LINEAR TWO-DIMENSIONAL INVERSE BOUNDARY VALUE PROBLEM FOR A THIRD-ORDER PSEUDOHYPERBOLIC EQUATION WITH AN ADDITIONAL INTEGRAL CONDITION

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The paper studies an inverse boundary value problem with an unknown right-hand side for a third-order pseudohyperbolic equation with an additional integral condition. First, the definition of the classical solution for the considered problem is introduced. The main goal of the problem is to simultaneously determine the solution and the unknown coefficient. To investigate the solvability of the original problem, we first consider an auxiliary inverse boundary value problem and prove its equivalence to the original problem in a certain sense. Then using the Fourier method the existence and uniqueness of a solution to an auxiliary problem is proved. Furthermore, based on the equivalency of these problems, the existence and uniqueness theorem for the classical solution of the original inverse problem is proved.

Keywords: Inverse problem, pseudohyperbolic equation of third order, integral overdetermination condition, classical solution, existence, uniqueness.

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1. Introduction and problem statement

It is known that the practical requirements often lead to the problem of determining the coefficients or the right hand side of the partial differential equations for some known data about its solutions. Such problems are called inverse boundary value problems in mathematical physics. Inverse problems arise in various fields of human activity such as seismology, mineral exploration, biology, medical visualization, computed tomography, Earth remote sensing, spectral analysis, nondestructive control, etc.

Fundamentals of the theory and practice of research of inverse problems were established and developed in the pioneering works by A.N.Tikhonov [29], M.M.Lav-rent'ev et al. [16], V.K.Ivanov et al. [11], V.G.Romanov [26], A.M.Denisov [8], M.I.Ivanchov [10], S.I. Kabanikhin [12], A.I.Kozhanov [13], and the references therein.

From the point of view of physical applications, third-order partial differential equations are of great interest. These equations are considered when solving problems of the theory of nonlinear acoustics and in the hydrodynamic theory of cosmic plasma, modeling fluid filtration in porous media. Studies of wave propagation in cold plasma and magnetohydrodynamics also reduce to the partial differential equations of third order (see [7, 23, 27]). To the study of nonlocal boundary value problems (including integral conditions) for partial differential equations of the third order are devoted large number of works (see, for example, [3, 4, 6, 9, 17, 25, 28], and the references therein).

Pseudohyperbolic equations arise in the theory of non-stationary flow of a viscous gas during the propagation of initial densifications in a viscous gas [30, 31], in the theory of

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solitons [18] when describing the process of electron motion in the system "superconductor - dielectric with tunneling conductivity – superconductor".

It should be noted that the integral conditions are completely natural and they arise in mathematical modeling in cases where it is impossible to obtain information about the process occurring at the boundary of the region of its flow using direct measurements or when it is possible to measure only some averaged (integral) characteristics of the desired quantity.

The solvability of inverse problems in different formulations, with certain overdetermination conditions for pseudohyperbolic equations, was the subject of researches in many papers (see, for example, [1, 2, 5, 14, 15, 19, 20, 21, 22, 24], and the bibliography therein).

The purpose of this paper is to prove the existence and uniqueness of the classical solution of an inverse boundary value problem for a third-order pseudohyperbolic equation (1), with boundary conditions (1)-(4) and integral overdetermination condition (12).

Let $D_T = Q_{xy} \times \{0 \leq t \leq T\}$ be a parallelepiped, where $Q_{xy} = \{(x,y) : 0 < x < 1, 0 < y < 1\}$ and let $F(x,y,t), g(x,y,t), \varphi(x,y)$, and $\psi(x,y)$, be given functions of $x, y \in [0,1]$ and $t \in [0,T]$. Consider the following boundary value problem of identifying an unknown function $u(x,y,t)$, that satisfies the equation

$$u_{tt}(x,y,t) - \alpha\Delta u_t(x,y,t) - \beta\Delta u(x,y,t) = F(x,y,t) \ (x,y,t) \in D_T, \tag{1}$$

with the initial conditions

$$u(x,y,0) = \varphi(x,y), \ u_t(x,y,0) = \psi(x,y), \ 0 \leq x, y \leq 1, \tag{2}$$

the boundary conditions

$$u_x(0,y,t) = u(1,y,t) = 0, \ 0 \leq y \leq 1, \ 0 \leq t \leq T, \tag{3}$$

$$u(x,0,t) = u_y(x,1,t) = 0, \ 0 \leq x \leq 1, \ 0 \leq t \leq T, \tag{4}$$

where α and β are given positive numbers and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

We introduce the following set of functions

$$\tilde{C}^{2,2,2}(D_T) = \{u(x,t) : u(x,t) \in C^{2,2,2}(D_T), u_{txx}(x,t), u_{tyy}(x,t) \in C(D_T)\}.$$

Definition 1.1. *The function $u(x,y,t) \in \tilde{C}^{2,2,2}(\bar{D}_T)$ is said to be a classical solution of the problem (1)-(4), if this function satisfy Equation (1) in D_T, the condition (2) on $[0,1]$, and the statements (3)-(4) on the interval $[0,T]$.*

Theorem 1.1. *If problem (1) - (4) has a solution, then it is unique in the class $\tilde{C}^{2,2,2}(\bar{D}_T)$.*

Proof. Assume that there are two solutions to the considered problem as $u_1(x,y,t)$ and $u_2(x,y,t)$. Let us denote the difference of these solutions by $v(x,y,t) = u_1(x,y,t) - u_2(x,y,t)$.

