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SOME FURTHER RESULTS ON THE ANNIHILATOR IDEAL GRAPH OF A COMMUTATIVE RING

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Let R be a commutative ring with unity. The annihilator ideal graph of R, denoted by $\Gamma_{\mathrm{Ann}}(R)$, is a graph whose vertices are all non-trivial ideals of R and two distinct vertices I and J are adjacent if and only if $I \cap \mathrm{Ann}(J) \neq (0)$ or $J \cap \mathrm{Ann}(I) \neq (0)$. In this paper, we show that if R is reduced, then $\Gamma_{\mathrm{Ann}}(R)$ is complete multipartite. Also, some results on the annihilator ideal graphs with finite clique numbers are given. Moreover, some properties such as connectivity, diameter, girth and etc. of a subgraph induced by ideals with non-zero annihilators are studied. Moreover, we characterize all rings for which this subgraph and annihilating-ideal graphs are identical.

Keywords: Annihilator ideal graph; Annihilating-ideal graph; Complete bipartite graph; Clique number.

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1. Introduction

One of the most important and active research area in algebraic combinatorics is applying graph theory and combinatorics in abstract algebra. There are a lot of papers which apply combinatorial methods to obtain algebraic results in ring theory (see for instance [1], [3], [5] and [10]). Moreover, for the most recent study in this field see [6], [13] and [17].

Throughout this paper, all rings are assumed to be non-domain commutative rings with identity. The sets of all zero-divisors, nilpotent elements, non-trivial ideals, non-zero ideals with non-zero annihilator and minimal prime ideals of R are denoted by $Z(R)$, $\mathrm{Nil}(R)$, $\mathbb{I}(R)$, $\mathbb{I}^{'}(R)$ and $\mathrm{Min}(R)$, respectively. A non-zero ideal I of R is called *essential*, denoted by $I \leq_e R$, if I has a non-zero intersection with any non-zero ideal of R. The ring R is said to be *reduced* if it has no non-zero nilpotent element. A proper ideal I of R is said to be an *annihilator ideal*, if $I = \mathrm{Ann}(J)$, for some $J \in \mathbb{I}(R)$. The *socle* of an R-module M, denoted by $soc(M)$, is the sum of all simple submodules of M. If there are no simple submodules, this sum is defined to be zero. It is well-known $\mathrm{soc}(M)$ is the intersection of all essential submodules (see [16, 21.1]). We write $\mathrm{depth}(R) = 0$ if and only if every non-unit element of a ring R is zero-divisor. We say x is a *regular element* of R if x is non-unit and non zero-divisor. For any undefined notation or terminology in ring theory, we refer the reader to [4].

Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By $\overline{G}$, $\mathrm{diam}(G)$ and $\mathrm{girth}(G)$, we mean the complement, the diameter and the girth of G, respectively. The graph $H = (V_0, E_0)$ is a *subgraph* of G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced subgraph* by V_0, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E \,|\, u, v \in V_0\}$. For two vertices u and v in G, the notation $u - v$ means that u and v are adjacent. A complete bipartite graph with part sizes m and n is denoted by $K_{m,n}$. If the size of one of the parts is 1, then the graph is said to be a *star graph*. A *clique* of G is a complete subgraph of G and the number of vertices in a largest clique of G, denoted by $\omega(G)$, is called the *clique number* of G. The *chromatic number* of G, denoted by $\chi(G)$, is the minimal number of colors which can be assigned to the

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vertices of G in such a way that every two adjacent vertices have different colors. A graph G is said to be *weakly perfect* if $\omega(G) = \chi(G)$. Let G_1 and G_2 be two disjoint graphs. The *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with the vertex set $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv \,|\, u \in V(G_1), v \in V(G_2)\}$. For any undefined notation or terminology in graph theory, we refer the reader to [15].

Let R be a commutative ring with unity. The *annihilator ideal graph* of R, denoted by $\Gamma_{\mathrm{Ann}}(R)$, is a graph whose vertices are all non-trivial ideals of R and two distinct vertices I and J are adjacent if and only if $I \cap \mathrm{Ann}(J) \neq (0)$ or $J \cap \mathrm{Ann}(I) \neq (0)$. The annihilator ideal graph was first introduced and study in [2]. Many of basic properties of annihilator ideal graph may be found in [2]. In this paper, we continue this study. Indeed, we show that annihilator ideal graph associated with a ring contains the annihilating-ideal graph as a subgraph. The *annihilating-ideal graph* of R, denoted by $\mathbb{AG}(R)$, is a graph with the vertex set $\mathbb{I}^{'}(R)$, and two distinct vertices I and J are adjacent if and only if $IJ = (0)$. The story of annihilating-ideal graphs goes back to [8, 9]. Because of the interesting properties of annihilating-ideal graphs, many papers have been devoted to study different parameters of this graph. For instance, the coloring of annihilating-ideal graphs in [1], the domination number of annihilating-ideal graphs in [12] and the complement graph of annihilating-ideal graphs in [14] were studied by different authors. Also, in [18], annihilating-ideal graphs whose cores consist of only triangles were characterized. In Section 2, we complete the study of basic properties of $\Gamma_{\mathrm{Ann}}(R)$ which was started in [2]. It is shown that if R is reduced, then $\Gamma_{\mathrm{Ann}}(R)$ is a weakly perfect graph. Also, some results on the annihilator ideal graph with finite clique number are given. Then, in Section 3, we focus on a subgraph of annihilator ideal graph induced by ideals with non-zero annihilator. Some parameters of this subgraph such as diameter, girth and condition under which this subgraph is star or complete are studied. Finally, in Section 4, we apply our results in Sections 2,3 to investigate the affinity between annihilator ideal graphs and annihilating-ideal graphs.

2. Annihilator Ideal Graphs with Finite Clique Numbers

Our main aim in this section is to study the annihilator ideal graphs with finite clique numbers. But first, it is shown that if R is a reduced ring, then $\Gamma_{\mathrm{Ann}}(R)$ is weakly perfect (Indeed, we show that $\Gamma_{\mathrm{Ann}}(R)$ is a complete multipartite graph).

The following lemma will be used frequently in this paper.

