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ON GENERALIZED ROUGH (m,n)-BI-Γ-HYPERIDEALS IN Γ-SEMIHYPERGROUPS

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Recently, in [12], Yaqoob et al. introduced the notion of (m,n)-bi-Γ-hyperideals and applied the concept of rough set theory to (m,n)-bi-Γ-hyperideals, which is a generalization of (m,n)-bi-Γ-hyperideals of Γ-semihypergroups. In this paper, applying the rough set theory based on an arbitrary binary relation (not an equivalent relation) we extend and generalize these notions, introducing the notion of generalized rough $(m,0)$-Γ-hyperideals (generalized rough $(0,n)$-Γ-hyperideals), generalized rough (m,n)-quasi and bi-Γ-hyperideals and generalized rough m-left Γ-hyperideals) and establish some of their basic properties in Γ-semihypergroups.

Keywords: (m,n)-quasi(bi-)Γ-hyperideal, rough (m,n)-bi-Γ-hyperideals, generalized rough (m,n)-(quasi)bi-Γ-hyperideals.

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1. Introduction and preliminaries

The applications of mathematics in other disciplines, for example in informatics, play a key role and they represent, in the last decades, one of the purpose of the study of the experts of Hyperstructures Theory all over the world. Hyperstructure theory was introduced in 1934 by a French mathematician F. Marty [8], at the 8th Congress of Scandinavian Mathematicians, where he defined hypergroups based on the notion of hyperoperation, began to analyze their properties and applied them to groups. In the following decades and nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics by many mathematicians. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Several books have been written on hyperstructure theory, see [1, 2, 3, 9, 11]. A recent book on hyperstructures [2] points out on their applications in rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. Another book [3] is devoted especially to the study of hyperring theory. Several kinds of hyperrings are introduced and analyzed. The volume ends with an outline of applications in chemistry and physics, analyzing

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several special kinds of hyperstructures: e-hyperstructures and transposition hypergroups. Recently, Davvaz et al. [7, 5, 13, 15, 16, 17, 6] introduced the notion of Γ-semihypergroup as a generalization of a semigroup, a generalization of a semihypergroup and a generalization of a Γ-semigroup. They presented many interesting examples and obtained a several characterizations of Γ-semihypergroups.

The rough set theory, proposed by Pawlak [18, 47] as a method for data mining in 1982 and as a new mathematical approach to deal with inexact, uncertain or vague knowledge, has attracted the interest of researchers and practitioners in various fields of science and technology. This technique has led to many practical applications in various areas such as, but not limited to, medicine, economics, finance, engineering, and even arts and culture [53, 54]. Combined with other complementary concepts such as fuzzy sets, statistics, and logical data analysis, rough sets have been exploited in hybrid approaches to improve the performance of data analysis tools. Rough Set Theory (RST) can be approached as an extension of the Classical Set Theory, for use when representing incomplete knowledge. Rough sets can be considered sets with fuzzy boundaries - sets that cannot be precisely characterized using the available set of attributes. A key notion in the Pawlak rough set model is the equivalence relation. The equivalence classes are the building blocks for the construction of the lower and upper approximations. The lower approximation of a given set is the union of all equivalence classes which are subsets of the set, and the upper approximation is the union of all equivalence classes which have a non-empty intersection with the set. It is a natural question to ask what happens if we substitute the universe set with an algebraic system. Studying the algebraic structure of a mathematical theory has proved itself effective in making the applications in the sciences more efficient. This is the inherent motivation for us to study the algebraic structures of these generalized rough sets. Such research may not only provide more insight into rough set theory, but also hopefully develop methods for applications. The algebraic approach of rough sets was studied by some authors, for example, [55], [20], [19], [21], [46], [22], [23], [25], [24], [26]-[31], [48], [50], [51] etc. In [10], [32]-[38], [6] etc., the concepts of approximation spaces and rough sets in the theory of algebraic hyperstructures are applied. An important generalization of rough set theory is the generalized rough set based on arbitrary binary relations on a universal set. Numerous papers have been published on rough sets. In comparison, however, relatively few results have been obtained for generalized rough sets based on arbitrary binary relations. Yao [39]-[43] introduced the concept of generalized rough sets. Further, Kondo [45], studied the structure of generalized rough sets. Kondo considered some fundamental properties of generalized rough sets induced by binary relations on algebras and do not restrict the universe to be finite and consider fundamental properties of generalized rough sets induced by binary relations (also see [44, 50]).