It is clear that the function $v(x,y,t)$ satisfies the following homogeneous equation

$$v_{tt}(x,y,t) - \alpha\Delta v_t(x,y,t) - \beta\Delta v(x,y,t) = 0 \ (x,y,t) \in D_T, \tag{5}$$

and the conditions

$$v(x,y,0) = 0, \ v_t(x,y,0) = 0, \ 0 \leq x, y \leq 1, \tag{6}$$

$$v_x(0,y,t) = v(1,y,t) = 0, \ 0 \leq y \leq 1, \ 0 \leq t \leq T, \tag{7}$$

$$v(x,0,t) = v_y(x,1,t) = 0, \ 0 \leq x \leq 1, \ 0 \leq t \leq T. \tag{8}$$

Let us prove that the function $v(x,y,t)$ is identically equal to zero.

Multiplying both sides of equation (5) by the function $2v_t(x,y,t)$ and integrating the resulting equality over Q_{xy}, we obtain

$$2\int_0^1\int_0^1 (v_{tt}(x,y,t) - \alpha\Delta v_t(x,y,t) - \beta\Delta v(x,y,t))v_t(x,y,t)dxdy = 0, \ 0 \leq t \leq T \tag{9}$$

Using the boundary conditions (7), (8), we have:

$$2\int_0^1\int_0^1 v_{tt}(x,y,t)v_t(x,y,t)dxdy = \frac{d}{dt}\int_0^1\int_0^1 v_t^2(x,y,t)dxdy,\ 0\le t\le T;$$

$$2\int_0^1\int_0^1 (v_{xx}(x,y,t)+v_{yy}(x,y,t))v_t(x,y,t)dxdy =$$

$$= -\frac{d}{dt}\int_0^1\int_0^1 (v_x^2(x,y,t)+v_y^2(x,y,t))dxdy,\ 0\le t\le T;$$

$$2\int_0^1\int_0^1 (v_{txx}(x,y,t)+v_{tyy}(x,y,t))v_t(x,y,t)dxdy =$$

$$= -2\int_0^1\int_0^1 (v_{tx}^2(x,y,t)+v_{ty}^2(x,y,t))dxdy,\ 0\le t\le T.$$

Then, from (9), we get

$$\frac{d}{dt}\int_0^1\int_0^1 (v_t^2(x,y,t)+v_x^2(x,y,t)+v_y^2(x,y,t))dxdy =$$

$$= -2\int_0^1\int_0^1 (v_{tx}^2(x,y,t)+v_{ty}^2(x,y,t))dxdy,\ 0\le t\le T. \tag{10}$$

We denote

$$y(t) = \int_0^1\int_0^1 (v_t^2(x,y,t)+v_x^2(x,y,t)+v_y^2(x,y,t))dxdy \ge 0.$$

From (6) it is obvious that $y(0)=0$, and in turn from (10) it is easy to see that $y'(t)\le 0$.From here we get:

$$y(t) = \int_0^1\int_0^1 (v_t^2(x,y,t)+v_x^2(x,y,t)+v_y^2(x,y,t))dxdy = 0.$$

Whence, it follows that

$$v_t(x,y,t)\equiv 0, v_x(x,y,t)\equiv 0, v_y(x,y,t)\equiv 0.$$

Thus, we get

$$v(x,y,t) = const = C_0.$$

Consequently, using conditions (6), we have:

$$v(x,y,0) = C_0 = 0.$$

This proves that $v(x,y,t)=0$. Thus, if there are two solutions $u_1(x,y,t)$ and $u_2(x,y,t)$ of problem (1)-(4) then $u_1(x,y,t)\equiv u_2(x,y,t)$. It follows that if a solution to problem (1)-(3) exists, then it is unique. □

Based on the direct problem (1)-(4) we consider the following inverse problem. Let

$$F(x,y,t) = a(t)g(x,y,t) + f(x,y,t), \tag{11}$$

where the functions $g(x,y,t)$ and $f(x,y,t)$ are given functions and the function $a(t)$ is unknown.

It is required to determine $a(t)$, if the following additional information about the solution of problem (1)-(4) is given:

$$\int_0^1\int_0^1 \omega(x,y)u(x,y,t)dxdy = h(t),\ 0 \le t \le T, \tag{12}$$

where the functions $\omega(x,y)$ and $h(t)$ are unknown functions.