Lemma 2.1. *Let R be a ring and $I, J \in \mathbb{I}(R)$. Then the following statements hold.*

(1) *If $I - J$ is not an edge of $\Gamma_{\mathrm{Ann}}(R)$, then $\mathrm{Ann}(I) = \mathrm{Ann}(J)$. Moreover, if R is a reduced ring, then the converse is also true.*

(2) *If $I \cap \mathrm{Ann}(I) \neq (0)$, then I is adjacent to every other vertex.*

(3) *If $\mathrm{Ann}(I) = (0)$ and $\mathrm{Ann}(J) \neq (0)$, then $I - J$ is an edge of $\Gamma_{\mathrm{Ann}}(R)$.*

Proof. (1) Since $I - J$ is not an edge of $\Gamma_{\mathrm{Ann}}(R)$, $I \cap \mathrm{Ann}(J) = (0)$ and $J \cap \mathrm{Ann}(I) = (0)$. Thus $\mathrm{Ann}(I) \subseteq \mathrm{Ann}(J)$ and $\mathrm{Ann}(J) \subseteq \mathrm{Ann}(I)$. Let R be a reduced ring and $\mathrm{Ann}(I) = \mathrm{Ann}(J)$. As $K \cap \mathrm{Ann}(K) = (0)$, for every $K \in \mathbb{I}(R)$, we can easily see that $I \cap \mathrm{Ann}(J) = (0)$ and $J \cap \mathrm{Ann}(I) = (0)$. Thus $I - J$ is not an edge of $\Gamma_{\mathrm{Ann}}(R)$.

(2) Assume to the contrary, I is not adjacent to J, for some $J \in \mathbb{I}(R)$. By part (1), $\mathrm{Ann}(I) = \mathrm{Ann}(J)$ and so $I \cap \mathrm{Ann}(J) \neq (0)$, a contradiction. Thus I is adjacent to every other vertex.

(3) Since $\mathrm{Ann}(I) = (0)$, $I\mathrm{Ann}(J) \neq (0)$ and thus $I \cap \mathrm{Ann}(J) \neq (0)$. □

Let R be a reduced ring. Using Lemma 2.1, we show that $\Gamma_{\mathrm{Ann}}(R)$ is a complete multipartite graph.

Theorem 2.1. *Let R be a reduced ring. Then $\omega(\Gamma_{\mathrm{Ann}}(R)) = \chi(\Gamma_{\mathrm{Ann}}(R)) \in \{k, k+1\}$, where k is the number of annihilator ideals of R.*

Proof. Define the relation $\sim$ on $V(\Gamma_{\mathrm{Ann}}(R))$ as follows: For ever $I, J \in \mathbb{I}(R)$ we write $I \sim J$ if and only if $\mathrm{Ann}(I) = \mathrm{Ann}(J)$. It is easily seen that $\sim$ is an equivalence relation on $V(\Gamma_{\mathrm{Ann}}(R))$. By

$[I]$, we mean the equivalence class of I. Therefore, the number of equivalence classes is equal to k or $k+1$ (this number is $k+1$ if $\mathrm{Ann}(I)=(0)$ for some $I\in\mathbb{I}(R)$). Now, suppose that $[I]$ and $[J]$ are two distinct arbitrary equivalence classes. By Lemma 2.1, there is no adjacency between two vertices contained in $[I]$, but every vertex contained in $[I]$ is adjacent to every vertex contained in $[J]$. Indeed, $\Gamma_{\mathrm{Ann}}(R)$ is either a complete k-partite graph or a complete $(k+1)$-partite graph, where k is the number of annihilator ideals of R. Thus $\omega(\Gamma_{\mathrm{Ann}}(R))=\chi(\Gamma_{\mathrm{Ann}}(R))\in\{k,k+1\}$. □

It is worthy to mention that the above theorem immediately generalizes [2, Theorem 14] to arbitrary (not necessary direct sum of finitely many integral domains) reduced rings.

In two next results, we study rings whose annihilator ideal graphs have finite clique numbers.

Theorem 2.2. *Let R be a non-reduced ring, $\omega(\Gamma_{\mathrm{Ann}}(R))<\infty$ and $I\leq_e R$, for some ideal $I\subset Z(R)$. Then the following statements are equivalent.*

(1) *R is a Noetherian ring.*

(2) *R is an Artinian ring.*

(3) *$\Gamma_{\mathrm{Ann}}(R)$ is a complete graph.*

Proof. (2) ⇒ (3) is obtained by [2, Theorem 10] and (3) ⇒ (1) is clear since $\omega(\Gamma_{\mathrm{Ann}}(R))<\infty$.

(1) ⇒ (2) Let $A=\{I\in\mathbb{I}(R)\mid I\subseteq\mathrm{Nil}(R)\}$. By [2, Lemma 4], the induced subgraph by A is complete and so $|A|<\infty$. This implies that $\mathrm{soc}(R)\neq(0)$. Let $B=\{I\subseteq Z(R)\mid I \text{ is essential in } R\}$. It is not hard to check that the induced subgraph by B is also complete and so $|B|<\infty$. Put $J=\cap_{I\in B}I$. By [16, 21.2], $\mathrm{soc}(R)\subseteq J$ and $\mathrm{soc}(R)=\mathrm{soc}(J)$. If the number of essential ideals in J is infinite, then $\omega(\Gamma_{\mathrm{Ann}}(R))=\infty$, a contradiction. This implies that $\mathrm{soc}(J)\leq_e J$. Now, by [16, 17.3], $\mathrm{soc}(R)\leq_e R$. Finally, it is well known that a commutative ring R is Noetherian and $\mathrm{soc}(R)\leq_e R$ if and only if R is Artinian, as desired. □

The following example shows that in Theorem 2.2, the condition "$I\subset Z(R)$ is an essential ideal for some $I\in\mathbb{I}(R)$" is needed and so can not be omitted.

Example 2.1. Let $D=\mathbb{Z}_2[X,Y,Z]$, $I=(X^2,Y^2,XY,XZ,YZ)D$ be an ideal of D, and let $R=D/I$. Also, let $x=X+I$, $y=Y+I$ and $z=Z+I$ be elements of R. Then $\mathrm{Nil}(R)=R(x,y)$ and $Z(R)=R(x,y,z)$. It is not hard to check that the set $\{Rx,Ry,\mathrm{Nil}(R),Z(R)\}$ is a clique and $\Gamma_{\mathrm{Ann}}(R)=K_4\vee\overline{K}_\infty$ and so $\omega(\Gamma_{\mathrm{Ann}}(R))=5$. But since there is no essential ideal in $Z(R)$ such that $I\neq Z(R)$, R is not an Artinian ring.