Recently, Hila and et. al. [4] introduced the notion of quasi-hyperideal in semihypergroups, and moreover, the notion of an (m, n)-quasi-hyperideal, n-right hyperideal, and m-left hyperideal in semihypergroups, and relations between them are studied. Different characterizations concerning different properties of (m, n)-quasi-hyperideals, minimal (m, n)-quasi-hyperideals, minimal m-left hyperideals and minimal n-right hyperideals are obtained, and relations between them are investigated. In [6] we have extended these notions introducing and studying (m, n)-quasi-Γ-hyperideals in Γ-semihypergroups. Applying the rough set theory, the notion of rough (m, n)-quasi-Γ-hyperideals is introduced and properties of them are investigated.

In [12], the notion of (m, n)-bi-Γ-hyperideals has been introduced and it is applied the concept of rough set theory to (m, n)-bi-Γ-hyperideals, which is a generalization of (m, n)-bi-Γ-hyperideals of Γ-semihypergroups.

In this paper, applying the rough set theory based on an arbitrary binary relation (not an equivalent relation) we extend and generalize these notions, introducing the notion of generalized rough $(m, 0)$-Γ-hyperideals (generalized rough $(0, n)$-Γ-hyperideals), generalized rough (m, n)-quasi and bi-Γ-hyperideals and generalized rough m-left Γ-hyperideals) and establish some of their basic properties in Γ-semihypergroups.

Recall first the basic terms and definitions from the hyperstructure theory and rough set theory.

A map $\circ : H \times H \to \mathcal{P}^*(H)$ is called *hyperoperation* or *join operation* on the set H, where H is a non-empty set and $\mathcal{P}^*(H) = \mathcal{P}(H)\backslash\{\emptyset\}$ denotes the set of all non-empty subsets of H. A *hyperstructure* is called the pair $(H, \circ)$ where $\circ$ is a hyperoperation on the set H. A hyperstructure $(H, \circ)$ is called a *semihypergroup* if for all $x, y, z \in H$, $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

If $x \in H$ and A, B are non-empty subsets of H, then

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, A \circ x = A \circ \{x\}, \text{ and } x \circ B = \{x\} \circ B.$$

A non-empty subset B of a semihypergroup H is called a *sub-semihypergroup* of H if $B \circ B \subseteq B$. Let $(H, \circ)$ be a semihypergroup. Then H is called a *hypergroup* if it satisfies the reproduction axiom, for all $a \in H$, $a \circ H = H \circ a = H$. A non-empty subset I of a semihypergroup H is called a *right* (*left*) *ideal* of H if for all $x \in H$ and $r \in I$, $r \circ x \subseteq I (x \circ r \subseteq I)$. An element e in a hypergroup H is called *identity* if $\forall x \in H$, $x \in e \circ x \cap x \circ e$.

Definition 1.1. [7, 13, 16] *Let H and Γ be two non-empty sets. Any map from $H \times \Gamma \times H \to \mathcal{P}^*(H)$ will be called a Γ-hypermultiplication in H and denoted by $(\cdot)_\Gamma$. The result of this hypermultiplication for $a, b \in H$ and $\alpha \in \Gamma$ is denoted by $a\alpha b$. A Γ-semihypergroup H is an ordered pair $(H, (\cdot)_\Gamma)$ where H and Γ are non-empty sets and $(\cdot)_\Gamma$ is a Γ-hypermultiplication on H which satisfies the following property*

$$\forall (a, b, c, \alpha, \beta) \in H^3 \times \Gamma^2, (a\alpha b)\beta c = a\alpha(b\beta c).$$

If every $\gamma \in \Gamma$ is an operation, then H is a Γ-semigroup. If (H, γ) is a hypergroup for every $\gamma \in \Gamma$, then H is called a Γ-hypergroup.

Let A and B be two non-empty subset of H. Then we define

$$A\Gamma B = \bigcup_{\gamma \in \Gamma} A\gamma B = \bigcup \{a\gamma b \mid a \in A,\ b \in B \text{ and } \gamma \in \Gamma\}.$$

Let $(H, \circ)$ be a semihypergroup and let $\Gamma = \{\circ\}$. Then H is Γ-semihypergroup. So every semihypergroup is Γ-semihypergroup.