Definition 1.2. *A pair $\{u(x,y,t), a(t)\}$ is called a classical solution to problem (1)-(4), (12) if the functions $u(x,y,t) \in \tilde{C}^{2,2,2}(\bar{D}_T)$ and $a(t) \in C[0,T]$ satisfy Equation (1) in D_T, the condition (2) in $\bar{Q}_{xy}$, the conditions (3) and (4) in $[0,1]\times[0,T]$, and the condition (12) on the interval $[0,T]$.*

Theorem 1.2. *Assume that $\varphi(x,y), \psi(x,y) \in C(\bar{Q}_{xy})$, $f(x,y,t), g(x,y,t) \in C(\bar{D}_T)$, $\omega(x,y) \in C(\bar{Q}_{xy})$, $\int_0^1\int_0^1 \omega(x,y)g(x,y,t)dxdy \neq 0\ (0 \le t \le T)$, $h(t) \in C^2[0,T]$, and the compatibility conditions*

$$\int_0^1\int_0^1 \omega(x,y)\varphi(x,y)dxdy = h(0),\quad \int_0^1\int_0^1 \omega(x,y)\psi(x,y)dxdy = h'(0), \tag{13}$$

holds. Then the problem of finding a classical solution of (1)-(4), (12) is equivalent to the problem of determining the functions $u(x,y,t) \in \tilde{C}^{2,2,2}(\bar{D}_T)$ and $a(t) \in C[0,T]$, satisfying the conditions (1)-(4) and the relation

$$h''(t) - \alpha\int_0^1\int_0^1 \omega(x,y)\Delta u_t(x,y,t)dxdt - \beta\int_0^1\int_0^1 \omega(x,y)\Delta u(x,y,t)dxdt =$$

$$= a(t)\int_0^1\int_0^1 \omega(x,t)g(x,y,t)dxdy + \int_0^1\int_0^1 \omega(x,t)f(x,y,t)dxdy,\ 0 \le t \le T. \tag{14}$$

Proof. Let $\{u(x,y,t), a(t)\}$ be a classical solution of (1)-(4), (12). Further, assuming $h(t) \in C^2[0,T]$ and differentiating (12) twice, we have

$$\int_0^1\int_0^1 \omega(x,y)u_{tt}(x,y,t)dxdy = h''(t),\ 0 \le t \le T. \tag{15}$$

Further, multiplying both sides of the Equation (1) by the functions $\omega(x,y)$ and integrating with respect to x and y over the interval $[0,1]$, we have

$$\frac{d^2}{dt^2}\int_0^1\int_0^1 \omega(x,y)u(x,y,t)dxdy - \alpha\int_0^1\int_0^1 \omega(x,y)\Delta u_t(x,y,t)dxdy -$$

$$-\beta\int_0^1\int_0^1 \omega(x,y)\Delta u(x,y,t)dxdy =$$

$$= a(t)\int_0^1\int_0^1 \omega(x,y)g(x,y,t)dxdy + \int_0^1\int_0^1 \omega(x,y)f(x,y,t)dxdy,\ 0 \le t \le T. \tag{16}$$

From (16), taking into account (15), it follows the fulfillment of (14).

Now, suppose that $\{u(x,y,t), a(t)\}$ is a solution to problem (1)-(4), (14). Then from (14) and (16) we find

$$\frac{d^2}{dt^2}\left(\int_0^1\int_0^1 \omega(x,y)u(x,y,t)dxdy - h(t)\right) = 0,\ 0 \le t \le T. \tag{17}$$

By virtue of (2) and compatibility conditions (13), we have

$$\begin{gathered}\int_0^1\int_0^1 \omega(x,y)u(x,y,0)dxdy - h(0) = \int_0^1\int_0^1 \omega(x,y)\varphi(x,y)dxdy - h(0) = 0,\\ \int_0^1\int_0^1 \omega(x,y)u_t(x,y,0)dxdy - h'(0) = \int_0^1\int_0^1 \omega(x,y)\psi(x,y)dxdy - h'(0 = 0.\end{gathered} \tag{18}$$

From (17), (18) we conclude that

$$\int_0^1\int_0^1 \omega(x,y)u(x,y,t)dxdy - h(t) = 0,\ 0 \le t \le T,$$

i.e. the condition (12) is satisfied. □

2. **Existence and uniqueness of the classical solution of the inverse boundary value problem**

We shall seek the first component $u(x,y,t)$ of classical solution $\{u(x,y,t), a(t)\}$ of the problem (1)-(4), (14) in the form

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} u_{k,n}(t)\cos\lambda_k x \sin\gamma_n y, \tag{19}$$

where

$$\lambda_k = \frac{\pi}{2}(2k-1),\ k = 1,2,...,\gamma_n = \frac{\pi}{2}(2n-1),\ n = 1,2,...,$$

$$u_{k,n}(t) = 4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x \sin\gamma_n y dxdy,\ k,n = 1,2,....$$

