WARDOWSKI TYPE α-F-CONTRACTIVE APPROACH FOR NONSELF MULTIVALUED MAPPINGS

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In this research article, the Wardowski type notion for α-F-contractive nonself multivalued mappings has been introduced. The existence and uniqueness of the fixed point on α-F-contractive mapping by using the proposed notion are established. Some fixed point results have been proved by using the notion of α-F-admissible pairs. The article is furbished with some examples to support the novel idea.

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1. Introduction

Since its publication in 1922, the Banach contraction principle has attracted the attention of researchers working in both pure and applied mathematics. Its importance and significance can be noticed easily due to its wide range of applications and extensions. The celeberated Banach contraction principle (BCP)[10] states that each contraction on a complete metric space has a unique fixed point.
Later on, several results are produced by various authors as generalizations and extensions of the famous BCP by changing either the space under consideration or the condition on the mapping (see for example [1, 2, 4, 5, 7–9, 18–30]). An important result was proved by using the concepts of α-admissible and $\alpha\psi$-admissible type mappings for metric space by Samet *et al.* [22]. Karapinar [15] presented some fixed point results by applying the cyclic contractions. This new notion of cyclic contraction was introduced by Kirk *et al.* [17]. Some results are achieved and unified by the investigation of the related fixed point results available in the literature. It is worth mentioning here that the cyclic contraction is not required to be continuous. This advantage is appreciated and used by various researchers [1, 4, 8, 11, 19–22]. These contraction conditions provide a ground to investigate the fixed point using the α-admissible mappings as a base and variations of the concepts of $\alpha\psi$-contractive and ψ- mappings [1, 4, 5, 11, 13, 19–22]. In [3], Hussain and Iqbal firstly define the notion of α-F-contraction on multi-valued mappings and prove the existence of fixed point (Theorem 2.6 in [3]). In the present article, we are proving some results by using α-F-contraction.

2. Preliminaries

There are some basic definitions, fundamental concepts and results which provide a base to achieve the present results. From now on, $\mathbb{R}$ is set of real numbers and $\mathbb{R}^+$ is set of positive

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reals. A mapping $F\colon \mathbb{R}^+ \to \mathbb{R}$ is said to be an F-mapping [16, 28], if the following conditions are observed:

(i): F is strictly increasing function, that is, as for all $z_1, z_2 \in \mathbb{R}^+$, if $z_1 < z_2$ then $F(z_1) < F(z_2)$.

(ii): For each sequence $\{z_n\}$ of the positive real numbers $\mathbb{R}^+$,

$$\lim_{n\to\infty} (z_n) = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} F(z_n) = -\infty.$$

(iii): There is a real number $c \in (0,1)$ such as

$$\lim_{z\to\ 0^+} z^c F(z) = 0.$$

Let (Ω, d) be a metric space and $f : \Omega \to \Omega$ is a self mapping [28] for a non-empty set Ω, then T is said to be an F-contraction if there exists a positive real number $\kappa > 0$ such that

$$d(fa_1, fa_2) > 0 \;\; \Rightarrow \;\; \kappa + F(d(fa_1, fa_2)) \leq F(d(a_1, a_2)) \;\; \forall\, a_1, a_2 \in \Omega.$$

A number of fixed point results had been evidenced by many authors using the Wardowski type perception of F-contractive mappings. Some authors introduced and used some new concepts of contractions to establish some new fixed point results, among which F-contraction is an important discovery. Using the F-contractions some very important extensions and generalizations of Banach contractions principle in various dimensions can be found in literature [12, 14, 27].