Theorem 2.3. *Let R be a ring and suppose that $\omega(\Gamma_{\mathrm{Ann}}(R))<\infty$. Then the following statements are equivalent.*

(1) $Z(R)=\mathrm{Nil}(R)$.

(2) *R is an Artinian local ring.*

Proof. (2) ⇒ (1) is clear.

(1) ⇒ (2) Let $I\in\mathbb{I}(R)$ and $I\subseteq\mathrm{Nil}(R)$. We claim that I is a nilpotent ideal of R. It suffices to show that I is finitely generated. Suppose that I is generated by $\{x_i\}_{i\in\Lambda}$, where $|\Lambda|=\infty$. Since for every $i\in\Lambda$ we have $Rx_i\cap\mathrm{Ann}(x_i)\neq(0)$, it follows from Lemma 2.1 that $\{Rx_i\}_{i\in\Lambda}$ is an infinite clique in $\Gamma_{\mathrm{Ann}}(R)$, a contradiction. Hence I is finitely generated and so the claim is proved. Thus $I\cap\mathrm{Ann}(I)\neq(0)$. Let $A=\{I\in\mathbb{I}(R)\mid I\subseteq\mathrm{Nil}(R)\}$. Then part (1) of Lemma 2.1 implies that the induced subgraph by A is complete and so $|A|<\infty$. This, together with $Z(R)=\mathrm{Nil}(R)$, imply that Rx and $\mathrm{Ann}(x)$ are Artinian R-modules, where $x\in Z(R)^*$. Since $Rx\cong R/\mathrm{Ann}(x)$, R is an Artinian ring. Finally, $Z(R)=\mathrm{Nil}(R)$ shows that R is a local ring. □

We close this section with the following result which shows that [2, Theorem 19] does not occur.

Theorem 2.4. *Let R be a ring and* $\mathrm{depth}(R)\neq0$. *Then* $\omega(\Gamma_{\mathrm{Ann}}(R))\neq2$.

Proof. Consider two following cases:

Case 1. R is a reduced ring. Suppose to the contrary, $\omega(\Gamma_{\mathrm{Ann}}(R)) = 2$. We show that $Z(R) = (0)$. Let $x \in Z(R)^*$. So $xy = 0$, fore some $y \in Z(R)^*$. Since R is a reduced ring, $Rx \neq Ry$ and thus $Rx - Ry$ is an edge of $\Gamma_{\mathrm{Ann}}(R)$. Now, let z be a regular element of R. By part (3) of Lemma 2.1, $Rz - Rx - Ry - Rz$ is a triangle in $\Gamma_{\mathrm{Ann}}(R)$, which is impossible. Thus $Z(R) = (0)$, i.e., $\omega(\Gamma_{\mathrm{Ann}}(R)) < 2$, a contradiction.

Case 2. R is a non-reduced ring. We claim that $Z(R) = \mathrm{Nil}(R)$. To see this, let $x \in \mathrm{Nil}(R)^*$ and $y \in Z(R) \setminus \mathrm{Nil}(R)$. Since $x \in \mathrm{Nil}(R)^*$, we conclude that $Rx \cap \mathrm{Ann}(x) \neq (0)$. By part (2) of Lemma 2.1, $Rx - Ry$ is an edge of $\Gamma_{\mathrm{Ann}}(R)$. A similar argument to proof of Case 1 leads to a contradiction. Hence $Z(R) = \mathrm{Nil}(R)$ and so the claim is proved. Now, by Theorem 2.3, $\mathrm{depth}(R) = 0$, a contradiction and so the proof is complete. □

3. A Main Subgraph of the Annihilator Graph of a Ring

The classic zero-divisor graph is a subgraph of Beck's graph induced by $Z(R) \setminus \{0\}$, see [3, 7]. On the other hand, obviously, the set of ideals with non-zero annihilators, has a key role in the structures of both rings and annihilator ideal graphs. Thus, in this section, we study a subgraph of the annihilator ideal graph induced by ideals with non-zero annihilators. For instance, it is shown that $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$ is connected with diameter at most two and girth at most four (if it contains a cycle). Also, all rings R with star $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$ are classified.

Recall that the annihilating-ideal graph of a ring R, denoted by $\mathbb{AG}(R)$, is a graph with the vertex set $\mathbb{I}^{'}(R)$, and two distinct vertices I and J are adjacent if and only if $IJ = (0)$.

Theorem 3.1. *Let R be a ring. Then* $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$ *is connected and* $\mathrm{diam}(\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]) \leq 2$. *Moreover, if* $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$ *contains a cycle, then* $\mathrm{girth}(\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]) \leq 4$.

Proof. First we show that $\mathbb{AG}(R)$ is a subgraph of $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$. If $I - J$ is an edge of $\mathbb{AG}(R)$, then $IJ = (0)$ and so $I \cap \mathrm{Ann}(J) \neq (0)$. This implies that $I - J$ is an edge of $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$. Hence $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$ is connected and so by [8, Theorem 2.1], $\mathrm{girth}(\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]) \leq 4$.

Now, we show that $\mathrm{diam}(\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]) \leq 2$. If $\mathrm{Nil}(R) \neq (0)$, then by part (2) of Lemma 2.1, $\mathrm{diam}(\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]) \leq 2$. So we may assume that $\mathrm{Nil}(R) = (0)$. If $d(I,J) \neq 1$, for some distinct vertices I, J, then by part (1) of Lemma 2.1, $\mathrm{Ann}(I) = \mathrm{Ann}(J)$. Since R is a reduced ring, $I \cap \mathrm{Ann}(I) = (0)$. Therefore, both I and J are adjacent to $\mathrm{Ann}(I)$. This completes the proof. □

The next theorem shows that $\mathrm{girth}(\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]) = 4$ may occur.

Theorem 3.2. *Suppose that* $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$ *contains a cycle. Then* $\mathrm{girth}(\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]) = 4$ *if and only if R is reduced with* $|\mathrm{Min}(R)| = 2$.

