \ \ j = 1, ..., N. \tag{28}$$

Now, we present the following theorem about the discrete projection operator with the MLS shape functions as the basis.

Lemma 4.1. [4] *Having in mind the assumptions of Theorem 2.2, suppose that Ω_N, $N \geq 1$ are the discrete orthogonal projections for the shape functions of the MLS approximation corresponding to nodal points $X = \{x_1, ..., x_N\} \subset [a,b]$. If $u \in C^{q+1}[a,b]^*$ then $\Omega_N u$ converges to u as $N \to \infty$ and moreover*

$$\|\Omega_N u - u\|_\infty \leq (1+m) C h_{X,D}^{q+1} |u|_{C^{q+1}[a,b]^*}, \tag{29}$$

where m is a constant.

Furthermore, a sequence of q_N-point Gauss-Legendre rule with M subdivisions operators T_N on $L^2[a,b]$ is introduced by

$$T_N u(x) = \sum_{r=1}^{q_N} \bar{w}_r \sum_{p=1}^{M} H(x, \theta_k r^p) \varphi(x, \theta_r^p, u(\theta_r^p)) + f(x), \ \ N \geq 1. \tag{30}$$

Note that $\{T_N\}$ is a collectively compact family which is pointwise convergent on $L^2[0,1]$ [12] and furthermore for every $u \in C^{2q_N}[0,1]$, we have [5, 12]

$$\|Tu - T_N u\|_\infty \leq \frac{C_N}{M^{2q_N}} \sup_{0\leq x\leq 1} |u^{(2q_N)}(x)|. \tag{31}$$

Now, we can rewrite the system (21) as

$$\Omega_N T_N \hat{u}_N = \hat{u}_N. \tag{32}$$

Define the iterated solution by

$$\bar{u}_N = T_k(\hat{u}_N) \;\Rightarrow\; \Omega_N \bar{u}_N = \hat{u}_N. \tag{33}$$

To establish the error analysis of the presented method, we present the convergence theorem of iterated Galerkin method from [6] as follows:

Theorem 4.1. [6] *Let u_0 be a solution of the nonlinear integral equation (9) and assume that 1 is not an eigenvalue of $T'(u_0)$, where $T'(u_0)$ denotes the Frechet derivative of T at u_0. There are $\varepsilon, \bar{N} > 0$ such that for every $N > \bar{N}$, $T_N\Omega_N$ has a unique fixed point $\bar{u}_N$ in $B(u_0,\varepsilon)$. Also, there is a constant $C > 0$ such that*

$$\|\bar{u}_N - u_0\|_\infty \leq \gamma_1 \|Tu_0 - T_N\Omega_N u_0\|_\infty,\ N > \bar{N}. \tag{34}$$

Now, we are ready to present the convergence theorem of the presented method.

Theorem 4.2. *Suppose that $u_0 \in C^{q+1}[0,1]^* \cap C^{2q_N}[0,1]$ is the exact solution of the nonlinear integral equation (9). Then based on the assumption of Lemma 4.1 and Theorem 4.1, for sufficiently large N the proposed method has a unique solution $\hat{u}_N$ on a ball around u_0 and*

$$\|\hat{u}_N - u_0\|_\infty \leq (1 + m\gamma_1\gamma_2)(1+m)Ch_{X,D}^{q+1}|u_0|_{C^{q+1}[0,1]^*} + \frac{C_N m\gamma_1}{M^{2q_N}} \sup_{0\leq x\leq 1} |u_0^{(2q_N)}(x)|,$$

where $\gamma_1, \gamma_2, m, C_N$ and C are constants.

Proof. This can be immediately obtained from Theorem 4.1 that there exists a unique iterated solution $\bar{u}_N \in B(u_0,\varepsilon)$ such that

$$\begin{aligned} \|\bar{u}_N - u_0\|_\infty &\leq \gamma_1 \|Tu_0 - T_N\Omega_N u_0\|_\infty \\ &\leq \gamma_1[\|Tu_0 - T_N u_0\|_\infty + \|T_k(u_0 - \Omega_N u_0)\|_\infty]. \end{aligned} \tag{35}$$