Then applying the formal scheme of the Fourier method, for determining of unknown coefficients $u_{k,n}(t)$ $(k = 1,2,...; n = 1,2,...)$, from (1) and (2) we have

$$u''_{k,n}(t) + \alpha\mu^2_{k,n}u'_{k.n}(t) + \beta\mu^2_{k,n}u_{k,n}(t) = F_{k,n}(t;a),\ k,n = 1,2,...,\ 0 \le t \le T, \tag{20}$$

$$u_{k,n}(0) = \varphi_{k,n}, u'_{k,n}(0) = \psi_{k,n},\ k,n = 1,2,..., \tag{21}$$

where

$$\mu^2_{k,n} = \lambda^2_k + \gamma^2_n,\ k,n = 1,2,...,$$

$$F_{k,n}(t;a) = f_{k,n}(t) + a(t)g_{k,n}(t),\ k,n = 1,2,...,$$

$$f_{k,n}(t) = 4\int_0^1\int_0^1 f(x,y,t)\cos\lambda_k x \sin\gamma_n y dxdy,\ k,n = 1,2,...,$$

$$g_{k,n}(t) = 4\int_0^1\int_0^1 g(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy,\ \ k,n=1,2,...,$$

$$\varphi_{k,n} = 4\int_0^1\int_0^1 \varphi(x,y)\cos\lambda_k x\sin\gamma_n y dxdy,\ \ k,n=1,2,...,$$

$$\psi_{k,n} = 4\int_0^1\int_0^1 \psi(x,y)\cos\lambda_k x\sin\gamma_n y dxdy,\ \ k,n=1,2,....$$

Now suppose that

$$\frac{\alpha^2\pi^2}{8} - \beta > 0.$$

Solving problem (20), (21), we find

$$u_{k,n}(t) = \frac{1}{\gamma_{k,n}}[(\mu_{2,k,n}e^{\mu_{1,k,n}t} - \mu_{1,k,n}e^{\mu_{2,k,n}t})\varphi_{k,n} + (e^{\mu_{2,k,n}t} - e^{\mu_{1,k,n}t})\psi_{k,n} +$$

$$\left. + \int_0^t F_{k,n}(\tau;a)(e^{\mu_{2,k,n}(t-\tau)} - e^{\mu_{1,k,n}(t-\tau)})d\tau\right], \tag{22}$$

where

$$\mu_{i,k,n} = -\frac{\alpha\mu_{k,n}^2}{2} + (-1)^i\mu_{k,n}\sqrt{\frac{\alpha^2\mu_{k,n}^2}{4} - \beta}\ \ (i=1,2),$$

$$\gamma_{k,n} = \mu_{2,k,n} - \mu_{1,k,n} = 2\mu_{k,n}\sqrt{\frac{\alpha^2\mu_{k,n}^2}{4} - \beta}.$$

After substituting expressions described by (22) into (19), we obtain

$$u(x,y,t) = \sum_{k=1}^{\infty}\left\{\frac{1}{\gamma_{k,n}}[(\mu_{2,k,n}e^{\mu_{1,k,n}t} - \mu_{1,k,n}e^{\mu_{2,k,n}t})\varphi_{k,n} +\right.$$

$$+(e^{\mu_{2,k,n}t} - e^{\mu_{1,k,n}t})\psi_{k,n} +$$

$$\left.\left. + \int_0^t F_{k,n}(\tau;a)(e^{\mu_{2,k,n}(t-\tau)} - e^{\mu_{2,k,n}(t-\tau)})d\tau\right]\right\}\cos\lambda_k x\sin\gamma_n y. \tag{23}$$

From (14), taking into account (19), we have

$$a(t)v(t) = h_2''(t) - \int_0^1\int_0^1 \omega(x,y)f(x,y,t)dxdy +$$

$$+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} p_{k,n}\mu_{k,n}^2(\alpha u_{k,n}'(t) + \beta u_{k,n}(t)),\ \ 0\le t\le T, \tag{24}$$

where

$$v(t) \equiv \int_0^1\int_0^1 \omega(x,y)g(x,y,t)dxdy, p_{k,n} = \int_0^1\int_0^1 \omega(x,y)\cos\lambda_k x\sin\gamma_n y dxdy. \tag{25}$$

Differentiating (22) twice, yields

$$u_{k,n}'(t) = \frac{1}{\gamma_{k,n}}[\mu_{1,k,n}\mu_{2,k,n}(e^{\mu_{1,k,n}t} - e^{\mu_{2k,n}t})\varphi_{k,n} +$$

$$+(\mu_{2,k,n}e^{\mu_{2,k,n}t}-\mu_{1,k,n}e^{\mu_{1,k,n}t})\psi_{k,n}+$$

$$+\int_0^t F_{k,n}(\tau;a)(\mu_{2,k,n}e^{\mu_{2,k,n}(t-\tau)}-\mu_{1,k,n}e^{\mu_{1,k,n}(t-\tau)})d\tau\Bigg], \tag{26}$$

$$u''_{k,n}(t)=\frac{1}{\gamma_{k,n}}[\mu_{1,k,n}\mu_{2,k,n}(\mu_{1,k,n}e^{\mu_{1,k,n}t}-\mu_{2,k,n}e^{\mu_{2,k,n}t})\varphi_{k,n}+$$

$$+(\mu^2_{2,k,n}e^{\mu_{2,k,n}t}-\mu^2_{1,k,n}e^{\mu_{1,k,n}t})\psi_{k,n}+$$

$$+\int_0^t F_{k,n}(\tau;a)(\mu^2_{2,k,n}e^{\mu_{2,k,n}(t-\tau)}-\mu^2_{1,k,n}e^{\mu_{1,k,n}(t-\tau)})d\tau\Bigg]+F_{k,n}(t;a). \tag{27}$$

