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A NOVEL ITERATIVE SCHEME OF OPERATORS WITH PROPERTY (E) AND ITS APPLICATIONS

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The goal of this note is to propose a new iterative scheme for finding the fixed points of generalized nonexpansive mappings with property (E), called MCS-iteration. We prove both weak and strong convergence properties in a uniformly convex Banach space. Furthermore, an application of signal recovery in a compressed sensing problem is shown as numerical examples of the iterative scheme.

Keywords: MCS-iteration, Garcia-Falset mapping, compressed sensing, convergence analysis.

MSC2020: 47H10, 54H25, 37C25.

1. Introduction and preliminaries

A number of real world problems can be redeveloped by method of fixed point theory. For instance, they are applicable to solve differential equation, classification, regression, signal recovery, and image restoration, see [3, 4, 8, 10, 15, 16, 20, 21, 22, 23]. We consider that C is a nonempty subset of a real Banach space X. A mapping $T : C \to X$ is said to be

(i) nonexpansive if for all $x, y \in C$,

$$\|Tx - Ty\| \leq \|x - y\|,$$

(ii) quasi-nonexpansive if $Fix(T) \neq \emptyset$, and for any $x \in C$ and $p \in Fix(T)$,

$$\|Tx - p\| \leq \|x - p\|.$$

The fixed point problems, that is, problems which look for a point

$$x \in C \quad \text{such that} \quad Tx = x. \tag{1}$$

$Fix(T)$ denote the set of all fixed points of T. The various iterative schemes for fixed points numerical approximation have been introduced by assorted authors. Some research works of iterative schemes were originated and generally recognized for estimate fixed points of nonexpansive mappings, in particular [1, 9, 11, 12, 14, 27]. The Noor iteration [12] was defined as follows: $x_0 \in C$ and

$$\begin{aligned} z_n &= (1 - \gamma_n)x_n + \gamma_n T x_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T z_n, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n, \end{aligned} \tag{2}$$

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and the SP-iteration [14] was defined as follows: $x_0 \in C$ and

$$\begin{aligned} z_n &= (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n &= (1-\beta_n)z_n + \beta_n T z_n, \\ x_{n+1} &= (1-\alpha_n)y_n + \alpha_n T y_n, \end{aligned} \tag{3}$$

$\forall n \geq 0$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$. A generalization for the class of nonexpansive mappings on Banach spaces was proposed by Suzuki [24] in 2008. He entitled this property condition (C) (also labelled the class of Suzuki generalized nonexpansive mappings, or simplified the class of Suzuki mappings), but it is appropriately contained into the class of quasi-nonexpansive mappings. With regard to the numerical reckoning of fixed points for Suzuki mappings, some convergence theorems were provided. Furthermore, based on some three steps iteration schemes, the necessary and sufficient conditions for the existence of fixed points were also proved, for example, see, [2, 26, 29]. Garcia-Falset et al. [6] in 2011 proposed a general class of Suzuki mappings, and condition (E) is the resulting property. But it still remains stronger than quasi-nonexpansiveness. The class of mappings satisfying condition (E) will be henceforward referred to as Garcia-Falset-generalized nonexpansive mappings or Garcia-Falset mappings. Recently, Usurelu et al. [28] provided a consequence regarding the existence of fixed points for Garcia-Falset mappings in the framework of uniformly convex Banach spaces, based on the TTP-iteration (or the Thakur et al. iterative scheme) was introduced in [25]. Additionally, several convergence outcomes concerning this iterative scheme have been presented. The TTP-iteration was defined as follows: $x_0 \in C$ and

$$\begin{aligned} z_n &= (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n &= (1-\beta_n)z_n + \beta_n T z_n, \\ x_{n+1} &= (1-\alpha_n)T z_n + \alpha_n T y_n, \end{aligned} \tag{4}$$

$\forall n \geq 0$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$. To precede inspirational research, this work will suggest the new iterative scheme, called the MCS-iteration. The iteration method is defined as follows: $x_0 \in C$ and

$$\begin{aligned} z_n &= (1-d_n)x_n + d_n T x_n, \\ y_n &= (1-c_n)T x_n + c_n T z_n, \\ x_{n+1} &= (1-a_n-b_n)T x_n + a_n T y_n + b_n T z_n, \end{aligned} \tag{5}$$