Let H be a Γ-semihypergroup and $\gamma \in \Gamma$. A non-empty subset A of H is called a sub-Γ-semihypergroup of H if $x\gamma y \subseteq A$ for every $x, y \in A$. A Γ-semihypergroup H is called *commutative* if for all $x, y \in H$ and $\gamma \in \Gamma$, we have $x\gamma y = y\gamma x$.

Definition 1.2. [16] *A non-empty subset A of a Γ-semihypergroup H is a right (left) Γ-hyperideal of H if $A\Gamma S \subseteq A$ ($S\Gamma A \subseteq A$), and is a Γ-hyperideal of H if it is both a right and a left Γ-hyperideal.*

Different examples of Γ-semihypergroups and Γ-hyperideals can be found in [7, 5, 13, 15, 16].

For the sake of simplicity, throughout the paper, we denote
$H^n = H\Gamma H\Gamma....\Gamma H = (H\Gamma)^{n-1}H$.

Let H be a Γ-semihypergroup. If ρ is an equivalence relation on H, then, for every $x \in H$, $[x]_\rho$ stands for the equivalence class of x with the represent ρ. Let A and B be two non-empty subset of H. We define $(A, B) \in \overline{\rho}$ if for every $a \in A$ there exists $b \in B$ such that $(a, b) \in \rho$ and for every $d \in B$ there exists $c \in A$ such that $(c, d) \in \rho$. Now, we define the notion of a regular equivalence relation on a Γ-semihypergroup [14, 12].

Let H be a Γ-semihypergroup. An equivalence relation ρ on H is called *regular* on H if, for every $x \in H$ and $\gamma \in \Gamma$, we have

$$(a, b) \in \rho \Rightarrow (a\gamma x, b\gamma x) \in \overline{\rho} \text{ and } (x\gamma a, x\gamma b) \in \overline{\rho}.$$

In addition, ρ on H is called *congruence* if, for every $(x, y) \in S$ and $\gamma \in \Gamma$, we have

$$z \in [x]_\rho\gamma[y]_\rho \Rightarrow [z] \subseteq [x]_\rho\gamma[y]_\rho.$$

ρ is called *complete* if, for every $x, y \in S$ and $\gamma \in \Gamma$, $[x]_\rho\gamma[y]_\rho = [x\gamma y]_\rho$.

For an equivalence relation ρ on a set U, the set of the elements of U that are related to $x \in U$, is called the equivalence class of x, and is denoted by $[x]_\rho$. Moreover, let U/ρ denote the family of all equivalence classes induced on U by ρ. For any $X \subseteq U$, we write X^c to denote the complement of X in U, that is the set U/X. A pair (U, ρ) where $U \neq \emptyset$; and ρ is an equivalence relation on U, is called an approximation space. The interpretation of rough set is that our knowledge of the objects in U extends only up to a membership in the class of ρ, and our knowledge about a subset X of U is limited to the class of ρ and their unions. This leads to the following definition.

For an approximation space (U, ρ), by a rough approximation in (U, ρ) we mean a mapping $Apr : \mathcal{P}(U) \to \mathcal{P}(U) \times \mathcal{P}(U)$ defined for every $X \in \mathcal{P}(U)$ by

$$Apr(X) = (\underline{Apr}(X); \overline{Apr}(X));$$

where $\underline{Apr}(X) = \{x \in U | [x]_\rho \subseteq X\}$, $\overline{Apr}(X) = \{x \in U | [x]_\rho \cap X \neq \emptyset\}$. $\underline{Apr}(X)$ is called a *lower rough approximation* of X in (U, ρ), where as $\overline{Apr}(X)$ is called *upper rough approximation* of X in (U, ρ).

Given an approximation space (U, ρ), a pair $(A, B) \in \mathcal{P}(U) \times \mathcal{P}(U)$ is called a *rough subset* in (U, ρ) if and only if $(A, B) = Apr(X)$ for some $X \in \mathcal{P}(U)$. Note that a rough subset is also called a *rough set*. If the map Apr is surjective, then any pair (A, B) of subsets of U is a rough set. The rough set $Apr(X)$ denotes the description of X under the present knowledge, i.e., the classification of U.