Proof. First suppose that $\mathrm{girth}(\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]) = 4$. If $\mathrm{Nil}(R) \neq (0)$, then by part (2) of Lemma 2.1, $\mathrm{girth}(\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]) = 3$, a contradiction. Thus $\mathrm{Nil}(R) = (0)$. Now, let $I \in \mathbb{I}^{'}(R)$. We show that $\mathrm{Ann}(I)$ is a prime ideal of R. To see this, assume that $ab \in \mathrm{Ann}(I)$ such that $a \notin \mathrm{Ann}(I)$ and $b \notin \mathrm{Ann}(I)$. This implies that $aI \neq 0$ and $bI \neq 0$ but $aIbI = 0$. So for every $0 \neq c \in \mathrm{Ann}(I)$, it is easy to see that $Rc - aI - bI - Rc$ is a triangle, a contradiction (note that since $\mathrm{Nil}(R) = (0)$, $aI \neq bI$). Hence $\mathrm{Ann}(I)$ is a prime ideal. Since R is reduced, [11, Corollary 2.2] implies that $\mathrm{Ann}(I)$ is a minimal prime ideal. By using a similar argument, $\mathrm{Ann}(y)$ is a minimal prime ideal, for every $0 \neq y \in \mathrm{Ann}(I)$. Now, we prove that $\mathrm{Min}(R) = \{\mathrm{Ann}(I), \mathrm{Ann}(y)\}$. It is enough to show that $\mathrm{Ann}(I) \cap \mathrm{Ann}(y) = (0)$. Assume to the contrary, $0 \neq a \in \mathrm{Ann}(I) \cap \mathrm{Ann}(y)$. Thus $Ra - I - Ry - Ra$ is a triangle, a contradiction. Hence $\mathrm{Min}(R) = \{\mathrm{Ann}(I), \mathrm{Ann}(y)\}$.

Conversely, suppose that R is reduced and $|\mathrm{Min}(R)| = 2$. Let $\mathfrak{p}_1$, $\mathfrak{p}_2$ be the minimal prime ideals of R. Since R is reduced, we have $Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$ and $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (0)$, by [11, Corollary 2.4].

Let A,B be the sets of all non-zero ideals contained in $\mathfrak{p}_1,\mathfrak{p}_2$, respectively. It is not hard to see that $\Gamma_{\text{Ann}}(R)[\mathbb{I}^{'}(R)] = K_{|A|,|B|}$. As $\Gamma_{\text{Ann}}(R)[\mathbb{I}^{'}(R)]$ contains a cycle, girth$(\Gamma_{\text{Ann}}(R)[\mathbb{I}^{'}(R)]) = 4$. □

In order to characterize all rings R whose $\Gamma_{\text{Ann}}(R)[\mathbb{I}^{'}(R)]$ is star, the following lemma is needed.

Lemma 3.1. *Let R be a non-reduced ring. Suppose that $\Gamma_{\text{Ann}}(R)[\mathbb{I}^{'}(R)]$ is a star graph. Then the following statements hold.*

(1) *R is indecomposable.*

(2) $|\mathbb{I}^{'}(R)| = 2$.

Proof. (1) Let $R \cong R_1 \times R_2$, where R_i is a ring, for $1 \le i \le 2$. Then for every $a \in \text{Nil}(R)^*$, the vertices of the set $\{Ra, R_1 \times (0), (0) \times R_2\}$ forms a triangle, a contradiction. So R is indecomposable.

(2) We claim that $Z(R) = \text{Nil}(R)$. Let $a \in \text{Nil}(R)^*$ and $x \in Z(R) \setminus \text{Nil}(R)$. It is shown that $ax = 0$. Assume to the contrary, $ax \neq 0$. Since $Ra \cap \text{Ann}(a) \neq (0)$, by part (2) of Lemma 2.1, Ra is adjacent to every other vertex. Again, since $ax \neq 0$ and $x \in Z(R) \setminus \text{Nil}(R)$, $RxRy = (0)$ and $Rx \neq Ry$, for some $y \in \text{Ann}(x)$. This implies that $Ra - Rx - Ry - Ra$ is a triangle, a contradiction. If $a \neq b \in \text{Nil}(R)$ such that $Ra \neq Rb$, then $Ra - Rb - Rx - Rb$ is a triangle, a contradiction. Thus $\text{Nil}(R)$ is a minimal ideal of R. Therefore, $\text{Nil}(R) = Ra$ and hence either $Ra^2 = 0$ or $Ra^2 = Ra$. Since R is indecomposable, $Ra^2 = 0$. This means that $a \in \text{Ann}(Z(R))$ and thus $Z(R) \in V(\Gamma_{\text{Ann}}(R)[\mathbb{I}^{'}(R)])$. But this implies $Z(R)$ is adjacent to every other vertex, a contradiction. Hence $Z(R) = \text{Nil}(R)$ and so the claim is proved. Now, by Theorem 2.3, R is an Artinian ring and thus by Theorem 2.2, $\Gamma_{\text{Ann}}(R)[\mathbb{I}^{'}(R)]$ is a complete graph. Since $\Gamma_{\text{Ann}}(R)[\mathbb{I}^{'}(R)]$ is a star graph, we deduce that $|\mathbb{I}^{'}(R)| = 2$. □

Theorem 3.3. *Let R be a ring. Then $\Gamma_{\text{Ann}}(R)[\mathbb{I}^{'}(R)]$ is a star graph if and only if one of the following statements holds.*

(1) *$R \cong F \times D$, where F is a field and D is an integral domain.*

(2) *R is a local ring with exactly two non-trivial ideals.*

Proof. First suppose that $\Gamma_{\text{Ann}}(R)[\mathbb{I}^{'}(R)]$ is a star graph. We consider the following cases.

Case 1. R is a reduced ring. Suppose that the vertex I is adjacent to every other vertex. If $I \neq I^2$, then $I - I^2$ must be an edge of $\Gamma_{\text{Ann}}(R)[\mathbb{I}^{'}(R)]$. But since R is a reduced ring, $\text{Ann}(I) = \text{Ann}(I^2)$ and thus by part (1) of Lemma 2.1, I is not adjacent to I^2, a contradiction. Thus $I = I^2$. Now, let $J \subseteq I$. If $J \neq I$, then since $J\text{Ann}(I) = (0)$, we deduce that J is adjacent to $\text{Ann}(I)$, a contradiction. So I is a minimal ideal of R and thus by [16, 2.3 and 2.7], $R \cong Ra \times R(1-a)$, for an element $a \in R$. We may assume that $R \cong R_1 \times R_2$ with $R_1 \times (0)$ adjacent to every other vertex. If R_1 has a non-trivial ideal, say I. Then $I \times (0)$ is adjacent to $(0) \times R_2$, a contradiction. So R_1 is a field. Similarly, $Z(R_2) = (0)$. Therefore, $R \cong F \times D$, where F is a field and D is an integral domain.

Case 2. R is a non-reduced ring. By Lemma 3.1, it is easily seen that R is a local ring with exactly two non-trivial ideals.

