U.P.B. Sci. Bull., Series A, Vol. 76, Iss. 1, 2014 ISSN 1223-7027

LEGENDRE MULTI-WAVELETS TO SOLVE OSCILLATING MAGNETIC FIELDS INTEGRO-DIFFERENTIAL EQUATIONS

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In this paper, we consider an integro-differential equation which describes the charged particle motion for certain configurations of oscillating magnetic fields. We use the continuous linear Legendre multi-wavelets on the interval $[0,1)$ *to solve this equation. Illustrative examples are included to demonstrate the validity and applicability of the new technique.*

Keywords: Integro-differential equation, Legendre multi-wavelets, Operational matrix, Multiresolution of analysis (MRA)

MSC2010: 53C 05.

1. Introduction

In recent years, there has been an increase usage among scientists and engineers to apply wavelet technique to solve both linear and nonlinear problems [1-5]. The main advantage of the wavelet technique is its ability to transform complex problems into a system of algebraic equations. The overview of this method can be found in [6-15]. In this research, an integro-differential equation which describes the charged particle motion for certain configurations of oscillating magnetic fields is considered. We use linear Legendre multi-wavelets on the interval $[0,1)$ to solve this problem. Numerical examples are provided to show the high accuracy, simplicity and efficiency of this method.

2. Wavelets and Linear Legendre multi-wavelets

Wavelet constitutes a family of functions which is constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets as [16]:

$$\psi_{a,b}(t) = |a|^{-1}\,\psi\left(\frac{t-b}{a}\right),\ a,b\in\mathbb{R},\ a\neq 0.$$

If we restrict the parameters a and b to the discrete values $a = a_0^{-k},\ \ b = nb_0a_0^{-k}$, where $a_0 > 1,\ b_0 > 0$, n, and k are positive integers, we obtain the following discrete

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wavelets:

$$\psi_{n,k}(t) = |a|^{\frac{k}{2}}\, \psi\left(a_0^k t - n b_0\right),$$

which form a wavelet basis for $L^2(\mathbb{R})$. In particular, when $a_0 = 2$ and $b_0 = 1$ then $\psi_{n,k}(t)$ form an orthogonal basis [16]. The linear Legendre multi-wavelets are described in [6]. Khellat [6] used this kind of wavelets to solve an optimal control problem. To construct the linear Legendre multi-wavelets, we first define scaling functions $\phi_0(x)$ and $\phi_1(x)$ as:

$$\phi_0(t) = 1,\ \phi_1(t) = \sqrt{3}\,(2t-1),\ 0 \le t < 1.$$

Now let $\psi^0(t)$ and $\psi^1(t)$ be the corresponding mother wavelets, then by Multiresolution of analysis and applying suitable conditions [6] on $\psi^0(t)$ and $\psi^1(t)$ the explicit formula for linear Legendre mother wavelets is obtained as:

$$\psi^0(t) = \begin{cases} -\sqrt{3}\,(4t-1), & 0 \le t < \frac{1}{2}, \\ \sqrt{3}\,(4t-3), & \frac{1}{2} \le t < 1, \end{cases} \tag{1}$$

$$\psi^1(t) = \begin{cases} 6t-1, & 0 \le t < \frac{1}{2}, \\ 6t-5, & \frac{1}{2} \le t < 1. \end{cases} \tag{2}$$

The family $\left\{\psi_{kn}^j\right\} = \left\{2^{\frac{k}{2}}\psi^j\left(2^k t - n\right)\right\}$, where k is any nonnegative integer, $n = 0, 1, \cdots, 2^k - 1$ and $j = 0, 1$, forms an orthogonal basis for $L^2(\mathbb{R})$.