Let (Ω, d) be a metric space. A mapping $f\colon \Omega \to \Omega$ is called α-F-contractive mapping [6] if for the two mappings $\alpha\colon \Omega \times \Omega \to [0, \infty)$, $F\colon \mathbb{R}^+ \to \mathbb{R}$ and $\kappa > 0$, we have

$$\kappa + F(\alpha(a_1, a_2) d(fa_1, fa_2)) \leq F(d(a_1, a_2)) \;\; \forall\, a_1, a_2 \in \Omega.$$

Recall that a mapping $f\colon \Omega \to \Omega$ is said to be an α-admissible [15] if for all $a_1, a_2 \in \Omega$,

$$\alpha(a_1, a_2) \geq 1 \Rightarrow \alpha(fa_1, fa_2) \geq 1.$$

Let (Ω, d) be a complete metric space for a non-empty set Ω. We designate $N(\Omega)$ to the class of all non-empty subsets of the space Ω and $CL(\Omega)$ to the class of all non-empty closed subsets of space Ω. For any point $a \in \Omega$ and a subset $B \subset N(\Omega)$, the distance between the point a and B is defined as:

$$d_1(a, B) = \inf_{b\in B} \{d(a, b)\}.$$

Let us use H for the Hausdorff metric which is defined as:

$$H(A,B) = \begin{cases} \max\left\{ \sup\limits_{a\in A} d(a,B),\ \sup\limits_{b\in B} d(b,A) \right\} & \text{if the maximum exists;} \\ \infty & \text{otherwise.} \end{cases} \qquad (\mathbf{2}.1)$$

for all non-empty subsets $A, B \in CL(\Omega)$. Then $(CL(\Omega), H)$ is called a Hausdorff metric space.

3. Main Results

This section includes some fundamental definitions and novel results supported by valid examples. The notion of nonself α-admissible mapping is modified by Ali et. al [5] as follows:

Definition 3.1. *[5] Consider a complete metric space (Ω, d) for a non-empty set Ω. Let S be a nonempty subset of Ω, then a mapping $D : S \to CL(\Omega)$ is said to be an α-admissible mapping if there exists a map $\alpha\colon S \times S \to [0, \infty)$ such that $\alpha(a_1, a_2) \geq 1$ implies that $\alpha(m, n) \geq 1$ for each $m \in Da_1 \cap S$ and $n \in Da_2 \cap S$ for all $a_1,\ a_2 \in S$.*

Definition 3.2. *Consider a complete metric space* (Ω, d), *with* Ω *a non-empty set and* S *a non-empty subset of* Ω. *A mapping* $D : S \to CL(\Omega)$ *is said to be an* α-F*-contractive mapping if for a function* $\alpha\colon S \times S \to [0, \infty)$, *an* F*-mapping and* $\kappa > 0$, *the following conditions are satisfied:*

(i): $Da \cap S \neq \phi$ *for all* $a \in S$.

(ii): *For each* $a_1, a_2 \in S$, *we have*

$$\kappa + F\left(\alpha(a_1, a_2)H(Da_1 \cap S, Da_2 \cap S)\right) \leq F(M(a_1, a_2)), \tag{3.1}$$

where

$$\min\left\{\alpha(a_1, a_2)H(Da_1 \cap S, Da_2 \cap S)), M(a_1, a_2)\right\} > 0,$$

and

$$M(a_1, a_2) = \max\Big\{d(a_1, a_2), \frac{d(a_1, Da_1 \cap S) + d(a_2, Da_2 \cap S)}{2}, \frac{d(a_1, Da_2 \cap S) + d(a_2, Da_1 \cap S)}{2}\Big\}$$

Since F is a strictly increasing function, therefore $D : S \to CL(\Omega)$ is a strictly α-F-contractive type mapping on a complete sub-space S of Ω.

Theorem 3.1. *Consider a metric space* (Ω, d) *and a complete non-empty subset* S *of* Ω *induced with the metric* d. *Let* $D : S \to CL(\Omega)$ *be a strictly* α-F*-contractive type mapping on* S, *then* D *has a fixed point* $u \in S$ *if the conditions given below are satisfied:*

(i): D *is an* α*-admissible mapping.*

(ii): *There exists* $a_0 \in S$ *and* $a_1 \in Da_0 \cap S$ *such that* $\alpha(a_0, a_1) \geq 1$.

(iii): D *is a continuous mapping.*

Proof. By the condition (ii), there is $a_0 \in S$ and $a_1 \in Da_0 \cap S$ such that $\alpha(a_0, a_1) \geq 1$. For $a_0 = a_1$, the proof is obvious. Now suppose that $a_0 \neq a_1$. If $a_1 \in Da_1 \cap S$, then a_1 is straightforwardly a fixed point. Suppose that $a_1 \notin Da_1 \cap S$.