The converse is clear. □

To prove Theorem 3.4, we state the following lemma.

Lemma 3.2. *Suppose that $R = R_1 \times R_2$, where R_1 and R_2 are two rings. Then $\Gamma_{\text{Ann}}(R)[\mathbb{I}^{'}(R)]$ is complete if and only if $\Gamma_{\text{Ann}}(R_i)[\mathbb{I}^{'}(R_i)]$ is complete, for $i = 1,2$.*

Proof. Suppose that $\Gamma_{\text{Ann}}(R)[\mathbb{I}^{'}(R)]$ is a complete graph and I,J are to distinct vertices of $\Gamma_{\text{Ann}}(R_2)[\mathbb{I}^{'}(R_2)]$. With no loss of generality, one may suppose that $(R_1 \times I) \cap \text{Ann}(R_1 \times J) \neq 0$. Thus $I \cap \text{Ann}(J) \neq 0$ and so I and J are adjacent together. Hence $\Gamma_{\text{Ann}}(R_2)[\mathbb{I}^{'}(R_2)]$ is complete. Similarly, $\Gamma_{\text{Ann}}(R_1)[\mathbb{I}^{'}(R_1)]$ is complete. To prove the converse, let $I_1 \times I_2, J_1 \times J_2 \in \mathbb{I}^{'}(R)$. Then without loss of generality, we may assume that $\text{Ann}(I_1) \neq (0)$ in R_1. If $\text{Ann}(J_1) \neq (0)$ in R_1, then since $\Gamma_{\text{Ann}}(R_1)[\mathbb{I}^{'}(R_1)]$ is complete, we conclude that $I_1 \times I_2 - J_1 \times J_2$ is an edge of $\Gamma_{\text{Ann}}(R)[\mathbb{I}^{'}(R)]$. If $\text{Ann}(J_1) = (0)$ in R_1, then J_1 is an essential ideal of R_1. Thus $J_1 \cap \mathrm{Ann}(I_1) \neq (0)$ and so $I_1 \times I_2 - J_1 \times J_2$ is an edge of $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$, as desired. $\square$

Let R be a ring and $I, J \in \mathbb{I}^{'}(R)$. We say that I contains a J-regular element say, x, if $x \notin \mathrm{Ann}(J)$ and $RxJ \neq J$.

We now state our last result in this section.

Theorem 3.4. *Let R be a Noetherian ring. Then $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$ is complete if and only if either there exists $x \in I^*$ such that x is not a J-regular element or there exists $y \in J^*$ such that y is not an I-regular element, for every pair of distinct vertices I, J.*

Proof. If $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$ is complete, then it directly follows from the definition of annihilator ideal graph that for every pair of distinct vertices $I, J \in \mathbb{I}^{'}(R)$ either there exists an element $x \in I^*$ such that x is not J-regular or there exists $y \in J^*$ such that y is not an I-regular element.

To prove the other side, let I, J be two distinct vertices of $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$. By Lemma 3.2, we may assume that R is indecomposable. Without loss of generality, assume that $x \in I^*$ is not an J-regular element. If $x \in \mathrm{Ann}(J)$, then there is nothing to prove. If $x \notin \mathrm{Ann}(J)$, then $RxJ = J$, and so, by [4, Corollary 2.5], there exists an element $a \in Rx$ such that $(1-a)J = 0$. Thus $1-a \in \mathrm{Ann}(J)$, and hence $Rx + \mathrm{Ann}(J) = I + \mathrm{Ann}(J) = R$. Since R is indecomposable, $I \cap \mathrm{Ann}(J) \neq (0)$. Hence $I - J$ is an edge of $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$. This completes the proof. $\square$

4. When $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$ and $\mathbb{AG}(R)$ Are Identical?

As we have seen in the previous section, $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$ and $\mathbb{AG}(R)$ are close to each other. So, it is interesting to characterize rings R whose $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$ and $\mathbb{AG}(R)$ are identical. This characterization also make some of properties of $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$ (and $\mathbb{AG}(R)$) clear. First we study the case when R is reduced.

Theorem 4.1. *Let R be a reduced ring. Then the following statements are equivalent.*

(1) $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)] = \mathbb{AG}(R)$.

(2) $|\mathrm{Min}(R)| = 2$.

(3) $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$ *is a complete bipartite graph.*

(4) $\mathbb{AG}(R)$ *is a complete bipartite graph.*

Proof. (1) $\Rightarrow$ (2) Suppose to the contrary, $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$ are three distinct minimal prime ideals. Let $a \in \mathfrak{p}_1 \setminus \mathfrak{p}_2 \cup \mathfrak{p}_3$. Thus $\mathfrak{p}_2 \cup \mathfrak{p}_3 \nsubseteq \mathrm{Ann}(a)$ (as $\mathrm{Ann}(a) \subseteq \mathfrak{p}_2 \cap \mathfrak{p}_3$). So one may assume that $ab \neq 0$, for some $b \in \mathfrak{p}_2 \cup \mathfrak{p}_3 \setminus \mathfrak{p}_1$. With no loss of generality, assume that $b \in \mathfrak{p}_2 \setminus \mathfrak{p}_1$. Obviously, $\mathrm{Ann}(b) \subseteq \mathfrak{p}_1$. Also, it follows from [11, Corollary 2.2], there exists an element $x \in \mathrm{Ann}(a)$ such that $x \notin \mathfrak{p}_1$. Therefore, $\mathrm{Ann}(a) \neq \mathrm{Ann}(b)$, and so by part (1) of Lemma 2.1, $Ra - Rb$ is an edge of $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$ that is not an edge of $\mathbb{AG}(R)$, a contradiction. Hence $|\mathrm{Min}(R)| = 2$.

(2) $\Rightarrow$ (3) Suppose that $|\mathrm{Min}(R)| = 2$. Let $\mathfrak{p}_1$, $\mathfrak{p}_2$ be minimal prime ideals of R. Since R is a reduced ring, we have $Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$ and $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (0)$. Let A, B be the sets of all non-zero ideals contained in $\mathfrak{p}_1, \mathfrak{p}_2$, respectively. It is not hard to check that $V(\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]) = A \cup B$ and for every $I \in A$ and $J \in B$, $\mathrm{Ann}(I) = \mathfrak{p}_2$ and $\mathrm{Ann}(J) = \mathfrak{p}_1$. Now, it is easily seen that $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)] = K_{|A|,|B|}$.