By virtue of (20) and (27) we have:

$$\alpha\mu^2_{k,n}u'_{k,n}(t)+\beta\mu^2_{k,n}u_{k,n}(t)=F_{k,n}(t;u,a)-u''_{k,n}(t)=$$

$$=-\frac{1}{\gamma_{k,n}}[\mu_{1,k,n}\mu_{2,k,n}(\mu_{1,k,n}e^{\mu_{1,k,n}t}-\mu_{2,k,n}e^{\mu_{2,k,n}t})\varphi_{k,n}+$$

$$+(\mu^2_{2,k,n}e^{\mu_{2,k,n}t}-\mu^2_{1,k,n}e^{\mu_{1,k,n}t})\psi_{k,n}+$$

$$+\int_0^t F_{k,n}(\tau;a)(\mu^2_{2,k,n}e^{\mu_{2,k,n}(t-\tau)}-\mu^2_{1,k,n}e^{\mu_{1,k,n}(t-\tau)})d\tau\Bigg]. \tag{28}$$

In order to obtain an expression for the second component $a(t)$ of the solution $\{u(x,y,t),a(t)\}$ of problem (1)-(4), (14), we substitute expression (28) into (24):

$$a(t)=[v(t)]^{-1}\Bigg\{h''(t)-\int_0^1\int_0^1\omega(x,y)f(x,y,t)dxdy+$$

$$+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_{k,n}}{\gamma_{k,n}}[\mu_{1,k,n}\mu_{2,k,n}(\mu_{1,k,n}e^{\mu_{1,k,n}t}-\mu_{2,k,n}e^{\mu_{2,k,n}t})\varphi_{k,n}+$$

$$+(\mu^2_{2,k,n}e^{\mu_{2,k,n}t}-\mu^2_{1,k,n}e^{\mu_{1,k,n}t})\psi_{k,n}+$$

$$+\int_0^t F_{k,n}(\tau;a)(\mu^2_{2,k,n}e^{\mu_{2,k,n}(t-\tau)}-\mu^2_{1,k,n}e^{\mu_{1,k,n}(t-\tau)})d\tau\Bigg]\Bigg\}. \tag{29}$$

Thus, the solution of problem (1) - (4), (14) was reduced to the solution of system (23), (29) with respect to unknown functions $u(x,y,t)$ and $a(t)$.

It is easy to see that $a(t)$ is a solution to equation (29), then a pair $\{u(x,y,t),a(t)\}$ of functions $u(x,y,t)$ and $a(t)$ will be a solution to problem (1)-(4), (14). Therefore, the problem posed is reduced to determining $a(t)$ from equation (29).

The following theorem is valid

Theorem 2.1. *Suppose that the data of problem (1)-(4), (14) satisfy the following conditions:*

A) $\alpha>0,\ \beta>0,\ \frac{\alpha^2}{8}-\beta>0$;

B) $\varphi(x,y),\varphi_x(x,y),\varphi_{xx}(x,y),\varphi_y(x,y),\varphi_{xy}(x,y),\varphi_{yy}(x,y)\in C(\bar{Q}_{xy})$,
$\varphi_{xxy}(x,y),\varphi_{xyy}(x,y),\varphi_{xxx}(x,y),\varphi_{yyy}(x,y)\in L_2(Q_{xy})$,
$\varphi_x(0,y)=\varphi(1,y)=\varphi_{xx}(1,y)=0,\ 0\le y\le 1$,
$\varphi(x,0)=\varphi_y(x,1)=\varphi_{yy}(x,0)=0,\ 0\le x\le 1$;

C) $\psi(x,y), \psi_x(x,y), \psi_y(x,y), \psi_{xx}(x,y), \psi_{xy}(x,y), \psi_{yy}(x,y) \in C(\bar{Q}_{xy})$,
$\psi_{xxy}(x,y), \psi_{xyy}(x,y), \psi_{xxx}(x,y), \psi_{yyy}(x,y) \in L_2(Q_{xy})$,
$\psi_x(0,y) = \psi(1,y) = \psi_{xx}(1,y) = 0, 0 \le y \le 1$,
$\psi(x,0) = \psi_y(x,1) = \psi_{yy}(x,1) = 0, 0 \le x \le 1$;