A subset X of Γ-semihypergroup H is called *definable* if $\overline{Apr}_\rho(X) = \underline{Apr}_\rho(X)$. If $X \subseteq H$ is given by a predicate P and $x \in H$, then

(1) $x \in \underline{Apr}_\rho(X)$ means that x certainly has property P,

(2) $x \in \overline{Apr}_\rho(X)$ means that x possibly has property P,

(3) $x \in S \backslash \overline{Apr}_\rho(X)$ means that x definitely does not have property P.

Let H be a Γ-semihypergoup and ρ a congruence relation on H. Then a non-empty subset A of a Γ-semihypergoup H is called ρ-lower (ρ-upper) rough sub-Γ-semihypergroup of H if $\underline{Apr}_\rho(A)$ ($\overline{Apr}_\rho(A)$) is a sub-Γ-semihypergroup of H, and a ρ-lower (ρ-upper) rough right (left, two-sided) Γ-hyperideal of H if $\underline{Apr}_\rho(A)$ ($\overline{Apr}_\rho(A)$) is a right (left, two-sided) Γ-hyperideal of H.

Further definitions and results can be found in [14] and [12].

Let ρ be a regular relation on a Γ-semihypergroup H. A non-empty subset A of H is called a *ρ-upper rough m-left Γ-hyperideal* (*n-right Γ-hyperideal*) of H if $\overline{Apr}_\rho(A)$ is a m-left Γ-hyperideal (n-right Γ-hyperideal) of H. Similarly, a non-empty subset A of a Γ-semihypergroup H is called a ρ-lower rough m-left Γ-hyperideal (n-right Γ-hyperideal) of H if $\underline{Apr}_\rho(A)$ is a m-left Γ-hyperideal (n-right Γ-hyperideal) of H. A non-empty subset Q of H is called ρ-lower (ρ-upper) rough (m,n)-(quasi)bi-Γ-hyperideal of H if the $\underline{Apr}_\rho(Q)$ ($\overline{Apr}_\rho(Q)$) is a (m,n)-(quasi)bi-Γ-hyperideal of H.

The following theorem is well-known (cf. [21]).

Theorem 1.1. *Let ρ be a reagular relation on a Γ-semihypergroup H and let A, B be non-empty subsets of H. Then*

(1) $\underline{Apr}_\rho(A) \subseteq A \subseteq \overline{Apr}_\rho(A)$.
(2) $\underline{Apr}_\rho(H) = H = \overline{Apr}_\rho(H)$.
(3) $\underline{Apr}_\rho(\underline{Apr}_\rho(A)) = \underline{Apr}_\rho(A)$ *and* $\overline{Apr}_\rho(\overline{Apr}_\rho(A)) = \overline{Apr}_\rho(A)$
(4) $\overline{Apr}_\rho(A \cup B) = \overline{Apr}_\rho(A) \cup \overline{Apr}_\rho(B)$.
(5) $\underline{Apr}_\rho(A \cap B) = \underline{Apr}_\rho(A) \cap \underline{Apr}_\rho(B)$.
(6) $A \subseteq B$ *implies* $\underline{Apr}_\rho(A) \subseteq \underline{Apr}_\rho(B)$ *and* $\overline{Apr}_\rho(A) \subseteq \overline{Apr}_\rho(B)$.
(7) $\overline{Apr}_\rho(A \cap B) \subseteq \overline{Apr}_\rho(A) \cap \overline{Apr}_\rho(B)$.
(8) $\underline{Apr}_\rho(A) \cup \underline{Apr}_\rho(B) \subseteq \underline{Apr}_\rho(A \cup B)$.
(9) $\overline{Apr}_\rho(A)\Gamma\overline{Apr}_\rho(B) \subseteq \overline{Apr}_\rho(A\Gamma B)$.
(10) *If ρ is complete, then* $\underline{Apr}_\rho(A)\Gamma\underline{Apr}_\rho(B) \subseteq \underline{Apr}_\rho(A\Gamma B)$.

2. Generalized rough subsets in Γ-semihypergroups

Let X be a non-empty set and θ be a binary relation on X. For all $A \subseteq X$, we define θ_- and $\theta_+ : \mathcal{P}(X) \to \mathcal{P}(X)$ by

$$\theta_-(A) = \{x \in X : \forall y, x\theta y \Rightarrow y \in A\} = \{x \in X : \theta N(x) \subseteq A\}$$

and

$$\theta_+(A) = \{x \in X : \exists y \in A, \text{such that } x\theta y\} = \{x \in X : \theta N(x) \cap A \neq \emptyset\},$$

where $\theta N(x) = \{y \in X : x\theta y\}$. $\theta_-(A)$ and $\theta_+(A)$ are called the θ-lower approximation and the θ-upper approximation operations, respectively [45]. For all $A \subseteq X$, by $\theta N(A)$ we mean $\theta N(A) = \{y \in X : x\theta y, \forall x \in A\}$.