3. Linear Legendre multi-wavelets operational matrix of integration

Let us define:

$$\Psi(t) = \left[\phi_0(t), \phi_1(t), \psi_{00}^0(t), \psi_{00}^1(t), \cdots, \psi_{M0}^0(t), \psi_{M1}^0(t), \cdots,\right. \tag{3}$$

$$\left.\psi_{M(2^M-1)}^0(t), \cdots \psi_{M0}^1(t), \psi_{M1}^1(t), \cdots \psi_{M(2^M-1)}^1(t)\right]^T,$$

where M is a nonnegative integer. The integration of the vector $\Psi(t)$ defined in (3) can be obtained as:

$$\int_0^t \Psi(\tau)\, d\tau \approx P\Psi(t), \tag{4}$$

where P is a $2^{M+2} \times 2^{M+2}$ matrix given by [6]:

$$P = \begin{bmatrix} P_{2^{M+1}\times 2^{M+1}} & Q_{2^{M+1}\times 2^{M+1}} \\ -Q_{2^{M+1}\times 2^{M+1}}^T & R_{2^{M+1}\times 2^{M+1}} \end{bmatrix}. \tag{5}$$

The submatrix $P_{2^{M+1}\times 2^{M+1}}$ in equation (5) is generated by:

$$P_{2\times 2} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{6} & 0 \end{bmatrix}, \tag{6}$$

and the submatrix $R_{2^{M+1}\times 2^{M+1}}$ is generated by the formula:

$$R_{2^{M+1}\times 2^{M+1}} = \frac{\sqrt{3}}{24} \times \frac{1}{2^M} \begin{bmatrix} O & I \\ -I & O \end{bmatrix}, \tag{7}$$

for $M = 0, 1, 2, \cdots$, where O and I are $2^M \times 2^M$ zero and identity matrices, respectively. To generate the submatrix $Q_{2^{M+1}\times 2^{M+1}}$ ($M = 1, 2, \cdots$), suppose it has the block form:

$$Q_{2^{M+1}\times 2^{M+1}} = \begin{bmatrix} S & O \\ T & O \end{bmatrix}, \tag{8}$$

where S and T are $2^M \times 2^M$ matrices and O is a zero matrix. To characterize S, let $Q_{2^M\times 2^M}$ has the form:

$$Q_{2^M\times 2^M} = [C_1 \; C_2 \cdots C_{2^{M-1}} \; O \; O \cdots O], \tag{9}$$

where $C_i (1 \le i \le 2^{M-1})$ and O is a $2^M \times 1$ column matrix. Then S can be obtained by:

$$S = \frac{\sqrt{2}}{8} [C_1 \; C_1 \; C_2 \; C_2 \cdots C_{2^{M-1}} \; C_{2^{M-1}}]. \tag{10}$$

Hence, we need $Q_{2\times 2}$ which has the following matrix

$$Q_{2\times 2} = \frac{1}{8} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{11}$$

To obtain matrix T, we begin by:

$$T_{2\times 2} = \frac{\sqrt{2}}{23} \begin{bmatrix} -1 & 1 \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, \tag{12}$$

and for $M \ge 2$, we consider:

$$K_1 = \tfrac{1}{2} \begin{bmatrix} I & O \\ O & O \end{bmatrix}, \; K_2 = \tfrac{1}{2} \begin{bmatrix} O & I \\ O & O \end{bmatrix}, K_3 = \tfrac{1}{2} \begin{bmatrix} O & O \\ I & O \end{bmatrix}, K_4 = \tfrac{1}{2} \begin{bmatrix} O & O \\ O & I \end{bmatrix},$$

where I is the identity matrix and O is a zero matrix of dimension $2^{M-2} \times 2^{M-2}$. If we put $H = T_{2^{M-1}\times 2^{M-1}}$, then T can be characterized as:

$$T = \begin{bmatrix} K_1 H & K_3 H \\ K_2 H & K_4 H \end{bmatrix}. \tag{13}$$

Hence, the matrix P in Equation (5) is obtained by using Equations (7) and (8).

4. Applying linear Legendre multi-wavelets to the problem

In this section we use the linear Legendre multi-wavelets to approximate the functions. Then by substituting of these approximations in the linear integro-differential equation and using the collocation points, the equation will be transformed into a system of algebraic equations.