D) $f(x,y,t), f_x(x,y,t), f_y(x,y,t), f_{xx}(x,y,t), f_{xy}(x,y,t), f_{yy}(x,y,t) \in C(D_T)$,
$f_{xxx}(x,y,t), f_{xxy}(x,y,t), f_{xyy}(x,y,t), f_{yyy}(x,y,t) \in L_2(D_T)$,
$f_x(0,y,t) = f(1,y,t) = f_{xx}(0,y,t) = 0,\ 0 \le y \le 1,\ 0 \le t \le T$,
$f(x,0,t) = f_y(x,1,t) = f_{yy}(x,1,t) = 0,\ 0 \le x \le 1,\ 0 \le t \le T$;

E) $g(x,y,t), g_x(x,y,t), g_y(x,y,t), g_{xx}(x,y,t), g_{xy}(x,y,t), g_{yy}(x,y,t) \in C(D_T)$,
$g_{xxx}(x,y,t), g_{xxy}(x,y,t), g_{xyy}(x,y,t), g_{yyy}(x,y,t) \in L_2(D_T)$,
$g_x(0,y,t) = g(1,y,t) = g_{xx}(0,y,t) = 0,\ 0 \le y \le 1,\ 0 \le t \le T$,
$g(x,0,t) = g_y(x,1,t) = g_{yy}(x,1,t) = 0,\ 0 \le x \le 1,\ 0 \le t \le T$;

F) $h(t) \in C^2[0,T],\ v(t) \equiv \int_0^1 \int_0^1 \omega(x,t) g(x,y,t) dxdy \neq 0,\ 0 \le t \le T$

Then problem (1)-(4), (14) has a unique solution.

Proof. Equation (29) can be written as:

$$a(t) = b(t) + \int_0^t G(t,\tau) a(\tau) d\tau, \tag{30}$$

where

$$b(t) = [v(t)]^{-1} \left\{ h''(t) - \int_0^1 \int_0^1 \omega(x,y) f(x,y,t) dxdy + \right.$$

$$+ \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_{k,n}}{\gamma_{k,n}} [\mu_{1,k,n}\mu_{2,k,n}(\mu_{1,k,n} e^{\mu_{1,k,n}t} - \mu_{2,k,n} e^{\mu_{2,k,n}t})\varphi_{k,n} +$$

$$+(\mu_{2,k,n}^2 e^{\mu_{2,k,n}t} - \mu_{1,k,n}^2 e^{\mu_{1,k,n}t})\psi_{k,n} +$$

$$\left. + \int_0^t f_{k,n}(\tau)(\mu_{2,k,n}^2 e^{\mu_{2,k,n}(t-\tau)} - \mu_{1,k,n}^2 e^{\mu_{1,k,n}(t-\tau)}) d\tau \Bigg] \right\},$$

$$G(t,\tau) = -[v(t)]^{-1} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_{k,n} g_{k,n}(\tau)}{\gamma_{k,n}} (\mu_{2,k,n}^2 e^{\mu_{2,k,n}(t-\tau)} - \mu_{1,k,n}^2 e^{\mu_{1,k,n}(t-\tau)}).$$

It is easy to see that

$$\mu_{k,n}^3 \le (\lambda_k^2 + \gamma_k^2)(\lambda_k + \gamma_n) = \lambda_k^3 + \lambda_k^2 \gamma_n + \gamma_n^2 \lambda_k + \gamma_n^3,$$

$$|\gamma_{k,n}| > \frac{\alpha}{\sqrt{2}} \mu_{k,n}^2, \quad |\mu_{i,k,n}| \le \alpha \mu_{k,n}^2 \ (i = 1,2),$$

$$|\mu_{1,k,n}\mu_{2,k,n}| = \beta \mu_{k,n}^2 (i = 1,2), |p_{k,n}| \le \|\omega(x,y)\|_{C(\bar{Q}_{xy})}.$$

Taking into account these relations, taking into account the conditions of *Theor*em 1.2, we can show that

$$\|b(t)\|_{C[0,T]} \le \left\|[v(t)]^{-1}\right\|_{C[0,T]} \left\{ \left\| h''(t) - \int_0^1 \int_0^1 \omega(x,y) f(x,y,t) dxdy \right\|_{C[0,T]} + \right.$$

$$+2\sqrt{2} \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu_{k,n}^{-2} \right)^{\frac{1}{2}} \|\omega(x,y)\|_{C[\bar{Q}_{xy}]} [\beta \|\varphi_{xxx}(x,y)\|_{L_2(Q_{xy})} +$$

$$+\beta \|\varphi_{xxy}(x,y)\|_{L_2(Q_{xy})} + \beta \|\varphi_{xyy}(x,y)\|_{L_2(Q_{xy})} + \beta \|\varphi_{yyy}(x,y)\|_{L_2(Q_{xy})} +$$

$$+\alpha\left\|\psi_x(x,y)\right\|_{L_2(Q_{xy})}+\alpha\left\|\varphi_y(x,y)\right\|_{L_2(Q_{xy})}+$$
$$+\alpha\sqrt{T}(\left\|f_{xxx}(x,y,t)\right\|_{L_2(D_T)}+\left\|f_{xxy}(x,y,t)\right\|_{L_2(D_T)}+$$
$$+\left\|f_{xyy}(x,y,t)\right\|_{L_2(D_T)}+\left\|f_{yyy}(x,y,t)\right\|_{L_2(D_T)})]\}, \tag{31}$$