Theorem 2.1. *Let θ be a reflexive, transitive and compatible relation on a Γ-semihypergroup H. If A and B are non-empty subsets of H, then*

$$\theta_-(A \cap B) = \theta_-(A) \cap \theta_-(B).$$

Proof. let $a \in \theta_-(A \cap B)$. Then $\theta N(a) \subseteq A \cap B$. Thus $\theta N(a) \subseteq A$ and $\theta N(a) \subseteq B \Leftrightarrow a \in \theta_-(A)$ and $a \in \theta_-(B) \Leftrightarrow a \in \theta_-(A) \cap \theta_-(B)$.

Thus $\theta_-(A \cap B) = \theta_-(A) \cap \theta_-(B)$. □

Theorem 2.2. *Let θ be a reflexive, transitive and compatible relation on a Γ-semihypergroup H. If A and B are non-empty subsets of H, then*

$$\theta_+(A)\Gamma\theta_+(B) = \theta_+(A\Gamma B).$$

Proof. Let $c \in \theta_+(A)\Gamma\theta_+(B)$. Then $c \in a\gamma b$ where $a \in \theta_+(A), b \in \theta_+(B)$ and $\gamma \in \Gamma$. Thus there exist elements $x, y \in S$ such that $x \in A$ and $a\theta x$ and $y \in B$ and $b\theta y$. Since θ is compatible relation on H, so $a\gamma b\theta x\gamma y$. As $x\gamma y \subseteq A\Gamma B$, so we have $c \in a\gamma b \subseteq \theta_+(A\Gamma B)$. Thus $\theta_+(A)\Gamma\theta_+(B) \subseteq \theta_+(A\Gamma B)$. □

Definition 2.1. *Let θ be a transitive and compatible relation on a Γ-semihypergroup H. Then for every $a, b \in S, \gamma \in \Gamma, \theta N(a)\gamma\theta N(b) \subseteq \theta N(a\gamma b)$. If $\theta N(a)\gamma\theta N(b) = \theta N(a\gamma b)$, then θ is called complete compatible relation.*

Theorem 2.3. *Let θ be a reflexive, transitive and complete compatible relation on a Γ-semihypergroup H. If A and B are non-empty subsets of H, then*

$$\theta_-(A)\Gamma\theta_-(B) \subseteq \theta_-(A\Gamma B).$$

Proof. Let $c \in \theta_-(A)\Gamma\theta_-(B)$. Then $c \in a\gamma b$ where $a \in \theta_-(A), b \in \theta_-(B)$ and $\gamma \in \Gamma$. Thus we have $\theta N(a) \subseteq A$ and $\theta N(b) \subseteq B$. Since θ is complete compatible relation on H, so we have for all $\gamma \in \Gamma$, $\theta N(a\gamma b) = \theta N(a)\gamma\theta N(b) \subseteq A\Gamma B$, which implies that $a\gamma b \subseteq \theta_-(A\Gamma B)$. Thus $\theta_-(A)\Gamma\theta_-(B) \subseteq \theta_-(A\Gamma B)$. □

Let θ be a reflexive, transitive and compatible relation on a Γ-semihypergroup H. Then a non-empty subset A of H is called a *generalized θ-upper rough sub-Γ-semihypergroup* of H if $\theta_+(A)$ is a sub-Γ-semihypergroup of H and A is called a *generalized θ-lower rough sub-Γ-semihypergroup* of H if $\theta_-(A)$ is a sub-Γ-semihypergroup of H.

Theorem 2.4. *Let θ be a reflexive, transitive and compatible relation on a Γ-semihypergroup H. Then*

(1) *If A is a sub-Γ-semihypergroup of H, then A is a generalized θ-upper rough sub-Γ-semihypergoup of H.*
(2) *If θ is complete, then for a sub-Γ-semihypergroup A of H, $\theta_-(A)$ is, if it is non-empty, a sub-Γ-semihypergroup of H.*

Proof. (1) Let A be a sub-Γ-semihypergroup of H. Then by Theorem 2.2, we have

$$\theta_+(A)\Gamma\theta_+(A) \subseteq \theta_+(A\Gamma A) \subseteq \theta_+(A).$$

Thus $\theta_+(A)$ is a sub-Γ-semihypergroup of H, that is A is a generalized θ-upper rough sub-Γ-semihypergroup of H.