Since D is a strictly α-F-contractive type mapping on S, the following holds.

$$\kappa + F(\alpha(a_0, a_1)H(Da_0 \cap S, Da_1 \cap S)) \leq F\Big((\max\Big\{d(a_0, a_1), \frac{d(a_0, Da_0 \cap S) + d(a_1, Da_1 \cap S}{2}, \frac{d(a_0, Da_1 \cap S) + d(a_1, Da_0) \cap S}{2}\Big\}\Big) \leq F\Big(\max\Big\{d(a_0, a_1), d(a_1, Da_1)\Big\}\Big)$$
$$\leq F(d(a_0, Da_0)) \leq F(d(a_0, a_1)) \ \ \forall\, a_0, a_1 \in S.$$

Therefore, we have

$$\kappa + F(H(Da_1 \cap S, Da_0 \cap S)) \leq F(d(a_1, a_0)) \ \ \forall\, a_0, a_1 \in S.$$

This implies for $a_2 \in Da_1 \cap S$, we have

$$\kappa + F(d(a_2, a_1)) \leq F(d(a_1, a_0)) \ \ \forall\, a_2, a_1 \in S.$$

Now using the α-admissibility, we have $\alpha(a_0, a_1) \geq 1 \Rightarrow \alpha(a_1, a_2) \geq 1$, if $a_1 \in Da_0 \cap S$ and $a_2 \in Da_1 \cap S$. Continuing in this way, the following can be easily claimed for $a_{n+1} \in Da_n \cap S$.

$$\alpha(a_n, a_{n+1}) \geq 1, \quad \forall\, a_n, a_{n+1} \in S \text{ and } n \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

By running iteratively, $\kappa + F(d(a_{n+1}, a_n)) \leq F(d(a_n, a_{n-1}))$. Inductively, we have

$$F(d(a_n, a_{n+1})) \leq F(d(a_0, a_1)) - n\kappa \tag{3.3}$$

Taking limit $n \to \infty$ on both sides

$$\lim_{n\to\infty} F(d(a_n, a_{n+1})) = -\infty \tag{3.4}$$

By using the definition of F-mapping, we have

$$\lim_{n\to\infty} d(a_n, a_{n+1}) = 0 \tag{3.5}$$

Furthermore, denote $d(a_n, a_{n+1})$ by d_n. By using definition of F-function, there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} d_n^k F(d_n) = 0. \tag{3.6}$$

With the new notation, (**3**.3) may be expressed as

$$F(d_n) - F(d_0) \leq -n\kappa \Rightarrow d_n^k F(d_n) - d_n^k F(d_0) \leq d_n^k (F(d_0) - n\kappa) - d_n^k F(d_0) = -n d_n^k \kappa \leq 0$$

$$\Rightarrow \lim_{n\to\infty} [d_n^k F(d_n) - d_n^k F(d_0)] \leq \lim_{n\to\infty} -d_n^k n\kappa$$

$$\Rightarrow \lim_{n\to\infty} -n d_n^k \kappa \geq 0 \Rightarrow \lim_{n\to\infty} n d_n^k = 0 \quad as \quad \kappa > 0 \quad \text{(using (3.5) and (3.6))}.$$

There exists $n_0 \in \mathbb{N}$ such that $n d_n^k \leq 1$ for all $n \geq n_0$.

$$d_n^k \leq 1/n \Rightarrow d_n \leq \frac{1}{n^{1/k}} \tag{3.7}$$

To show that $\{a_n\}$ is a Cauchy sequence, proceed as follows:

$$\begin{aligned} d(a_n, a_m) &\leq d(a_n, a_{n+1}) + d(a_{n+1}, a_{n+2}) + \ldots + d(a_{m-1}, a_m) \\ &\leq \sum_{i=n}^{\infty} d_i \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned} \tag{3.8}$$