$$|G(t,\tau)|\le 2\sqrt{2T}\alpha\left\|[v(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu_{k,n}^{-2}\right)^{\frac{1}{2}}\left\|\omega(x,y)\right\|_{C[\bar{Q}_{xy}]}\times$$
$$\times(\left\|g_{xxx}(x,y,t)\right\|_{L_2(D_T)}+\left\|g_{xxy}(x,y,t)\right\|_{L_2(D_T)}+$$
$$+\left\|g_{xyy}(x,y,t)\right\|_{L_2(D_T)}+\left\|g_{yyy}(x,y,t)\right\|_{L_2(D_T)}). \tag{32}$$

Due to estimates (31) and (32), the Voltaire-type linear integral equation (30) has a unique solution from $C[0,T]$.

Let us show that $u(x,y,t)\in\tilde{C}^{2,2,2}(\bar{D}_T)$. In fact, from (22), (26) implies that $u_{k,n}(t), u'_{k,n}(t)\in C[0,T]$ and

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^3\left\|u_{k,n}(t)\right\|_{C[0,T]})^2\right\}^{\frac{1}{2}}\le 4\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^3\left|\varphi_{k,n}\right|)^2\right)^{\frac{1}{2}}+$$
$$+4\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\gamma_n\left|\varphi_{k,n}\right|)^2\right)^{\frac{1}{2}}+4\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k\gamma_n^2\left|\varphi_{k,n}\right|)^2\right)^{\frac{1}{2}}+$$
$$+4\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^3\left|\varphi_{k,n}\right|)^2\right)^{\frac{1}{2}}+\frac{4}{\alpha}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k\left|\psi_{k,n}\right|)^2\right)^{\frac{1}{2}}+\frac{4}{\alpha}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n\left|\psi_{k,n}\right|)^2\right)^{\frac{1}{2}}+$$
$$+\frac{4\sqrt{T}}{\alpha}\left(\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k\left|f_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n\left|f_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}\right)+$$
$$+\frac{4\sqrt{T}}{\alpha}\left\|a(t)\right\|_{C[0,T]}\left(\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k\left|g_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+\right.$$
$$\left.+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n\left|g_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}\right),$$

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^3\left\|u'_{k,n}(t)\right\|_{C[0,T]})^2\right\}^{\frac{1}{2}}\le\frac{4\sqrt{2}\beta}{\alpha}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^3\left|\varphi_{k,n}\right|)^2\right)^{\frac{1}{2}}+$$
$$+\frac{4\sqrt{2}\beta}{\alpha}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\gamma_n\left|\varphi_{k,n}\right|)^2\right)^{\frac{1}{2}}+\frac{4\sqrt{2}\beta}{\alpha}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k\gamma_n^2\left|\varphi_{k,n}\right|)^2\right)^{\frac{1}{2}}+$$
$$+\frac{4\sqrt{2}\beta}{\alpha}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^3\left|\varphi_{k,n}\right|)^2\right)^{\frac{1}{2}}+$$
$$+4\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^3\left|\psi_{k,n}\right|)^2\right)^{\frac{1}{2}}+4\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\gamma_n\left|\psi_{k,n}\right|)^2\right)^{\frac{1}{2}}+$$
$$+4\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k\gamma_n^2\left|\psi_{k,n}\right|)^2\right)^{\frac{1}{2}}+4\sqrt{2}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^3\left|\psi_{k,n}\right|)^2\right)^{\frac{1}{2}}+$$

$$+4\sqrt{2T}\left(\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^3\left|f_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\gamma_n\left|f_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+\right.$$

$$\left.+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k\gamma_n^2\left|f_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^3\left|f_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}\right)+$$

$$+4\sqrt{2T}\left\|a(t)\right\|_{C[0,T]}\left(\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^3\left|g_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+\right.$$

$$+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\gamma_n\left|g_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+$$

$$\left.+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k\gamma_n^2\left|g_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^3\left|g_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}\right)$$

or

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^3\left\|u_{k,n}(t)\right\|_{C[0,T]})^2\right\}^{\frac{1}{2}}=$$

$$=4\left\|\varphi_{xxx}(x,y)\right\|_{L_2(Q_{xy})}+4\left\|\varphi_{xxy}(x,y)\right\|_{L_2(Q_{xy})}+4\left\|\varphi_{xyy}(x,y)\right\|_{L_2(Q_{xy})}+$$

$$+4\left\|\varphi_{yyy}(x,y)\right\|_{L_2(Q_{xy})}+\frac{4}{\alpha}\left\|\psi_x(x,y)\right\|_{L_2(Q_{xy})}+$$

$$+\frac{4}{\alpha}\left\|\psi_y(x,y)\right\|_{L_2(Q_{xy})}+\frac{4\sqrt{T}}{\alpha}(\left\|f_x(x,y,t)\right\|_{L_2(D_T)}+\left\|f_y(x,y,t)\right\|_{L_2(D_T)})+$$

$$+\frac{4\sqrt{T}}{\alpha}\left\|a(t)\right\|_{C[0,T]}(\left\|g_x(x,y,t)\right\|_{L_2(D_T)}+\left\|g_y(x,y,t)\right\|_{L_2(D_T)}),$$