$\forall n \geq 0$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ and $\{a_n + b_n\} \subset (0,1)$. By using this iterative scheme for Garcia-Falset mappings in a uniformly convex Banach space, some weak and strong convergence theorems are shown. Moreover, a result relating to the existence of fixed points for these mappings is established. In the next section, the weak and strong convergence theorems for our iterative scheme can be obtained by some control assumptions. The proofs of our lemmas and theorems are provided. In section 3, an application of the compressive sensing signal reconstruction will be considered and so compared with the previous results. In the last section, we briefly conclude our work.

2. **Convergence analysis**

We first state the conditions that we will assume to hold for both weak and strong convergence properties. Let $T : C \to X$ be a mapping on a subset C of a Banach space X. We say that T is satisfied

(i) Condition (C) [24] if

$$\frac{1}{2}\|x - Ty\| \leq \|x - y\| \Longrightarrow \|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C,$$

(ii) Condition (E_μ) [6] if there exists $\mu \geq 1$ such that

$$\|x - Ty\| \leq \mu\|x - Tx\| + \|x - y\|, \forall x, y \in C.$$

Ausuume that C is convex, so $T : C \to C$ is said to satisfy

(iii) Condition (I) [18] if there is a nondecreasing function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and $g(s) > 0$, $\forall s > 0$, with

$$g(d(x, Fix(T))) \leq d(x, Tx),$$

$\forall x \in C$, where $d(x, Fix(T)) = \inf_{x \in Fix(T)} \|x - p\|$.

2.1. Weak convergence results

Throughout this subsection, we suppose that

- C is a nonempty closed convex subset of a real Banach space X.
- $T : C \to C$ is a mapping which satisfies condition (E).

Next, the following lemma is stated and verified below.

Lemma 2.1. *Let $Fix(T)$ is nonempty and let $\{x_n\}$ be a sequence defined by the MCS-iteration* (5) *where $x_0 \in C$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in Fix(T)$.*

Proof. Given $p \in Fix(T)$. Since the mapping T is quasi-nonexpansive by [6, Proposition 1], we have

$$\|Tz_n - p\| \leq \|z_n - p\| \leq (1 - d_n)\|x_n - p\| + d_n\|Tx_n - p\| \leq \|x_n - p\|. \tag{6}$$

Applying (6), we get

$$\begin{aligned} \|Ty_n - p\| \leq \|y_n - p\| &\leq (1 - c_n)\|Tx_n - p\| + c_n\|Tz_n - p\| \\ &\leq (1 - c_n)\|x_n - p\| + c_n\|x_n - p\| = \|x_n - p\|. \end{aligned} \tag{7}$$

By combining (6) and (7), henceforth

$$\begin{aligned} \|x_{n+1} - p\| &\leq (1 - a_n - b_n)\|Tx_n - p\| + a_n\|Ty_n - p\| + b_n\|Tz_n - p\| \\ &\leq (1 - a_n - b_n)\|x_n - p\| + a_n\|x_n - p\| + b_n\|x_n - p\| = \|x_n - p\|. \end{aligned} \tag{8}$$

Obviously, $\{\|x_n - p\|\}$ is bounded and nonincreasing for all $p \in Fix(T)$. That is, $\lim_{n\to\infty} \|x_n - p\|$ exists. □

Now, the main theorems are presented and proved as follows.