Taking limit $n \to \infty$ on both sides of (**3**.8),

$$\lim_{n\to\infty} d(a_n, a_m) \leq \lim_{n\to\infty} \sum_{i=n}^{\infty} \frac{1}{i^{1/k}} = 0.$$

Therefore, $\{a_n\}$ is a Cauchy sequence. As S is complete, there exists $u \in S$ such that

$$\lim_{n\to\infty} d(a_n, u) = 0 \Rightarrow \lim_{n\to\infty} a_n = u.$$

Since D is continuous, therefore,

$$d(u, Du) \leq \lim_{n\to\infty} H(Da_n, Du) = 0.$$

Thus, we have $d(u, Du) = 0 \Rightarrow u \in Du$. Hence D has a fixed point. □

Theorem 3.2. *Consider a metric space (Ω, d) and assume that S be a complete non-empty subset of Ω induced with respect to the metric d. If $D : S \to CL(\Omega)$ be a strictly α-F-contractive type mapping on S then D has a fixed point if the following assertions are satisfied:*

(i): *D is an α-admissible mapping.*
(ii): *There exists $a_0 \in S$ and $a_1 \in Da_0 \cap S$ such that $\alpha(a_0, a_1) \geq 1$.*
(iii): *For any sequence $\{a_n\}$ in S with $a_n \to u$ and $\alpha(a_n, a_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, either*

(a) $\lim_{n\to\infty} \alpha(a_n, u) \geq 1$ *or*
(b) $\alpha(a_n, u) \geq 1$.

Proof. Following the proof of Theorem **3**.1, we conclude that $\{a_n\}$ in S is a Cauchy sequence such that

$$\lim_{n\to\infty} d(a_n, u) = 0,$$

and $\alpha(a_n, a_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Assume that $d(u, Du) \neq 0$. Using Definition **3**.2, we obtain

$$\begin{aligned}\kappa + F\Big(\alpha(a_n, u) d(a_{n+1}, Du \cap S)\Big) &\\ &\leq \kappa + F(\alpha(a_n, u) H(Da_n \cap S, Du \cap S))\\ &\leq F\Big(\max\Big\{d(a_n, u), \frac{d(a_n, Da_n \cap S) + d(u, Du \cap S)}{2},\\ &\quad \frac{d(a_n, Du \cap S) + d(u, Da_n \cap S)}{2})\Big\}\Big).\end{aligned}$$

Since $\alpha(a_n, a_{n+1}) \geq 1$ and F is an increasing function, it is easy to observe that

$$F(d(a_{n+1}, Du \cap S)) \leq F(d(a_0, Du \cap S)) - n\kappa.$$

Taking limit $n \to \infty$ on both sides,

$$\Rightarrow \lim_{n\to\infty} F(d(a_{n+1}, Du \cap S)) = -\infty.$$

By using the definition of F-function, we have

$$\lim_{n\to\infty} d(a_{n+1}, Du \cap S) = 0. \tag{3.9}$$

Using assertion **(iii)(a)** the following claim can be easily defended

$$d(u, Du \cap S) \leq \lim_{n\to\infty} \alpha(a_n, u) d(a_{n+1}, Du \cap S) = 0.$$

Furthermore, since it is obvious that $d(u, Du) \leq d(u, Du \cap S) \leq 0$, therefore $d(u, Du) = 0$. If assertion **(iii)(b)** is used as an argument

$$d(a_{n+1}, Du \cap S) \leq \alpha(a_n, u) d(a_{n+1}, Du \cap S) \leq \alpha(a_n, u) H(Da_n \cap S, Du \cap S) \tag{3.10}$$

using (**3**.9) one can deduce

$$d(u, Du) \leq d(u, Du \cap S) = 0.$$

Hence, it follows that

$$d(u, Du) = 0.$$

That is $u \in Du$. □

Example 3.1. *Consider $\Omega = (-\infty, -8) \cup \{\frac{1}{2^{n-1}} : n \in \mathbb{N}\} \cup \{0\}$, accompanied with the usual metric d, and $S = \{\frac{1}{2^n} : n \in \mathbb{N}\} \cup \{0, 1\}$, Now define $D : S \to 2^{\Omega}$ on metric space as*