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^3\left\|u'_{k,n}(t)\right\|_{C[0,T]})^2\right\}^{\frac{1}{2}}\leq\frac{4\sqrt{2}\beta}{\alpha}\left\|\varphi_{xxx}(x,y)\right\|_{L_2(Q_{xy})}+$$

$$+4\left\|\varphi_{xxx}(x,y)\right\|_{L_2(Q_{xy})}+4\left\|\varphi_{xxy}(x,y)\right\|_{L_2(Q_{xy})}+4\left\|\varphi_{xyy}(x,y)\right\|_{L_2(Q_{xy})}+$$

$$+\frac{4\sqrt{2}\beta}{\alpha}\left\|\varphi_{xxy}(x,y)\right\|_{L_2(Q_{xy})}+\frac{4\sqrt{2}\beta}{\alpha}\left\|\varphi_{xyy}(x,y)\right\|_{L_2(Q_{xy})}+$$

$$+\frac{4\sqrt{2}\beta}{\alpha}\left\|\varphi_{yyy}(x,y)\right\|_{L_2(Q_{xy})}+4\sqrt{2}\left\|\psi_{xxx}(x,y)\right\|_{L_2(Q_{xy})}+$$

$$+4\sqrt{2}\left\|\psi_{xxy}(x,y)\right\|_{L_2(Q_{xy})}+4\sqrt{2}\left\|\psi_{xyy}(x,y)\right\|_{L_2(Q_{xy})}+$$

$$+4\sqrt{2}\left\|\psi_{yyy}(x,y)\right\|_{L_2(Q_{xy})}+4\sqrt{2T}(\left\|f_{xxx}(x,y,t)\right\|_{L_2(D_T)}+\left\|f_{xxy}(x,y,t)\right\|_{L_2(D_T)}+$$

$$+\left\|f_{xyy}(x,y,t)\right\|_{L_2(D_T)}+\left\|f_{yyy}(x,y,t)\right\|_{L_2(D_T)})+$$

$$+4\sqrt{2T}\left\|a(t)\right\|_{C[0,T]}(\left\|g_{xxx}(x,y,t)\right\|_{L_2(D_T)}+\left\|g_{xxy}(x,y,t)\right\|_{L_2(D_T)}+$$

$$+\left\|g_{xyy}(x,y,t)\right\|_{L_2(D_T)}+\left\|g_{yyy}(x,y,t)\right\|_{L_2(D_T)}).$$

From the last relation it is clear that the function $u(x,y,t)$ is continuous and has continuous derivatives $u_x(x,y,t)$, $u_{xx}(x,y,t)$, $u_y(x,y,t)$, $u_{xy}(x,y,t)$, $u_{yy}(x,y,t)$, $u_{xxx}(x,y,t)$, $u_{yyy}(x,y,t)$, $u_t(x,y,t)$, $u_{tx}(x,y,t)$, $u_{ty}(x,y,t)$, $u_{txx}(x,y,t)$, and $u_{tyy}(x,y,t)$ in D_T.

Now, from (20) it is easy to see that

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}\left\|u''_{k,n}(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}} \leq 2\left[\alpha\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3\left\|u'_{k,n}(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}+\right.$$

$$+\beta\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3\left\|u_{k,n}(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}+\left\|\|f_x(x,y,t)+f_y(x,y,t)\|_{C[0,T]}\right\|_{L_2(Q_{xy})}+$$

$$\left.+\left\|\|a(t)(g_x(x,y,t)+g_y(x,y,t))\|_{C[0,T]}\right\|_{L_2(Q_{xy})}\right].$$

From this it is clear that $u_{tt}(x,y,t)$ is continuous in D_T. It is easy to check that equation (1) and conditions (2)-(4), (14) are satisfied in the usual sense. Thus, the solution to problem (1)-(4), (14) is a pair of $\{u(x,t),a(t)\}$. By virtue of Theorem 1.1, it is unique. □

Using Theorem 1.2 and Theorem 2.1, we obtain the unique solvability of problem (1)-(4), (12).

Theorem 2.2. *Let all the conditions of Theorem 1.2 and the compatibility conditions (13) be satisfied. Then problem (1)-(4), (12) has a unique classical solution.*

3. Conclusions

The aim of the work was to study the unique solvability of an inverse boundary value problem with an unknown right-hand side for a third-order pseudohyperbolic equation with an additional integral condition. For this purpose, first an auxiliary inverse boundary value problem was considered and proved its equivalence to the original problem in a certain sense. Then using the Fourier method the existence and uniqueness of a solution to an auxiliary problem is proved. Furthermore, based on the equivalency of these problems, the existence and uniqueness theorem for the classical solution of the original inverse problem is proved.

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