From the pointwise convergence of T_N, and the principle of uniform boundedness [5], we can assume that $\|T_N\| \leq \gamma_2$, so

$$\|\bar{u}_N - u_0\|_\infty \leq \gamma_1 \|Tu_0 - T_N u_0\|_\infty + \gamma_1\gamma_2 \|u_0 - \Omega_N u_0\|_\infty. \tag{36}$$

If $\hat{u}_N = \Omega_N \bar{u}_N$ then $\hat{u}_N$ is a unique solution for the presented method. Consider the decomposition

$$u_0 - \hat{u}_N = u_0 - \Omega_N \bar{u}_N = (u_0 - \Omega_N u_0) + \Omega_N(u_0 - \bar{u}_N). \tag{37}$$

By the inequality (36), we obtain

$$\begin{aligned} \|\hat{u}_N - u_0\|_\infty &\leq \|u_0 - \Omega_N u_0\|_\infty + \|\Omega_N\|\|u_0 - \bar{u}_N\|_\infty. \\ &\leq (1 + m\gamma_1\gamma_2)\|u_0 - \Omega_N u_0\|_\infty + m\gamma_1 \|Tu_0 - T_N u_0\|_\infty. \end{aligned} \tag{38}$$

Applying the error bounds (29) and (31), we have

$$\|\hat{u}_N - u_0\|_\infty \leq (1 + m\gamma_1\gamma_2)(1+m)Ch_{X,D}^{q+1}|u_0|_{C^{q+1}[0,1]^*} + \frac{C_N m\gamma_1}{M^{2q_N}} \sup_{0\leq x\leq 1} |u_0^{(2q_N)}(x)|.$$

Since $\|\hat{u}_N - u_0\|_\infty \to 0$ where $N \to \infty$, justified by the quasi-uniform condition on X, there is a constant $\hat{N} > 0$ such that for every $N \geq \hat{N}$, $\|\hat{u}_N - u_0\|_\infty < \hat{\varepsilon}$. This completes the proof. □

TABLE 1. Some numerical results for Example 5.1

N	h	$\|e_N\|_2$		$\|e_N\|_\infty$			
		$q=1$	$q=2$	$q=1$	Ratio	$q=2$	Ratio
5	0.250	2.17×10^{-3}	6.11×10^{-4}	4.19×10^{-3}	–	1.30×10^{-3}	–
9	0.125	4.78×10^{-4}	7.31×10^{-5}	1.12×10^{-3}	1.90	2.14×10^{-4}	2.60
17	0.062	1.05×10^{-4}	7.70×10^{-6}	2.88×10^{-4}	1.95	3.11×10^{-5}	2.78
33	0.031	2.38×10^{-5}	7.47×10^{-7}	7.32×10^{-5}	1.97	4.21×10^{-6}	2.88
69	0.016	5.60×10^{-6}	6.85×10^{-8}	1.84×10^{-5}	1.98	5.42×10^{-7}	2.96
129	0.008	1.35×10^{-6}	6.77×10^{-9}	4.45×10^{-6}	2.05	6.80×10^{-8}	2.99

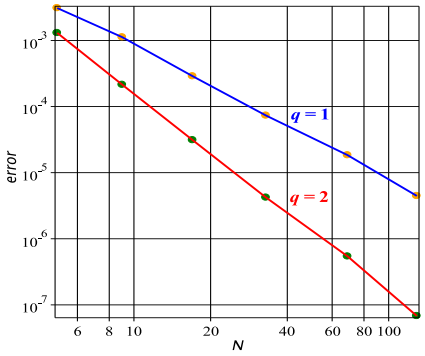

FIGURE 1. Absolute error distributions of Example 5.1

5. Numerical examples

In order to show the effectiveness of the new method two nonlinear integral equations with weakly singular kernels are solved. In computations, we put $h=\frac{1}{N-1}$, $\alpha=0.6\times h$ and $\delta=2\times h$ and $3\times h$ for the linear and quadratic cases, respectively. Here, we employ the composite Gauss-Legendre quadrature rule with $M=10$ and $q_N=10$ for the numerical integration. As we expected from Theorem 4.2, for q_N sufficiently large, the ratio of error remains approximately constant for the linear case (≈ 2) and for the quadratic case (≈ 3) so, the numerical results confirm the theoretical error estimates. All routines are written in MAPLE software and run on a Laptop with 2.10 GHz of Core 2 CPU and 4 GB of RAM. The FSOLVE command is used to solve the nonlinear system of algebraic equations.