(3) $\Rightarrow$ (4) Suppose that $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$ is a complete bipartite graph with parts V_1, V_2, i.e., $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)] = K_{|V_1|,|V_2|}$. If $I, J \in V_i$ for $i = 1, 2$, then since $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$ is a complete bipartite graph, $IJ \neq (0)$. Also, if $I \in V_1$ and $J \in V_2$, then $IJ = (0)$ (If $IJ \neq (0)$, then $I - J - \mathrm{Ann}(I) - I$ is a triangle, a contradiction).

(4) $\Rightarrow$ (1) is obtained by [9, Corollary 2.11] and proof of (2) $\Rightarrow$ (3). $\square$

Theorem 4.2. *Let R be a reduced ring. Then the following statements are equivalent.*

(1) $\Gamma_{\mathrm{Ann}}(R) = \mathbb{AG}(R) \vee \overline{K}_\infty$.

(2) $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)] = \mathbb{AG}(R)$ *and* $\mathrm{depth}(R) \neq 0$.

(3) $|\mathrm{Min}(R)| = 2$ *and* $\Gamma_{\mathrm{Ann}}(R) = \Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)] \vee \overline{K}_{\infty}$.
(4) $|\mathrm{Min}(R)| = 2$ *and* $V(\mathbb{AG}(R)) = \infty$.

Proof. (1) $\Rightarrow$ (2) Suppose that $I - J$ is an edge of $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$ that is not an edge of $\mathbb{AG}(R)$. This implies that $I - J$ is an edge of $\Gamma_{\mathrm{Ann}}(R)$ that is not an edge of $\mathbb{AG}(R) \vee \overline{K}_{\infty}$, a contradiction. So $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)] = \mathbb{AG}(R)$. If depth$(R) = 0$, then by Theorem 4.1, $|\mathrm{Min}(R)| = 2$. Let $\mathfrak{p}_1$, $\mathfrak{p}_2$ be the minimal prime ideals of R. Since R is reduced, we have $Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$ and $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (0)$. This, together with every non-unit element of R is zero-divisor, implies that $V(\Gamma_{\mathrm{Ann}}(R)) = V(\mathbb{AG}(R))$, a contradiction.

(2) $\Rightarrow$ (3) By Theorem 4.1, $|\mathrm{Min}(R)| = 2$ and so $\mathrm{Ann}(I) \neq (0)$ for every $I \subset Z(R)$. If we put $A = \{I \in V(\Gamma_{\mathrm{Ann}}(R)) \mid \mathrm{Ann}(I) = (0)\}$, then $\Gamma_{\mathrm{Ann}}(R)[A]$ is null. Since depth$(R) \neq 0$, $|A| = \infty$. Part (3) of Lemma 2.1 implies that every vertex of A is adjacent to all of $V(\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)])$. Hence $\Gamma_{\mathrm{Ann}}(R) = \Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)] \vee \overline{K}_{\infty}$.

(3) $\Rightarrow$ (4) If $V(\mathbb{AG}(R)) < \infty$, then by [8, Theorem 1.1], R is an Artinian ring. Since $|\mathrm{Min}(R)| = 2$ and R is reduced, R is isomorphic to the direct product of two fields and so $\Gamma_{\mathrm{Ann}}(R) = K_{1,1}$, a contradiction.

(4) $\Rightarrow$ (1) Since $|\mathrm{Min}(R)| = 2$, $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)] = \mathbb{AG}(R)$. Let $A = \{I \in V(\Gamma_{\mathrm{Ann}}(R)) \mid \mathrm{Ann}(I) = (0)\}$. If $|A| < \infty$, then depth$(R) = 0$ and so R is isomorphic to the direct product of two fields and so $\mathbb{AG}(R) = K_{1,1}$, a contradiction. Part (3) of Lemma 2.1 implies that every vertex of A is adjacent to all of $V(\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)])$. Hence $\Gamma_{\mathrm{Ann}}(R) = \Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)] \vee \overline{K}_{\infty}$. □

Theorem 4.3. *Let R be a reduced ring. Then the following statements are equivalent.*
(1) $\Gamma_{\mathrm{Ann}}(R) = \mathbb{AG}(R)$.
(2) $\Gamma_{\mathrm{Ann}}(R) = K_2$.
(3) $\mathbb{AG}(R) = K_2$.
(4) $|\mathrm{Min}(R)| = 2$ *and* depth$(R) = 0$.

Proof. (1) $\Rightarrow$ (2) Since $\Gamma_{\mathrm{Ann}}(R) = \mathbb{AG}(R)$, $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)] = \mathbb{AG}(R)$ and depth$(R) = 0$. By Theorem 4.1, $|\mathrm{Min}(R)| = 2$. Since R is reduced, R is isomorphic to the direct product of two fields and so $\Gamma_{\mathrm{Ann}}(R) = K_{1,1}$. (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) are clear. □

To prove Theorem 4.4, the following lemma is needed.

Lemma 4.1. [9, Conjecture 1.11] *Let R be a reduced ring with more than two minimal prime ideals. Then* girth$(\mathbb{AG}(R)) = 3$.

Proof. Since R is reduced, by [11, Corollary 2.4], $Z(R) = \cup_{\mathfrak{p} \in \mathrm{Min}(R)} \mathfrak{p}$. Suppose that $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$ are three distinct minimal prime ideals. If $x \in \mathfrak{p}_1 \setminus \mathfrak{p}_2 \cup \mathfrak{p}_3$, then $\mathrm{Ann}(x) \subset \mathfrak{p}_2 \cap \mathfrak{p}_3$. Let $0 \neq y \in \mathrm{Ann}(x)$. Since $Rx\mathfrak{p}_2 \neq (0)$, let $a \in Rx \cap \mathfrak{p}_2$. As R is reduced, $Rx \cap \mathrm{Ann}(x) = (0)$. This implies that $a \notin \mathrm{Ann}(x)$. Since $a, y \in \mathfrak{p}_2$, we have $a + y = z \in \mathfrak{p}_2$ and so $\mathrm{Ann}(z) \neq (0)$. By [11, Corollary 2.2], $hz = 0$ for some $h \notin \mathfrak{p}_2$. Since $\mathrm{Ann}(z) = \mathrm{Ann}(y) \cap \mathrm{Ann}(a)$, $Ra - Ry - Rh - Ra$ is a cycle of length 3. □

Theorem 4.4. *Let R be a reduced ring. Then the following statements are equivalent.*
(1) $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$ *is a star graph.*
(2) girth$(\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]) = \infty$.
(3) $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)] = \mathbb{AG}(R)$ *and* girth$(\mathbb{AG}(R)) = \infty$.
(4) girth$(\mathbb{AG}(R)) = \infty$.
(5) $|\mathrm{Min}(R)| = 2$ *and at least one of minimal prime ideals is a minimal ideal.*
(6) *Either* $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)] = K_{1,1}$ *or* $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)] = K_{1,\infty}$.
(7) *Either* $\mathbb{AG}(R) = K_{1,1}$ *or* $\mathbb{AG}(R) = K_{1,\infty}$.