$$Da = \begin{cases} \left\{\frac{1}{2^n}, 1\right\} & \text{if } a \in \{\frac{1}{2^{n-1}} : n \in \mathbb{N}\},\\ \{0\} & \text{if } a = 0,\end{cases}$$

and $\alpha : S \times S \to [0, \infty)$ as

$$\alpha(a, b) = \begin{cases} 1 & \text{if } a, b \in \{\frac{1}{2^{n-1}} : n \in \mathbb{N}\},\\ 0 & \text{otherwise.}\end{cases}$$

when

$$\min(\alpha(a, b) H(Da \cap S, Db \cap S)), M(a, b)) > 0.$$

It is clear that $Da \cap S \neq \emptyset$ for each $a \in S$. Let $F(a) = a + \ln a$ for all $a > 0$. As $\alpha(a, b) \geq 1$ and $a, b \in \{\frac{1}{2^{n-1}} : n \in \mathbb{N}\}$, so, D is a multivalued mapping. Now through the following *way, D can be easily seen as an α-F-contractive and α-admissible mapping. Let $a = \frac{1}{2^n}$ and $b = \frac{1}{2^m}$, such that $m > n \geq 1$. Then we have, by using Definition (**3**.2)*

$$\begin{aligned} F(\alpha(a,b)H(Da \cap S, Db \cap S)) - F(d(a,b)) &= \ln \left| \frac{2^{m-n}-1}{2^{m+1}} \right| - \ln \left| \frac{2^{m-n}-1}{2^{m}} \right| \\ &= \ln \frac{1}{2} < -\frac{1}{2} \ \ \forall\, a, b \in S. \end{aligned}$$

*By this way, D is a multivalued α-F-contractive mapping on S with $\kappa = \frac{1}{2}$. Therefore D has a fixed point since it satisfies all the conditions of Theorem **3**.1.*

Definition 3.3. *Let $(\Omega, \preceq, d)$ be an ordered metric space and $A,\ B \subseteq \Omega$. We say that $A \prec_r B$ if for each $a \in A$ and $b \in B$, we have $a \preceq b$.*

Corollary 3.1. *Consider an ordered metric space $(\Omega, \preceq, d)$ with $(S, \preceq)$ a complete nonempty subset of Ω induced with respect to the metric d. Let $D : S \to CL(\Omega)$ be a α-F-contractive mapping such that $Da \cap S \neq \phi$ for all $a \in S$ with $a \preceq b$, then we have*

$$\kappa + F(\alpha(a,b)H(Da \cap S, Db \cap S)) \leq F(M(a,b)) \ \ \forall\, a, b \in S,$$

where

$$\min \Big\{ \alpha(a_1,a_2)H(Da_1 \cap S, Da_2 \cap S)), M(a_1,a_2) \Big\} > 0,$$

and

$$M(a,b) = \max \left\{ d(a,b), \frac{d(a, Da \cap S) + d(b, Db \cap S)}{2}, \frac{d(a, Db \cap S) + d(b, Da \cap S)}{2} \right\}$$

and F is an increasing function. Here we also assume that the following conditions are satisfied:

(i): *there exists $a_0 \in S$ and $a_1 \in Da_0 \cap S$ such that $a_0 \preceq a_1$.*
(ii): *either*
(a)D is continuous, or
(b) for any sequence $\{a_n\}$ in S with $a_n \to u$ as $n \to \infty$ and $a_n \preceq a_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, such that as $n \to \infty$, $a_n \preceq u$, or
(c) for any sequence $\{a_n\}$ in S with $a_n \to u$ as $n \to \infty$ and $a_n \preceq a_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}, a_n \preceq u$ for all $n \in \mathbb{N} \cup \{0\}$.

Then D has a fixed point $u \in S$.

