U.P.B. Sci. Bull., Series A, Vol. 82, Iss. 4, 2020 ISSN 1223-7027

EXISTENCE AND STABILITY OF EQUILIBRIA OF SOME ODE SYSTEMS

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The existence and stability of equilibria of some ODE system connected with some models in biology are considered.

Keywords: System of ODE, equilibrium, stability.

Introduction

In the following the existence and stability of equilibria of two differential systems are considered. This kind of systems is used to model some enzymatic reactions in biology [1,3]. A similar analysis was given for a somewhat different system in [2]. The notations used have their origin in the biological problems and we keep them as such. For the interpretation of the constants see [1,2].

I. Consider the system of ODE:

$$\begin{cases} c' = -(k^- + \sigma)c + k^+ m f(c) \\ m' = b - dm - k^+ m f(c) + k^- c \end{cases} \qquad (1)$$

in $D = (0, Z) \times (0, \infty)$; b, d, k^+,k^-,$\sigma > 0$, $f \in C^1[0, Z]$, $Z > 0$, $f(c) > 0, f(Z) = 0$, f strictly decreasing, $c = c(t), m = m(t)$, $t \geq 0$.
All these conditions are natural in the biological case.

Example. $f(c) = Z - c$.

The study of the existence of global solutions of (1) can be done along the same lines of a similar discussion in [2] and we omit it.

Proposition. The system (1) is cooperative.

Proof. Let us denote by F, G the right hands of the system. In fact we have:

$$F_m' = k^+ f(c) > 0 \text{ in } D, G_c' = -k^+ m f'(c) + k^- > 0.$$

It follows, for example, that the sets D_{++} and D_{--} are invariant [5] (see also the figure below).

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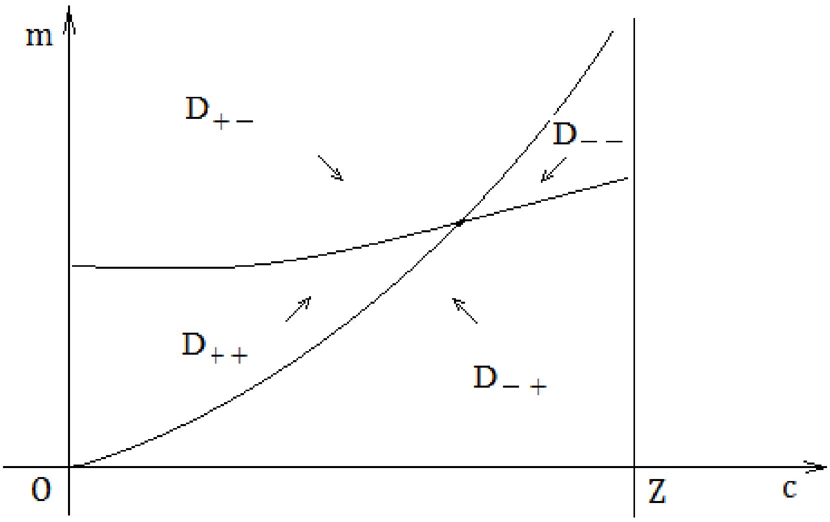

Fig.1 Phase portrait for the case $f(c) = Z - c$

Proposition. There are no periodic solutions of (1).

Proof. $F_c' = -(k^- + \sigma) + k^+ m f'(c) < 0, G_m' = -d - k^+ f(c) < 0,$ and the result follows from the Bendixon criterion.

Finding the equilibria.

In order to determine the equilibria of the system (1) we need to solve the (algebraic) system

$$\begin{cases} -(k^- + \sigma)c + k^+ m f(c) = 0 \\ b - dm - k^+ m f(c) + k^- c = 0 \end{cases} \tag{2}$$

By adding the equations we get $b = dm + \sigma c$ (a condition which is independent of f). So we have $m = \frac{b - \sigma c}{d}$ and from the first equation of (2) we get that $(k^- + \sigma)c = k^+ \frac{b - \sigma c}{d} f(c)$. Let us denote, for the moment, $a = \frac{(k^- + \sigma)d}{k^+}$ and define $g(c) = (b - \sigma c)f(c) - ac$.

Then, $g(0) = bf(0) > 0$ and $g(Z) = -aZ < 0$ so there is a point $c_0 \in (0, Z)$ such that $g(c_0) = 0$.