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) Since $\mathbb{AG}(R)$ is a subgraph of $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$, if $|\mathrm{Min}(R)| > 2$, then by Lemma 4.1, $\mathrm{girth}(\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]) = 3$, a contradiction. Thus $|\mathrm{Min}(R)| = 2$ and so $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)] = \mathbb{AG}(R)$, by Theorem 4.1. Also, this implies that $\mathrm{girth}(\mathbb{AG}(R)) = \infty$.

(3) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (5) By Lemma 4.1, $|\mathrm{Min}(R)| = 2$. By Theorem 4.1, $\mathbb{AG}(R)$ is complete bipartite and $\mathrm{girth}(\mathbb{AG}(R)) = \infty$. Let $\mathfrak{p}_1$, $\mathfrak{p}_2$ be the minimal prime ideals of R. Since R is reduced, we have $Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$ and $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (0)$. If $(0) \neq I \subset \mathfrak{p}_1$, $(0) \neq J \subset \mathfrak{p}_2$, then the cycle $I - J - \mathfrak{p}_1 - \mathfrak{p}_2 - I$ implies that $\mathrm{girth}(\mathbb{AG}(R)) = 4$, a contradiction.

(5) $\Rightarrow$ (6) Since R is reduced and contains a minimal ideal, we deduce that R is decomposable. Now, $|\mathrm{Min}(R)| = 2$ implies that $R = F \times D$, where F is a field and D is an integral domain and $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)] = \mathbb{AG}(R)$. Now, if D is a field, then $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)] = K_{1,1}$ otherwise $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)] = K_{1,\infty}$.

(6) $\Rightarrow$ (7) is clear since $\mathbb{AG}(R)$ is connected subgraph of $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$.

(7) $\Rightarrow$ (1) $\mathbb{AG}(R)$ is a star graph, by [8, Corollary 2.3], $|\mathrm{Min}(R)| = 2$ and thus $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$ is a star graph. □

Theorem 4.5. *Let R be a reduced ring. Then the following statements are equivalent.*

(1) $\mathrm{girth}(\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]) = 4$.

(2) $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)] = \mathbb{AG}(R)$ *and* $\mathrm{girth}(\mathbb{AG}(R)) = 4$.

(3) $\mathrm{girth}(\mathbb{AG}(R)) = 4$.

(4) $|\mathrm{Min}(R)| = 2$ *and both of minimal prime ideals of R are not minimal ideals.*

(5) $\mathbb{AG}(R) = K_{\infty,\infty}$.

(6) $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)] = K_{\infty,\infty}$.

Proof. (1) $\Rightarrow$ (2) By Theorems 3.2 and 4.1, $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)] = \mathbb{AG}(R)$, and so $\mathrm{girth}(\mathbb{AG}(R)) = 4$.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4) is obtained by Lemma 4.1 and Theorem 4.4.

(4) $\Rightarrow$ (5) Let $\mathfrak{p}_1$, $\mathfrak{p}_2$ be the minimal prime ideals of R. Since R is reduced, we deduce that $Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$ and $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (0)$. If either $\mathfrak{p}_1$ or $\mathfrak{p}_2$ contains a minimal ideal, then $R = F \times D$, where F is a field and D is an integral domain, a contradiction.

(5) $\Rightarrow$ (6) is obtained by Theorem 4.1.

(6) $\Rightarrow$ (1) is clear. □

In view of Theorems 4.1 and 3.2, we have the following corollary.

Corollary 4.1. *Let R be a reduced ring. Then the following statements are equivalent.*

(1) $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)] = \mathbb{AG}(R)$.

(2) $\mathrm{girth}(\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]) = \mathrm{girth}(\mathbb{AG}(R)) = \{4, \infty\}$.

In the rest of this section, we focus on non-reduced rings for which $\mathbb{AG}(R)$ and $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$ are identical.

Theorem 4.6. *Let R be a non-reduced ring. Then the following statements are equivalent.*

(1) $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)] = \mathbb{AG}(R)$.

(2) $\mathbb{AG}(R)$ *is a complete graph.*

(3) *Either* $\mathbb{AG}(R) = K_2$ *or* $Z(R)^2 = (0)$.

Proof. (1) $\Rightarrow$ (2) First we show that $\mathrm{Ann}(Z(R)) \neq (0)$. Let $a \in \mathrm{Nil}(R)^*$. If $ax \neq 0$, for some $x \in Z(R)$, then since $Ra \cap \mathrm{Ann}(a) \neq (0)$, by part 2 of Lemma 2.1, $Ra - Rx$ is an edge of $\Gamma_{\mathrm{Ann}}(R)[\mathbb{I}^{'}(R)]$ that is not an edge of $\mathbb{AG}(R)$, a contradiction. This implies that $\mathrm{Ann}(Z(R)) \neq (0)$ and so $Z(R) \in V(\mathbb{AG}(R))$. Let $I, J \in \mathbb{I}^{'}(R)$ and suppose that $I - J$ is not an edge of $\mathbb{AG}(R)$. Thus $IZ(R) \neq (0)$ and so $I - Z(R)$

is an edge of $\Gamma_{\text{Ann}}(R)[\mathbb{I}'(R)]$ which is not an edge of $\mathbb{AG}(R)$, a contradiction. Therefore, $\mathbb{AG}(R)$ is a complete graph.

(2) $\Rightarrow$ (3) is obtained by [8, Theorem 2.7].

(3) $\Rightarrow$ (1) is clear. □

Suppose that R is a non-reduced ring. The proof of [8, Theorem 2.2] shows that if there exists a vertex of $\mathbb{AG}(R)$ which is adjacent to every other vertex, then $\text{Ann}(Z(R)) \neq (0)$. By using this fact the following theorem is proved.