4.1. Function approximation

A function $f(t)$ defined over $[0, 1)$ may be expanded as:

$$f(t) = f_0 \phi_0(t) + f_1 \phi_1(t) + \sum_{k=0}^{\infty} \sum_{j=0}^{1} \sum_{n=0}^{\infty} f_{kn}^j \psi_{kn}^j(t), \tag{14}$$

where:

$$f_0 = \langle f(t), \phi_0(t) \rangle, \; f_1 = \langle f(t), \phi_1(t) \rangle, \; f_{kn}^j = \langle f(t), \psi_{kn}^j(t) \rangle. \tag{15}$$

In Equation (15), $\langle .,. \rangle$ denotes the inner product. If the infinite series of Equation (14) is truncated, then it can be written as:

$$f(t) \approx f_0\phi_0(t) + f_1\phi_1(t) + \sum_{k=0}^{M}\sum_{j=0}^{1}\sum_{n=0}^{2^k-1} f_{kn}^{j}\psi_{kn}^{j}(t) = F^T\Psi(t), \tag{16}$$

where $\Psi(t)$ is defined in (3) and F is given by:

$$F = \left[f_0, f_1, f_{00}^0, f_{00}^1, \cdots, f_{M0}^0, f_{M1}^0, \cdots,\right. \tag{17}$$

$$\left. f_{M(2^M-1)}^0, \cdots, f_{M0}^1, f_{M1}^1, \cdots f_{M(2^M-1)}^1\right]^T.$$

4.2. Oscillating magnetic field integro-differential equations

Consider the following integro-differential equation [17]:

$$\frac{d^2y}{dt^2} = -a(t)y(t) + b(t)\int_0^t \cos(w_p s)y(s)ds + g(t), \tag{18}$$

where $a(t)$, $b(t)$ and $g(t)$ are given periodic functions of time which may be easily found in the charged particle dynamics for some field configurations. Taking for instance the three mutually orthogonal magnetic field components $B_x = B_1\sin(w_p t)$, $B_y = 0$ and $B_z = B_0$, the nonrelativistic equations of motion for a particle of mass m and charge q in this field configuration are:

$$m\frac{d^2x}{dt^2} = q\left(B_0\frac{dy}{dt}\right), \tag{19}$$

$$m\frac{d^2y}{dt^2} = q\left(B_1\sin(w_p t)\frac{dz}{dt} - B_0\frac{dx}{dt}\right), \tag{20}$$

$$m\frac{d^2z}{dt^2} = q\left(-B_1\sin(w_p t)\frac{dy}{dt}\right). \tag{21}$$

By integration of (18) and (21) and replacement of the time first derivatives of z and x in (20) one gets (18) with:

$$a(t) = w_c^2 + w_f^2\sin^2(w_p t), \quad b(t) = w_f^2 w_p\sin(w_p t), \tag{22}$$

$$g(t) = w_f(\sin(w_p t))z'(0) + w_c^2 y(0) + w_c x'(0), \tag{23}$$

where $w_c = q\frac{B_0}{m}$ and $w_f = q\frac{B_1}{m}$. Making the additional simplification by seting $x'(0) = 0$ and $y(0) = 0$, Equation (18) is finally written as:

$$\frac{d^2y}{dt^2} = -\left(w_c^2 + w_f^2\sin^2(w_p t)y + w_f(\sin(w_p t))z'(0)\right) \tag{24}$$

$$+w_f^2 w_p\sin(w_p t)\int_0^t \cos(w_p s)y(s)ds.$$

In this paper, we consider the Equation (18) with the following initial conditions:

$$y(0) = \alpha, \quad y'(0) = \beta. \tag{25}$$

Second order derivative of the function $y(t)$ in Equation (18) exists, so:

$$y\left(t\right)=\int_{0}^{t}\left(\int_{0}^{x}y''\left(s\right)ds+y'\left(0\right)\right)dx+y\left(0\right). \tag{26}$$