But as $g'(c) = -\sigma f(c) + (b - \sigma c)f'(c) - a < 0$, g is strictly decreasing and so c_0 is the unique solution, in $(0, Z)$, of the equation $g(c) = 0$.
Remark that if $\frac{b}{\sigma} < Z$ we get $c_0 < \frac{b}{\sigma}$. We proved this way the following:

Theorem. The system (1) has an unique equilibrium in $(c_0, m_0), m_0 = \frac{b-\sigma c_0}{d}$, in D. Moreover $c_0 < min\left\{\frac{b}{\sigma}, Z\right\}$.

Remark. If $f \leq g$ and if c_{f0}, c_{g0} the solutions corresponding to f, g then $c_{f0} \leq c_{g0}$.

It would be useful to study the dependence of c_0 on b; for example, one can consider b a control parameter. In order to do that let us suppose, making a choice, that $b < \sigma Z$ and apply the implicit function theorem to the equation $h(c, b) = (b - \sigma c)f(c) - ac = 0$ (the first equality being a notation) . We have that: $h_c' = -\sigma f + (b - \sigma c)f' - a < 0$, $h_b' = f > 0$ so the globally defined function (see the previous proof) $c_0(b)$ is of class $C^1$and increasing.
Moreover we have that $0 < c_0(b) < \frac{b}{\sigma}$. Extend this function by putting $c_0(0) = 0$ and observe that we get a continuous function.

Stability.
The Jacobian matrix of the system (1) is:

$$J(c, m) = \begin{pmatrix} -(k^- + \sigma) + k^+ m f'(c) & k^+ f(c) \\ -k^+ m f'(c) + k^- & -d - k^+ f(c) \end{pmatrix}$$

As easy computation shows that det$J(c, m) > 0$ and tr$J(c, m) < 0$.
It follows that the matrix $J(c, m)$ is stable for every (c,m).
(It is the "form" of the matrix which matters). So we obtain the following:

Theorem. The unique echilibrium of the system (1) is asymptotically stable.
In fact, one can prove that the equilibrium is globally asymptotically stable.

II. Consider the system:

$$\begin{cases} c_1' = -(k_1^- + \sigma_1)c_1 + k_1^+ m f(c_1) \\ c_2' = -(k_2^- + \sigma_2)c_2 + k_2^+ m g(c_2) \\ m' = b - dm - k_1^+ m f(c_1) - k_2^+ m g(c_2) + k_1^- c_1 + k_2^- c_2 \end{cases} \tag{3}$$

The conditions on the coefficients are similar to those of (1). The functions $f \in C^1[0, Z_1]$, g $\in C^1[0, Z_2]$ and have similar properties of those in (1).
We shall consider the problem of the existence and stability of equilibria of (3).

Finding the equilibrium.
For existence of equilibria we need to solve the system:

$$\begin{cases} -(k_1^- + \sigma_1)c_1 + k_1^+ m f(c_1) = 0 \\ -(k_2^- + \sigma_2)c_2 + k_2^+ m g(c_2) = 0 \\ b - dm - k_1^+ m f(c_1) - k_2^+ m g(c_2) + k_1^- c_1 + k_2^- c_2 = 0 \end{cases} \qquad (4)$$

By adding the equations we get that $b = dm + \sigma_1 c_1 + \sigma_2 c_2$ (independent of f, g).
It follows that $m = \frac{b - \sigma_1 c_1 - \sigma_2 c_2}{d}$ and from the first two equations we obtain the system:

$$\begin{cases} a_1 c_1 = (b - \sigma_1 c_1 - \sigma_2 c_2) f(c_1) \\ a_2 c_2 = (b - \sigma_1 c_1 - \sigma_2 c_2) g(c_2) \end{cases}$$

where $a_1 = \frac{(k_1^- + \sigma_1)d}{k_1^+}$, $a_2 = \frac{(k_2^- + \sigma_2)d}{k_2^+}$.

Let us get rid of some indices and put $c_1 = x$, $c_2 = y$. The system becomes:

$$\begin{cases} a_1 x = (b - \sigma_1 x - \sigma_2 y) f(x) \\ a_2 y = (b - \sigma_1 x - \sigma_2 y) g(y) \end{cases} \qquad (5)$$

Obviously one can suppose $b > \sigma_1 x + \sigma_2 y$.
The system (5) will be solved by substitution. By using the result for the system (2) we can solve the first equation of (5) with respect to x, for every $0 \le y \le \frac{b}{\sigma_2}$.
We obtain a function $x = x(y)$ satisfying the equation
$a_1 x(y) = (b - \sigma_1 x(y) - \sigma_2 y) f(x(y))$ for every $y \in \left[0, \frac{b}{\sigma_2}\right]$.
This function is continous, nonincreasing and $x(y) < \frac{b - \sigma_2 y}{\sigma_1}$ for $0 \le y < \frac{b}{\sigma_2}$, $x(\frac{b}{\sigma_2}) = 0$.
Now let us introduce $x(y)$ in the second equation of (5). We get the equation:

$$(*)\ a_2 y = (b - \sigma_1 x(y) - \sigma_2 y) g(y).$$

We now prove that this equation has an unique solution in $(0, \frac{b}{\sigma_2})$.