(2) Let A be a sub-Γ-semihypergroup of H. Then by Theorem 2.3, we have

$$\theta_-(A)\Gamma\theta_-(A) \subseteq \theta_-(A\Gamma A) \subseteq \theta_-(A).$$

Thus $\theta_-(A)$, if it is non-empty, is a sub-Γ-semihypergroup of H. □

Lemma 2.1. *Let θ be a reflexive, transitive and compatible relation on a Γ-semihypergroup H. Then for a non-empty subset A of H*

(1) $(\theta_+(A))^n \subseteq \theta_+(A^n)$ *for all* $n \in N$.
(2) *If* θ *is complete, then* $(\theta_-(A))^n \subseteq \theta_-(A^n)$ *for all* $n \in N$.

3. Generalized rough (m,n)-bi-hyperideals in Γ-semihypergroups

Let θ be a reflexive, transitive and complete compatible relation on a Γ-semihypergroup H. A subset A of a Γ-semihypergroup H is called a generalized θ-upper rough $(m,0)$-Γ-hyperideal ($(0,n)$-Γ-hyperideal) of H if $\theta_+(A)$ is a $(m,0)$-Γ-hyperideal ($(0,n)$-Γ-hyperideal) of H. Similarly a subset A of a Γ-semihypergroup H is called a generalized θ-lower rough $(m,0)$-Γ-hyperideal ($(0,n)$-Γ-hyperideal) of H if $\theta_-(A)$ is a $(m,0)$-Γ-hyperideal ($(0,n)$-Γ-hyperideal) of H.

Theorem 3.1. *Let* θ *be a reflexive, transitive and compatible relation on a* Γ*-semihypergroup* H. *If* A *is a* $(m,0)$-Γ*-hyperideal* ($(0,n)$-Γ*-hyperideal) of* H. *Then*
(1) $\theta_+(A)$ *is a* $(m,0)$-Γ*-hyperideal* ($(0,n)$-Γ*-hyperideal) of* H.
(2) *If* θ *is complete, then* $\theta_-(A)$ *is empty or it is a* $(m,0)$-Γ*-hyperideal* ($(0,n)$-Γ*-hyperideal) of* H.

Proof. (1) Let A be a $(m,0)$-Γ-hyperideal of H, that is, $A^m\Gamma S \subseteq A$. Then by Theorem 2.2 and Lemma 2.1(1), we have

$$(\theta_+(A))^m\Gamma S \subseteq \theta_+(A^m)\Gamma\theta_+(S) \subseteq \theta_+(A^m\Gamma S) \subseteq \theta_+(A).$$

This shows that $\theta_+(A)$ is a $(m,0)$-Γ-hyperideal of H, that is, A is a generalized θ-upper rough $(m,0)$-Γ-hyperideal of H. Similarly we can show that generalized θ-upper approximation of a $(0,n)$-Γ-hyperideal is a $(0,n)$-Γ-hyperideal of H.

(2) Let A be a $(m,0)$-Γ-hyperideal of H, that is, $A^m\Gamma S \subseteq A$. Then by Theorem 2.3 and Lemma 2.1(2), we have

$$(\theta_-(A))^m\Gamma S \subseteq \theta_-(A^m\Gamma\theta_-(S) \subseteq \theta_-(A^m\Gamma S) \subseteq \theta_-(A).$$

This shows that $\theta_-(A)$ is a $(m,0)$-Γ-hyperideal of H, that is, A is a generalized θ-lower rough $(m,0)$-Γ-hyperideal of H. Similarly we can show that generalized θ-lower approximation of a $(0,n)$-Γ-hyperideal is a $(0,n)$-Γ-hyperideal of H. □

A subset A of a Γ-semihypergroup H is called a generalized θ-upper [generalized θ-lower] rough (m,n)-bi-Γ-hyperideal of H if $\theta_+(A)$ [$\theta_-(A)$] is a (m,n)-bi-Γ-hyperideal of H.