Approximating the functions $y\left(s\right)$ and $y''\left(s\right)$ with respect to the basis functions by (16) gives:

$$y\left(s\right)\approx Y^{T}\Psi\left(s\right),\qquad y''\left(s\right)=Y''^{T}\Psi\left(s\right). \tag{27}$$

Substituting Equation (27) into Equation (26) and using Equation (4), we obtain:

$$Y^{T}\Psi\left(t\right)\approx Y''^{T}P^{2}\Psi\left(t\right)+ty'\left(0\right)+y\left(0\right). \tag{28}$$

In Equation (28), two functions $ty'\left(0\right)$ and $y\left(0\right)$ can be approximated as:

$$ty'\left(0\right)\approx H^{T}\Psi\left(t\right),\qquad y\left(0\right)\approx K^{T}\ \Psi\left(t\right), \tag{29}$$

so:

$$Y^{T}\approx Y''^{T}P^{2}+H^{T}+K^{T}. \tag{30}$$

Combining Equations (18) and (28), yields:

$$Y''^{T}\left(\Psi\left(t\right)+a(t)P^{2}\Psi\left(t\right)-b(t)P^{2}\int_{0}^{t}\cos(w_{p}s)\Psi\left(s\right)ds\right)$$

$$=g(t)-a(t)\left(ty'(0)+y(0)\right)+b(t)\int_{0}^{t}\cos(w_{p}s)\left(sy'(0)+y(0)\right)ds. \tag{31}$$

Now, let $t_i=1,2,\cdots,2^{M+2}$ be 2^{M+2} appropriate points in interval $[0,1)$. Putting $t=t_i$ into (31), we have a linear system of 2^{M+2} algebraic equations of 2^{M+2} unknown coefficients corresponding to $y''\left(t\right)$. Solving this system of algebraic equations and substituting the result into Equation (30) lead us to find Y^T.

5. **Illustrative examples**

To reformulate the mentioned method and to prove its efficiency for solving the general Equation (18), we consider this equation for different values of $a(t)$, $b(t)$ and $g(t)$, where we can derive respective analytical solutions. In the considered cases, we choose the collocation points:

$$t_i=\frac{2i-1}{2^{M+3}},\qquad i=1,2,\cdots,2^{M+2}. \tag{32}$$

The computations for these examples were performed using Maple 14.

Example 1. Consider Equation (18) with:

$$\begin{aligned} w_p &= 2,\ \ a(t)=\cos(t),\quad b(t)=\sin\left(\frac{t}{2}\right),\\ g(t) &= \cos(t)-t\sin(t)+\cos(t)\left(t\sin(t)+\cos(t)\right)\\ &\quad -\sin\left(\frac{t}{2}\right)\left(\frac{2}{9}\sin(3t)-\frac{t}{6}\cos(3t)+\frac{t}{2}\cos(t)\right) \end{aligned}$$

and $\alpha=1$, $\beta=0$. The exact solution of this problem is given by $y(t)=t\sin(t)+\cos(t)$(see [18]). The numerical solution for Example 1 is obtained by the method in section 4 with $M=3$. Table 1 represents the numerical results of this example.

Example 2. Next, consider Equation (18) with:

$$\begin{aligned} w_p &= 1, \quad a(t) = -\sin(t), \quad b(t) = \sin(t), \\ g(t) &= \frac{1}{9}e^{-\frac{t}{3}} - \sin(t)\left(e^{-\frac{t}{3}} + t\right) \\ &\quad - \sin(t)\left(-\frac{3}{10}\cos(t)e^{-\frac{t}{3}} + \frac{9}{10}e^{-\frac{t}{3}}\sin(t) + \cos(t) + t\sin(t) - \frac{7}{10}\right). \end{aligned}$$