Theorem 4.7. *Let R be a non-reduced ring and let A be the set of all non-zero ideals contained in* $\text{Nil}(R)$. *If $Z(R) \neq \text{Nil}(R)$, then the following statements hold.*

(1) $\mathbb{AG}(R) = K_{|A|} \vee \overline{K}_\infty$ *if and only if* $\text{Ann}(Z(R))$ *is a prime ideal of R.*

(2) $\Gamma_{\text{Ann}}(R)[\mathbb{I}'(R)] = K_{|A|+1} \vee \overline{K}_\infty$ *and* $\text{Nil}(R) = \text{Ann}(Z(R))$ *if and only if* $\text{Ann}(Z(R))$ *is a prime ideal of R and for every $I \neq Z(R)$ and $I \notin A$, $I \cap \text{Ann}(Z(R)) = (0)$.*

Proof. (1) Since $\mathbb{AG}(R) = K_{|A|} \vee \overline{K}_\infty$, every vertex of $K_{|A|}$ is adjacent to all other vertices but there is no adjacency between two arbitrary vertices of $\overline{K}_\infty$. This implies that $\text{Ann}(Z(R)) = \text{Nil}(R)$ and $IJ \neq (0)$, for every $I, J \in V(\overline{K}_m)$. Now we show that $\text{Ann}(Z(R))$ is a prime ideal of R. To see this, let $IJ \subseteq \text{Ann}(Z(R))$, $I \nsubseteq \text{Ann}(Z(R))$ and $J \nsubseteq \text{Ann}(Z(R))$. We claim that $IJ \neq 0$. Suppose to the contrary, $IJ = (0)$. Since $I \nsubseteq \text{Nil}(R)$ and $J \nsubseteq \text{Nil}(R)$, $I - J$ is an edge of $V(\overline{K}_m)$, a contradiction unless $I = J$. Hence $I^2 = 0$, and so $I \subseteq \text{Ann}(Z(R))$, a contradiction. So $IJ \neq (0)$ and hence the claim is proved. Since $IJ \subseteq \text{Ann}(Z(R))$ and $I \nsubseteq \text{Ann}(Z(R))$, $KIJ = (0)$, $KI \neq (0)$ for some $K \in V(\mathbb{AG}(R))$. This implies that $J \in V(\mathbb{AG}(R))$, $IJJ = IJ^2 = (0)$, $J^2 \subseteq \text{Ann}(Z(R))$. Hence $J^2J = J^3 = (0)$, a contradiction. So $\text{Ann}(Z(R))$ is a prime ideal of R.

Conversely, since $\text{Ann}(Z(R))$ is a prime ideal of R, $\text{Ann}(Z(R)) = \text{Nil}(R)$. Let $B = V(\mathbb{AG}(R)) \setminus A$. So $IJ = (0)$, for all $I, J \in A$ and $IJ \neq (0)$, for all $I, J \in B$. Now, it is easy to see that $\mathbb{AG}(R)[A]$ and $\mathbb{AG}(R)[B]$ are two subgraphs of $\mathbb{AG}(R)$ such that $\mathbb{AG}(R)[A]$ is complete, $\mathbb{AG}(R)[B]$ is null and $\mathbb{AG}(R) = \mathbb{AG}(R)[A] \vee \mathbb{AG}(R)[B]$. We have only to prove that $|B| = \infty$. Suppose to the contrary, $|B| < \infty$ and let $x \in Z(R) \setminus \text{Nil}(R)$. Since $|B| < \infty$, $Rx^n = Rx^m$ for some positive integers $n < m$. So $Rx^n = Rx^nRx^{m-n}$. Now, by [4, Corollary 2.5], there exists an element $a \in Rx^n$ such that $(1-a)Rx^{m-n} = 0$. Thus $1 - a \in \text{Ann}(Rx^{m-n})$, and hence $Rx^n + \text{Ann}(Rx^{m-n}) = R$. On the other hand, $\text{Ann}(Rx^{m-n}) = \text{Nil}(R)$, a contradiction.

(2) Since $\text{Ann}(Z(R)) \neq (0)$, $Z(R) \in V(\mathbb{AG}(R))$ and so $Z(R)$ is adjacent to every other vertex in $\Gamma_{\text{Ann}}(R)[\mathbb{I}'(R)]$. This, together with $\Gamma_{\text{Ann}}(R)[\mathbb{I}'(R)] = K_{|A|+1} \vee \overline{K}_\infty$ and $\text{Nil}(R) = \text{Ann}(Z(R))$, imply that $\mathbb{AG}(R) = K_{|A|} \vee \overline{K}_\infty$. By part 1, $\text{Ann}(Z(R))$ is a prime ideal of R. Also, if $I \cap \text{Ann}(Z(R)) \neq (0)$ for some $I \notin A$ and $I \neq Z(R)$, then we can easily deduce that $\Gamma_{\text{Ann}}(R)[\mathbb{I}'(R)] = K_{|A|+n} \vee \overline{K}_\infty$, where $n \in \{\mathbb{N}, \infty\}$, a contradiction.

The converse, is obtained by part 1. Indeed, since $\text{Ann}(Z(R))$ is a prime ideal of R, $\mathbb{AG}(R) = K_{|A|} \vee \overline{K}_\infty$. Also, since for every $I \neq Z(R)$ and $I \notin A$, $I \cap \text{Ann}(Z(R)) = (0)$, for $V(\mathbb{AG}(R)) \setminus Z(R)$ we have $\mathbb{AG}(R) = \Gamma_{\text{Ann}}(R)[\mathbb{I}'(R)]$. This, together with $Z(R)$ is adjacent to every other vertex in $\Gamma_{\text{Ann}}(R)[\mathbb{I}'(R)]$, implies that $\Gamma_{\text{Ann}}(R)[\mathbb{I}'(R)] = K_{|A|+1} \vee \overline{K}_\infty$. □

We finish this paper with the following example which explains Theorem 4.7.

Example 4.1. Let $R = \mathbb{Z}_2[X,Y]/(XY, X^2)$ and let $x = X + (XY, X^2)$, $y = Y + (XY, X^2)$. Then $\text{Ann}(Z(R)) = \text{Nil}(R) = \{0, x\}$ is a prime ideal of R, $\text{Ann}(Z(R)) \neq Z(R)$ and $Z(R) = (x, y)R$. It is clear that $\mathbb{AG}(R) = K_1 \vee \overline{K}_\infty$ and $\Gamma_{\text{Ann}}(R)[\mathbb{I}'(R)] = K_2 \vee \overline{K}_\infty$.

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