In order to do this consider the continuous function

$$\varphi(y) = (b - \sigma_1 x(y) - \sigma_2 y) g(y) - a_2 y.$$

Remark that $\varphi(0) > 0$ and $\varphi\left(\frac{b}{\sigma_2}\right) < 0$ so at least one solution exists. In order to prove uniqueness consider the derivative of φ on $(0, \frac{b}{\sigma_2})$; one obtains

$$\varphi'(y) = (-\sigma_1 x'(y) - \sigma_2) g(y) + (b - \sigma_1 x(y) - \sigma_2 y) g'(y) - a_2;$$

by using the implicit function theorem we have that

$$x'(y) = -\frac{\sigma_2 f(y)}{a_1 + \sigma_1 f(y) - (b - \sigma_1 x(y) - \sigma_2 y) f'(y)}$$

and we see that if $-\sigma_1 x'(y) - \sigma_2 \leq 0$ then $\varphi'(y) < 0$ and so the uniqueness of the solution will follow.

But

$$\frac{\sigma_1 \sigma_2 f(y)}{a_1 + \sigma_1 f(y) - (b - \sigma_1 x(y) - \sigma_2 y) f'(y)} - \sigma_2 =$$

$$= \frac{\sigma_2 (b - \sigma_1 x(y) + \sigma_2 y) f'(y) - a_1 \sigma_2}{a_1 + \sigma_1 f(y) - (b - \sigma_1 x(y) - \sigma_2 y) f'(y)} < 0$$

and so we have the following:

Theorem. The system (4) has an unique solution (c_{1_0}, c_{2_0}, m_0).

Stability.

The jacobian matrix of the system (3) is

$$J = \begin{pmatrix} -(k_1^- + \sigma_1) + k_1^+ m f' & 0 & k_1^+ f \\ 0 & -(k_2^- + \sigma_2) + k_2^+ m g' & k_2^+ g \\ k_1^- - k_1^+ m f' & k_2^- - k_2^+ m g' & -d - k_1^+ f - k_2^+ g \end{pmatrix}$$

Let us make the notations :

$A = (k_1^- + \sigma_1) - k_1^+ m f'$, $B = (k_2^- + \sigma_2) - k_2^+ m g'$, $C = d + k_1^+ f + k_2^+ g$,
$u = k_1^+ f, v = k_2^+ g, \beta = k_1^- - k_1^+ m f', \gamma = k_2^- - k_2^+ m g'$.

Clearly $A, B, C, u, v, \beta, \gamma > 0$ and $A > \beta, B > \gamma, C > u + v$.

The matrix becomes,

$J = \begin{pmatrix} -A & 0 & u \\ 0 & -B & v \\ \beta & \gamma & -C \end{pmatrix}$ which is similar to $\begin{pmatrix} -A & 0 & -u \\ 0 & -B & -v \\ -\beta & -\gamma & -C \end{pmatrix}$.

In order to prove the stability of J is enough to prove the positive stability of the matrix

$M = \begin{pmatrix} A & 0 & u \\ 0 & B & v \\ \beta & \gamma & C \end{pmatrix}$. This is a P-matrix (all principal minors are positive).

Indeed the only not obvious fact is that $\det M > 0$.

But $\det M = A(BC - v\gamma) - Bu\beta > A(B(u + v) - v\gamma) - Bu\beta =$

$$= ABu + ABv - Av\gamma - Bu\beta = Bu(A - \beta) + Av(B - \gamma) > 0 .$$

For a 3× 3 P-matrix (x_{ij}) the condition for positive stability is

$$x_{11}x_{22}x_{33} > \frac{x_{13}x_{32}x_{21} + x_{12}x_{23}x_{31}}{2} .$$

The matrix M obviously satisfies this condition.

Theorem. The unique equilibrium of the system (3) is asymptotically stable.

R E F E R E N C E S.

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