Theorem 3.2. *Let* θ *be a reflexive, transitive and compatible relation on a* Γ*-semihypergroup* H. *If* A *is a* (m,n)*-bi-*Γ*-hyperideal of* H, *then it is a generalized* θ*-upper rough* (m,n)*-bi-*Γ*-hyperideal of* H.

Proof. Let A be a (m,n)-bi-Γ-hyperideal of H. Then by Theorem 2.2 and Lemma 2.1(1), we have

$$\begin{aligned}(\theta_+(A))^m\Gamma S\Gamma(\theta_+(A))^n &\subseteq \theta_+(A^m)\Gamma\theta_+(S)\Gamma\theta_+(A^n)\\ &\subseteq \theta_+(A^m\Gamma S)\Gamma\theta_+(A^n)\\ &\subseteq \theta_+(A^m\Gamma S\Gamma A^n)\\ &\subseteq \theta_+(A)\end{aligned}$$

From this and Theorem 2.4(1), we obtain that $\theta_+(A)$ is a (m,n)-bi-Γ-hyperideal of H, that is, A is a generalized θ-upper rough (m,n)-bi-Γ-hyperideal of H. □

Theorem 3.3. *Let θ be a reflexive, transitive and complete compatible relation on a Γ-semihypergroup H. If A is a (m,n)-bi-Γ-hyperideal of H, then $\theta_-(A)$ is empty or it is a (m,n)-bi-Γ-hyperideal of H.*

Proof. Let A be a (m,n)-bi-Γ-hyperideal of H. Then by Theorem 2.3 and Lemma 2.1(2), we have

$$\begin{aligned}(\theta_-(A))^m\Gamma S\Gamma(\theta_-(A))^n &\subseteq \theta_-(A^m)\Gamma\theta_-(S)\Gamma\theta_-(A^n)\\ &\subseteq \theta_-(A^m\Gamma S)\Gamma\theta_-(A^n)\\ &\subseteq \theta_-(A^m\Gamma S\Gamma A^n)\\ &\subseteq \theta_-(A)\end{aligned}$$

From this and Theorem 2.4(2), we obtain that $\theta_-(A)$ is empty or it is a (m,n)-bi-Γ-hyperideal of H. □

A subset A of a Γ-semihypergroup H is called a generalized θ-upper rough m-left Γ-hyperideal (generalized θ-upper rough n-right Γ-hyperideal) of H if $\theta_+(A)$ is a m-left Γ-hyperideal (n-right Γ-hyperideal) of H. Similarly a subset A of a Γ-semihypergroup H is called a generalized θ-lower rough m-left Γ-hyperideal (generalized lower rough n-right Γ-hyperideal) of H if $\theta_-(A)$ is a m-left Γ-hyperideal (n-right Γ-hyperideal) of H.

Theorem 3.4. *Let θ be a reflexive, transitive and compatible relation on a Γ-semihypergroup H. If A is a m-left Γ-hyperideal (n-right Γ-hyperideal) of H. Then*

(1) *$\theta_+(A)$ is a m-left Γ-hyperideal (n-right Γ-hyperideal) of H.*
(2) *If θ is complete, then $\theta_-(A)$ is empty or it is a m-left Γ-hyperideal (n-right Γ-hyperideal) of H.*

Proof. (1) Let A be a m-left Γ-hyperideal of H, that is, $S^m\Gamma A \subseteq A$. Then by Theorem 2.2 and Lemma 2.1(1), we have

$$S^m\Gamma\theta_+(A) = (\theta_+(S))^m\Gamma\theta_+(A) \subseteq \theta_+(S^m)\Gamma\theta_+(A) \subseteq \theta_+(S^m\Gamma A) \subseteq \theta_+(A).$$

This shows that $\theta_+(A)$ is a m-left Γ-hyperideal of H, that is, A is a generalized θ-upper rough m-left Γ-hyperideal of H. Similarly we can show that generalized θ-upper approximation of a n-right Γ-hyperideal is a n-right Γ-hyperideal of H.