Proof. Define $\alpha : S \times S \to [0, \infty)$ as

$$\alpha(a,b) = \begin{cases} 1 \text{ if } a \preceq b, \\ 0 \text{ otherwise,} \end{cases}$$

We get $\alpha(a_0, a_1) = 1$, which follows from condition (i) and from the definition of α-mapping, and from **(ii)**, we have that $a \preceq b$ implies that $Da \cap S \prec_r Db \cap S$ and hence we get $\alpha(a,b) = 1$ implies that $\alpha(u,v) = 1$ for all $u \in Da \cap S$ and $v \in Db \cap S$ which follows from definitions of $\prec$ ordered metric space and α-mapping. Furthermore we can easily verify that D is a strictly α-F-contractive type mapping on the subset S of Ω. So D has a fixed point as it satisfies all the conditions of previous theorem. □

Remark 3.1. *It is worth mentioning that in Theorem **3**.2, condition (a) was introduced by by Samet et al. [22] and condition (b) was introduced by Ali et al. [5] and we have introduced these conditions for different contraction. We can verify that both conditions (a) and (b) are independent with the help of following examples.*

Example 3.2. *Let* $\Omega = \left\{\frac{1}{m} : m \in \mathbb{N}\right\} \cup \{0\}\}$ *and* $a_m = \frac{1}{m+2}$ $\forall$ $m \in \mathbb{N} \cup \{0\}$*, then* $\{a_m\}$ *converges to* u^**. Define* $\alpha : \Omega \times \Omega \to [0, \infty)$ *as*

$$\alpha(a, b) = \begin{cases} \max\left\{\frac{1}{a}, \frac{1}{b}\right\} & \text{if } a \neq 0 \text{ and } b \neq 0 \\ \frac{1}{a+b} & \text{if either } a = 0 \text{ or } b = 0, \\ 1 & \text{if } a = 0 = b. \end{cases}$$

Since $\alpha(a_m, a_{m+1}) = \alpha\left(\frac{1}{m+2}, \frac{1}{m+3}\right) = m + 3 > 1$ *for all* $m \in \mathbb{N} \cup \{0\}$,
and $\alpha\left(\frac{1}{m+2}, 0\right) = m + 2 \geq 1$ *for all* $m \in \mathbb{N} \cup \{0\}$*, therefore, the condition* ***(iii)(b)*** *of Theorem* **3***.2 is satisfied but* $\lim_{n\to\infty} \alpha(a_m, u^*) = \lim_{m\to\infty}(m + 2) = \infty$,*, which means that (iii)(a) is not satisfied.*

Example 3.3. *Let* $\Omega = \left\{\frac{1}{m} : m \in \mathbb{N}\right\} \cup \{0\}$*. Let* $a_m = \frac{1}{m+2}$ *for all* $m \in \mathbb{N} \cup \{0\}$*, then* $\{a_m\}$ *converges to* u^**. Define* $\alpha : \Omega \times \Omega \to [0, \infty)$ *as*

$$\alpha(a, b) = \begin{cases} \max\left\{\frac{1}{a}, \frac{1}{b}\right\} & \text{if } a \neq 0 \text{ and } b \neq 0 \\ \frac{2}{a+b+2} & \text{if either } a = 0 \text{ or } b = 0, \\ 1 & \text{if } a = 0 = b. \end{cases}$$

Since $\alpha(a_m, a_{m+1}) = \alpha\left(\frac{1}{m+2}, \frac{1}{m+3}\right) = m + 3 > 1$ *for all* $m \in \mathbb{N} \cup \{0\}$ *and* $\alpha\left(\frac{1}{m+2}, 0\right) = \frac{2m+4}{2m+5}$.
Therefore

$$\lim_{m\to\infty} \alpha(a_m, u^*) = \lim_{m\to\infty} \frac{2m + 4}{2m + 5} = 1.$$

So, condition ***(iii)(a)*** *of Theorem* **3***.2 is satisfied but in this scenario is obviously not meeting the requirement of condition* ***(iii)(b)****.*

4. Application

In this section, we apply our main results for the existence of the solution of certain integral equations.

Definition 4.1. *Consider a complete metric space* (Ω, d)*, where* Ω *is a non-empty set and* S *a non-empty subset of* Ω*. A mapping* $D : S \to CL(\Omega)$ *is said to be an* α-F*-contractive mapping if for a function* α: $S \times S \to [0, \infty)$*, an* F*-mapping and* $\kappa > 0$*, the following conditions are satisfied:*

(i): $Da \cap S \neq \phi$ *for all* $a \in S$.