Theorem 2.1. *Let X be uniformly convex and let $\{x_n\}$ be a sequence defined by the MCS-iteration* (5)*, where $x_0 \in C$, $\{d_n\}$ is bounded away from* 0 *and* 1 *for all $n \geq 0$. Then $Fix(T)$ is nonempty if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.*

Proof. Assume that $Fix(T)$ is nonempty and let $p \in Fix(T)$. Then, by Lemma 2.1, there exists $r \geq 0$ such that $r = \lim_{n\to\infty} \|x_n - p\|$ and the sequence $\{x_n\}$ is bounded. Next, we will show that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Taking lim sup in (6), so we get

$$r = \limsup_{n\to\infty} \|x_n - p\| \geq \limsup_{n\to\infty} \|z_n - p\|. \tag{9}$$

By the quasinonexpansiveness of T, we have

$$r = \limsup_{n\to\infty} \|x_n - p\| \geq \limsup_{n\to\infty} \|Tx_n - p\|. \tag{10}$$

On the others hand,

$$\begin{aligned}\|x_{n+1}-p\| &\leq (1-a_n-b_n)\|Tx_n-p\|+a_n\|Ty_n-p\|+b_n\|Tz_n-p\| \\ &\leq \|x_n-p\|+b_n\left(\|z_n-p\|-\|x_n-p\|\right).\end{aligned}$$

This follows that

$$\|z_n-p\| \geq \|x_{n+1}-p\|.$$

Taking lim inf in the above inequality, we obtain

$$r \geq \limsup_{n\to\infty}\|z_n-p\| \geq \liminf_{n\to\infty}\|z_n-p\| \geq r.$$

That is,

$$r=\lim_{n\to\infty}\|z_n-p\|=\lim_{n\to\infty}\|(1-d_n)(x_n-p)+d_n(Tx_n-p)\|. \tag{11}$$

Applying (9)-(11) together with [17, Lemma 1.3], we can conclude that $\lim_{n\to\infty}\|Tx_n-x_n\|=0$.

Consequently, assume that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|Tx_n-x_n\|=0$. Next, suppose that $p\in A(C,\{x_n\})$. Since T satisfies condition (E), the following relation is obtained:

$$\begin{aligned}r(Tp,\{x_n\}) &= \limsup_{n\to\infty}\|Tx_n-p\| \leq \limsup_{n\to\infty}\left(\mu\|Tx_n-x_n\|+\|x_n-p\|\right) \\ &= \limsup_{n\to\infty}\|x_n-p\| = r(Tp,\{x_n\}).\end{aligned}$$

This follows that $Tp\in A(C,\{x_n\})$. By the uniqueness of asymptotic centers, we have $p=Tp$, i.e., $Fix(T)$ is nonempty. The proof is completed. □

Theorem 2.2. *Let X be a uniformly convex with Opial's property. Let T and $\{x_n\}$ be the same in Theorem 2.1 and $Fix(T)$ is nonempty. Then $\{x_n\}$ weakly converges to a point in $Fix(T)$.*

Proof. First, we have $\{x_n\}$ is bounded sequence, $\lim_{n\to\infty}\|x_n-p\|$ exists for all $p\in Fix(T)$, and $\lim_{n\to\infty}\|x_n-Tx_n\|=0$ by Lemma 2.1 and Theorem 2.1. Second, let $\{x_{n_i}\}$ and $\{x_{m_i}\}$ be subsequences of $\{x_n\}$ weakly converging to z_1 and z_2, respectively. Then $\lim_{i\to\infty}\|x_{n_i}-Tx_{n_i}\|=\lim_{i\to\infty}\|x_{m_i}-Tx_{m_i}\|=0$. Then we get $z_1,z_2\in C$ since C is closed and convex, also by Mazur's theorem. By the demiclosedness at zero of $I-T$ from [6, Theorem 1], we have $z_1,z_2\in Fix(T)$. Finally, we can conclude that $z_1=z_2$ by [19, Lemma 2.7]. Therefore, $\{x_n\}$ weakly converges to a fixed point of T. □

2.2. Strong convergence results

In this subsection, we present strong convergence theorems for a mapping satisfying condition (E).