(2) Let A be a m-left Γhyperideal of H, that is, $S^m\Gamma A \subseteq A$. Then by Theorem 2.3 and Lemma 2.1(2), we have

$$S^m\Gamma\theta_-(A) = (\theta_-(S))^m\Gamma\theta_-(A) \subseteq \theta_-(S^m)\Gamma\theta_-(A) \subseteq \theta_-(S^m\Gamma A) \subseteq \theta_-(A).$$

This shows that $\theta_-(A)$ is a m-left Γ-hyperideal of H, that is, A is a generalized θ-lower rough m-left Γ-hyperideal of H. Similarly we can show that generalized θ-lower approximation of a n-right Γ-hyperideal is a n-right Γ-hyperideal of H. □

4. Generalized rough (m,n)-quasi-hyperideals in Γ-semihypergroups

A non-empty subset Q of a semihypergroup H is called a quasi-Γ-hyperideal of H if $S\Gamma Q \cap Q\Gamma S \subseteq Q$. A subset Q of a Γ-semihypergroup H is called a ρ-lower [generalized θ-lower] rough quasi-Γ-hyperideal of H if $\underline{Apr}_\rho(Q)$ $[\theta_-(Q)]$ is a quasi-Γ-hyperideal of H.

Theorem 4.1. *Let θ be a reflexive, transitive and complete compatible relation on a Γ-semihypergroup H. If Q is a quasi-Γ-hyperideal of H, then Q is a generalized θ-lower rough quasi-Γ-hyperideal of H.*

Proof. Let Q be a quasi-Γ-hyperideal of H. Now by Theorems 2.1 and 2.3, we get

$$\begin{aligned}\theta_-(Q)\Gamma S \cap S\Gamma\theta_-(Q) &= \theta_-(Q)\Gamma\theta_-(S) \cap \theta_-(S)\Gamma c\theta_-(Q)\\ &\subseteq \theta_-(Q\Gamma S) \cap \theta_-(S\Gamma Q)\\ &\subseteq \theta_-(Q\Gamma S \cap S\Gamma Q)\\ &\subseteq \theta_-(Q)\end{aligned}$$

Thus we obtain that $\theta_-(Q)$ is a quasi-Γ-hyperideal of H, that is, Q is a generalized θ-lower rough quasi-Γ-hyperideal of H. □

Corollary 4.1. *Let ρ be a complete congruence relation on a Γ-semihypergroup H. If Q is a quasi-Γ-hyperideal of H, then Q is a ρ-lower rough quasi-Γ-hyperideal of H.*

Proof. This follows from Theorems 1.1(5) and 1.1(10). □

Definition 4.1. [6] *A non-empty subset Q of a Γ-semihypergroup H is called a (m,n)- quasi-Γ-hyperideal of H if $S^m\Gamma Q \cap Q\Gamma S^n \subseteq Q$.*

A subset Q of a Γ-semihypergroup H is called a ρ-lower [generalized ρ-lower] rough (m,n)-quasi-Γ-hyperideal of H if $\underline{Apr}_\rho(Q)[\underline{Apr}_\rho(Q)]$ is a (m,n)-quasi-Γ-hyperideal of H.

Theorem 4.2. *Let θ be a reflexive, transitive and complete compatible relation on a Γ-semihypergroup H. If Q is a (m,n)-quasi-Γ-hyperideal of H, then Q is a generalized θ-lower rough (m,n)-quasi-Γ-hyperideal of H.*

Proof. Let Q be a (m,n)-quasi-Γ-hyperideal of H. Now by Theorems 2.1 and 2.3 and Lemma 2.1(2), we get

$$\begin{aligned}S^m\Gamma\theta_-(Q) \cap \theta_-(Q)\Gamma S^n &= (\theta_-(S))^m\Gamma\theta_-(Q) \cap \theta_-(Q)\Gamma(\theta_-(S))^n\\ &\subseteq \theta_-(S^m)\Gamma\theta_-(Q) \cap \theta_-(Q)\Gamma\theta_-(S^n)\\ &\subseteq \theta_-(S^m\Gamma Q) \cap \theta_-(Q\Gamma S^n)\\ &= \theta_-(S^m\Gamma Q \cap Q\Gamma S^n)\\ &\subseteq \theta_-(Q).\end{aligned}$$

Thus we obtain that $\theta_-(Q)$ is a (m,n)-quasi-Γ-hyperideal of H, that is, Q is a generalized θ-lower rough (m,n)-quasi-Γ-hyperideal of H. □

Corollary 4.2. *Let ρ be a complete congruence relation on a Γ-semihypergroup H. If Q is a (m,n)-quasi-Γ-hyperideal of H, then Q is a ρ-lower rough (m,n)-quasi-Γ-hyperideal of H.*

Proof. This follows from Theorems 1.1(5) and 1.1(10), and Lemma 4.1(2)[12]. □

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