(ii): *For each* $a_1, a_2 \in S$*, we have*

$$\kappa + F\left(\alpha(a_1, a_2)H(Da_1 \cap S, Da_2 \cap S)\right) \leq F(M(a_1, a_2)), \tag{4.1}$$

where $\min\Big\{\alpha(a_1, a_2)H(Da_1 \cap S, Da_2 \cap S)), M(a_1, a_2)\Big\} > 0$*, and*

$$M(a_1, a_2) \quad = \max\Big\{d(a_1, a_2), \frac{d(a_1, Da_1 \cap S) + d(a_2, Da_2 \cap S)}{2} \frac{d(a_1, Da_2 \cap S) + d(a_2, Da_1 \cap S)}{2}\Big\}.$$

Here, we give existence theorem for Volterra-type integral equation. For this, assume that $\Omega = C([0, 1], R)$ and S be a nonempty set of Ω such that $S = C([0, 1], R_+)$ be the space of all continuous real valued functions on $[0, 1]$. Consider a complete metric space Ω with $d(a, b) = \sup\limits_{t\in[0,1]} |a(t) - b(t)|$. Consider the Volterra-type integral inclusion as

$$a(t) = \int_0^t N(t, s, a(s))ds + f(t), \text{ for all } t, s \in [0, 1], \tag{4.2}$$

along with the continuous functions $f\colon [0,1] \to \mathbb{R}_+$ and $N\colon [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}_+$. For each $a \in C([0,1], R)$, the operator $N(t,s,a(s))$ is lower semi continuous. For the integral equation as given above, we define a multivalued operator $D\colon S \to CL(\Omega)$ by as below: $D(a(t)) = \Big\{u \in C([0,1],\mathbb{R}) : u \in \int_0^t N(t,s,a(s))ds + f(t), t \in [0,1]\Big\}$. Let $a \in C([0,1], R)$, and denote $N_a = N(t,s,a(s))$ for each $t,s \in [0,1]$. Now for $N_a\colon [0,1] \times [0,1] \to P_{cv}(\mathbb{R}_+)$, by Michael,s selection Theorem, there exists a continuous operator $n_a\colon [0,1] \times [0,1] \to \mathbb{R}_+$ such that $n_a(t,s) \in N_a(t,s)$ for all $t,s \in [0,1]$. This shows that $\int_0^t n_a(t,s)ds + f(t) \in D(a(t))$. Thus the operator Da is nonempty, and operator Da is closed. If multi-valued operator D has a fixed point, then $Da = a$.

Theorem 4.1. *Let $\Omega = C([0,1], R)$ and S be a nonempty set of Ω such that $S = C([0,1], R_+)$ be the space of all continuous real valued functions on $[0,1]$. Let $D\colon S \to CL(\Omega)$ be a multi-valued operator defined by as below: $D(a(t)) = \{u \in C([0,1],\mathbb{R}) : u \in \int_0^t N(t,s,a(s))ds + f(t), t \in [0,1]\}$, with a continuous functions $f\colon [0,1] \to \mathbb{R}_+$ and a multivalued function $N\colon [0,1] \times [0,1] \times \mathbb{R} \to P_{cv}(\mathbb{R}_+)$ are such that for each $a \in C([0,1], R)$, the operator $N(t,s,a(s))$ is lower semi continuous. Suppose that the conditions given below are satisfied:*