Table 1. Numerical results of Example 1

t	Exact Solution	Approximate Solution	Absolute Error
0	1.	0.9958923638	4.1076×10^{-3}
0.1	1.004987507	1.006711649	1.7241×10^{-3}
0.2	1.019800444	1.019729643	7.0801×10^{-5}
0.3	1.043992551	1.043967189	2.5363×10^{-5}
0.4	1.076828331	1.078484444	1.6561×10^{-3}
0.5	1.117295331	1.116769872	5.2546×10^{-4}
0.6	1.164121099	1.164117090	4.0093×10^{-6}
0.7	1.215794568	1.216007719	2.1315×10^{-4}
0.8	1.270591582	1.270604841	1.3259×10^{-5}
0.9	1.326604187	1.326616801	1.2614×10^{-5}

and $\alpha = 1$, $\beta = \frac{2}{3}$. $y(t) = e^{-\frac{t}{3}} + t$ is the exact solution of this Equation [18]. We solve this example using the proposed method with $M = 3$. Table 2 indicates the numerical results of this example.

Table 2. Numerical results of Example 2

t	Exact Solution	Approximate Solution	Absolute Error
0	1.	0.9995324854	4.6752×10^{-4}
0.1	1.067216100	1.067409867	1.9377×10^{-4}
0.2	1.135506985	1.135498873	8.1101×10^{-6}
0.3	1.204837418	1.204831254	6.1641×10^{-6}
0.4	1.275173319	1.275363888	1.9057×10^{-4}
0.5	1.346481725	1.346379637	1.0209×10^{-4}
0.6	1.418730753	1.418729558	1.195×10^{-6}
0.7	1.491889566	1.491932333	4.2767×10^{-5}
0.8	1.565928338	1.565927886	4.5232×10^{-7}
0.9	1.640818221	1.640830867	1.2646×10^{-5}

Example 3. Finally, we consider Equation (18) [18], with:

$$\begin{aligned} w_p &= 3, \quad a(t) = 1, \quad b(t) = \sin(t) + \cos(t), \\ g(t) &= -t^3 + t^2 - 11t + 4 - (\sin(t) + \cos(t)) \\ &\quad \left(-\frac{t^3}{3}\sin(3t) - \frac{t^3}{3}\cos(3t) - \frac{13}{27}\cos(3t) - \frac{13}{9}t\sin(3t) \right. \\ &\quad \left. + \frac{t^2}{3}\sin(3t) + \frac{16}{27}\sin(3t) + \frac{2}{9}t\cos(3t) + \frac{13}{27} \right), \end{aligned}$$

and $\alpha = 2$, $\beta = -5$. $y(t) = -t^3 + t^2 - 5t + 2$ is the exact solution of this equation. We apply the method with $M = 3$. The exact solution, approximate solution and absolute error are listed in Table 3.

Table 3. Numerical results of Example 3

t	Exact Solution	Approximate Solution	Absolute Error
0	2.	1.996800526	3.1995×10^{-3}
0.1	1.509000000	1.510232801	1.2328×10^{-3}
0.2	1.032000000	1.031768407	2.3159×10^{-4}
0.3	0.5630000000	0.5631496030	1.496×10^{-4}
0.4	0.09600000000	0.09662690505	6.2691×10^{-4}
0.5	−0.3750000000	−0.3731667483	1.8333×10^{-3}
0.6	−0.8560000000	−0.8560124373	1.2437×10^{-5}
0.7	−1.353000000	−1.353869376	8.6938×10^{-4}
0.8	−1.872000000	−1.871984718	1.5282×10^{-5}
0.9	−2.419000000	−2.419488059	4.8806×10^{-4}

6. Conclusions

The aim of the present work is to propose an efficient method for solving the integro-differential equation which describes the charged particle motion for certain configurations of oscillating magnetic fields. The linear Legendre multi-wavelets and collocation points have been applied for solving the problem by reducing the given integro-differential equation into a system of algebraic equations. The method is computationally attractive and applications are demonstrated through several illustrative examples.

7. Acknowledgment

This research was partially supported by the Center of Excellence for Mathematics, University of Shahrekord, Iran.

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