Example 5.1. Consider the weakly singular integral equations kernels

$$u(x)+\int_0^1\left\{\frac{y^2+y}{\sqrt{1+x}}\ln|x-y|+\frac{x}{y+1}\right\}u^2(y)\mathrm{d}y=f(x),\ 0\le x\le 1, \tag{39}$$

where

$$f(x)=\frac{15x-6x^3\ln\left(\frac{1-x}{x}\right)-6x^2+6\ln(1-x)-2}{18\sqrt{1+x}}+(x^2+1)\left(\frac{3}{2}-2\ln(2)\right),$$

with the exact solution $u_{ex}(x)=\frac{x}{\sqrt{1+x}}$. The numerical results in terms of $\|e_N\|_2$ and $\|e_N\|_\infty$ at different numbers of N and the rate of convergence for the linear and quadratic basis functions are presented in Tables 1. The obtained errors for different numbers of N are drawn in the logarithmic mode in Figure 1.

Example 5.2. In this example, we solve the following integral equation:

$$u(x)-15\int_0^1\left\{\frac{y(x+1)}{\sqrt{|x-y|}}+e^{x+y}\right\}e^{u(y)}\mathrm{d}y=f(x),\ 0\le x\le 1, \tag{40}$$

TABLE 2. Some numerical results for Example 5.1

N	h	$\|e_N\|_2$		$\|e_N\|_\infty$			
		$q=1$	$q=2$	$q=1$	Ratio	$q=2$	Ratio
5	0.250	2.24×10^{-3}	5.79×10^{-4}	4.25×10^{-3}	–	1.22×10^{-3}	–
9	0.125	4.92×10^{-4}	6.73×10^{-5}	1.12×10^{-3}	1.91	1.96×10^{-4}	2.64
17	0.062	1.08×10^{-4}	6.98×10^{-6}	2.89×10^{-4}	1.96	2.81×10^{-5}	2.80
33	0.031	2.47×10^{-5}	6.72×10^{-7}	7.33×10^{-5}	1.98	3.79×10^{-6}	2.89
69	0.016	5.84×10^{-6}	6.13×10^{-8}	1.14×10^{-5}	1.98	4.39×10^{-7}	2.96
129	0.008	1.41×10^{-6}	6.17×10^{-9}	4.56×10^{-6}	2.01	6.08×10^{-8}	2.99

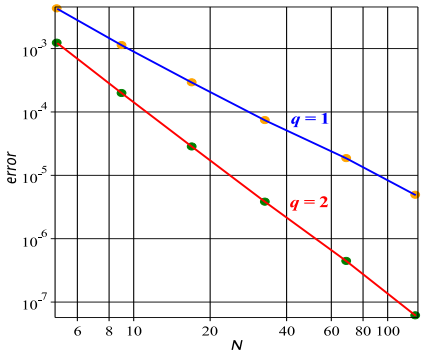

FIGURE 2. Absolute error distributions of Example 5.2

where

$$f(x)=\ln(x+1)-\frac{16}{7}x^{\frac{9}{2}}-\frac{44}{7}x^{\frac{7}{2}}-4x^{\frac{5}{2}}-(1-x)^{\frac{3}{2}}\left(\frac{16}{7}x^3-\frac{68}{7}x^2-\frac{124}{7}x-\frac{72}{7}\right)-15e^{x+1},$$

with the exact solution $u_{ex}(x)=\ln(1+x)$. Table 2 shows $\|e\|_\infty$, $\|e\|_2$ and the values of ratio at different numbers of N for for linear ($q=1$) and quadratic ($q=2$) basis functions. The obtained errors for different numbers of N are drawn in the logarithmic mode in Figure 2.

6. Conclusions

In this paper, we have studied a computational method for solving Fredholm integral equations of the second kind with weakly singular kernels. The method has been established on the discrete Galerkin method with the shape functions of the moving least squares (MLS) approximation constructed on scattered points as basis. The singular integrals in the scheme have been computed utilizing the composite non-uniform Gauss-Legendre integration rule. The proposed method is meshless, since it does not require any cell structures. We also have investigated the error estimate of the new approach and found that the rate of convergence of proposed method is of $O(h^q)$. The convergence accuracy has been examined in two weakly singular integral equations.

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