- **(i):** *there exists a continuous mapping $p\colon S \to [0,\infty)$. such that*
$$(N(t,s,a(s)) - N(t,s,b(s))) \leq p(s)|a(s) - b(s))|$$
for each $t,s \in [0,1]$ and for all $a,b \in S$;
- **(i):** *there exists $\kappa > 0$ and $\alpha\colon S \times S \to [0,\infty)$ for each $a,b \in S$, we have $\int_0^t p(s)ds \leq \dfrac{e^{-\kappa}}{\alpha(a,b)}, t \in [0,1]$;*
- **(iii):** *there exists $a_0 \in S$ and $a_1 \in Da_0 \cap S$ with $\alpha(a,b) \geq 1$;*
- **(iv):** *if $a \in S$ and $b \in Da \cap S$ such that $\alpha(a,b) \geq 1$, then we have $\alpha(b,c) \geq 1$ for each $c \in Db \cap S$;*
- **(v):** *for any sequence $a_n \to u$ as $n \to \infty$ and $\alpha(a_{n+1}, a_n) \geq 1$ for each $n \in N$, we have $\alpha(a_n, u) \geq 1$ for each $n \in N$;*

Then Volterra-type integral inclusion has a solution.

Proof. We show that the operator D satisfy all conditions of Theorem (**3**.1). To see (2), let $a,b \in S$ such that $u \in Da \cap S$, then we have $n_a(t,s) \in N_a(t,s)$ for all $t,s \in [0,1]$ such that $u(t) = \int_0^t N(t,s)ds + f(t)$, and on the other hand, from hypothesis (i), it ensures that there exists $v(t,s) \in N_b(t,s)$ such that $|n_a(t,s) - v(t,s)| \leq p(s)|a(s) - b(s)|$ for all $t,s \in [0,1]$ *and* $a \in S$. Consider the multivalued operator D_1 defined as: $D_1(t,s) = N_b(t,s) \cap \{w \in \mathbb{R} : |n_a(t,s) - w| \leq p(s)|a(s) - b(s)|\}$, for all $t,s \in [0,1]$ and $a \in S$.

Since the operator D is lower semi continuous, so there exists a mapping $n_b\colon [0,1] \times [0,1] \to \mathbb{R}_+$ such that $n_b(t,s) \in D_1(t,s)$ for all $t,s \in [0,1]$. Thus, we get $r(t) = \int_0^t n_b(t,s)ds + f(t) \in \int_0^t N(t,s)ds + f(t)$, for all $t,s \in [0,1]$ and we have

$$\begin{aligned}|u(t) - r(t)| &\leq (\int_0^t |n_a(t,s) - n_b(t,s)|ds) \leq (\int_0^t p(s)|a(s) - b(s)|ds)\\ &\leq (\sup_{t\in[0,1]} |a(t) - b(t)| \int_0^t p(s)ds) \leq d(a,b)(\int_0^t p(s)ds)\\ &\leq \frac{e^{-\kappa}}{\alpha(a,b)} d(a,b) \text{ for all } t,s \in [0,1]\end{aligned}$$

Consequently, we have $\alpha(a,b)d(u,r) \leq e^{-\kappa} d(a,b)$.
Now, if we replace the role of a and b, we get that $\alpha(a,b)H(Da,Db) \leq e^{-\kappa} d(a,b)$ for all $a,b \in$

S, whenever $\min\{\alpha(a,b)H(Da\cap S, Db\cap S)), M(a,b)\} > 0$. As the natural logarithm belongs to Υ_s, applying it on above inequality and doing some simplification, we have $\kappa + \ln(\alpha(a,b)H(Da,Db)) \leq \ln(d(a,b))$ for all $a,b \in S$. So, D is an α-F-contractive mapping and $F(a) = \ln a; a > 0$. In this way, Theorem's(**3**.1) all conditions follows from hypothesis. Hence the mapping D has a fixed point and integral inclusion has a solution.

□

5. Conclusion

Wardowski [28] gave the idea of F-contraction and proved some fixed point results. These results generalizes the conventional Banach contraction principle. Following Wardowski many authors contributed a lot towards the fixed point theory[14, 27]. In [5] Ali *et al.* introduced a new approach of (α, ψ)-contractive non self multivalued mappings. Combining these approaches ([5],[28]) a new notion of α -F nonself multivalued mappings has been introduced in this article. Using new concept we established Theorem **3**.1. By relaxing condition **(iii)** in Theorem **3**.1 a new fixed point Theorem **3**.2 is proved as well. These theorems together with the endorsing examples can be a good contributions towards fixed point theory.

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