Theorem 2.3. *Let X be a uniformly convex Banach space and C a nonempty, compact and convex subset of X. Let T and $\{x_n\}$ be as same as in Theorem 2.1. If $Fix(T)$ is nonempty, then $\{x_n\}$ strongly converges to a point in $Fix(T)$.*

Proof. This proof is identical with the proof of [28, Theorem 3.4]. □

Theorem 2.4. *Let X be a uniformly convex Banach space and C a nonempty, closed and convex subset of X. Define T and $\{x_n\}$ similarly with Theorem 2.1. If T satisfies the condition (I) and $Fix(T)$ is nonempty, then $\{x_n\}$ strongly converges to a point in $Fix(T)$.*

Proof. The proof is similar to the proof of [28, Theorem 3.5]. □

3. Signal recovery and its numerical results

In this section, we apply our iterative scheme to the problem of recovering the original signal from compressive measurements. Let us begin by given that $\bar{x} \in \mathbb{R}^N$ and $y \in \mathbb{R}^M$ are the original signals and the observed data, respectively. Consider

$$y = A\bar{x} + \varepsilon, \tag{12}$$

where $A \in \mathbb{R}^{M\times N}$ $(M < N)$, and $\varepsilon \in \mathbb{R}^M$ represents the Gaussian noise with $N(0, \sigma^2)$. We wish to solve the compressive sensing signal reconstruction modeled as in the above equation. However, there is known fact to solve (12) is equivalent to the following LASSO problem:

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|Ax - y\|_2^2 \text{ subject to } \|x\|_1 \leq \zeta, \tag{13}$$

where $\zeta > 0$. The problem (13) can be seen as the fixed point problem through the following settings:

$$T = P_D(I - \frac{1}{\|A\|^2}\nabla g), \text{ where } g(x) = \frac{1}{2}\|Ax - y\|_2^2 \text{ and } D = \{x \in \mathbb{R}^N : \|x\|_1 \leq \zeta\}.$$

We have known that $(I - \kappa\nabla g)$ is nonexpansive for any $0 < \kappa < \frac{2}{\|A\|^2}$ (see [7]). In addition, P_D has closed forms which is the projection onto the closed l_1 ball in $\mathbb{R}^N$ (see [5]).

Next, we present a numerical result for the problem (13). In particular, we investigate the behavior of the MCS-iteration (5) and then compare it with three iterative schemes: the Noor iteration (2), the SP-iteration (3) and the TTP-iteration (4). Let $N = 2^{12}$ and $M = 2^{11}$ be the size of signal. Assume that the original signal has m nonzero elements, then generate the Gaussian matrix A by using $randn(M, N)$ and $\zeta = m$. Next, select $x_0 = A^t y$ as the initial point. For any $n \geq 0$, let $\alpha_n = a_n = \frac{3n+3}{4n+12}$, $\beta_n = c_n = \frac{\sqrt{15n+10}-(3n+3)^{\frac{1}{4}}}{10\sqrt{15n+10}}$, $\gamma_n = d_n = \sqrt{\frac{n+1}{16n+15}}$, and $b_n = \frac{n+2}{6n+6}$. Here, comparing the accuracy between the recovered signals with the mean-squared error: $\text{MSE}_n = \frac{1}{N}\|x_n - \bar{x}\|^2 < 5 \times 10^{-5}$. By using Matlab R2015b and running on a MacBook Pro with a 2.7 GHz Intel Core i5 processor and 8 GB of RAM, we obtain the results are shown as follows.

In Table 1, the numerical experiments have been done in the different numbers of nonzero elements: $m = 2^5, 2^6, 2^7$. For these three cases, the elapsed times and number of iterations are recorded for each iterative scheme. As shown below, the MCS-iteration averagely spends less elapsed time than the other three iterative schemes. Similarly, the number of iterations of the MCS-iteration is also less than the others. We further display the recovery signals when $m = 2^7$ and $\sigma = 0.1$ in Figure 1.

Iterative schemes		m Nonzero Elements		
		$m = 2^5$	$m = 2^6$	$m = 2^7$
Noor	Elapsed Time (s)	1.1692	1.9590	4.1187
	No. of Iter.	48	80	151
SP	Elapsed Time (s)	0.8833	1.5606	3.8822
	No. of Iter.	36	60	115
TTP	Elapsed Time (s)	0.6355	1.0918	3.5876
	No. of Iter.	26	45	92
MCS	Elapsed Time (s)	0.4912	0.8426	2.8445
	No. of Iter.	20	35	70

Table 1. Numerical comparison of four iterative schemes for $\sigma = 0.1$.

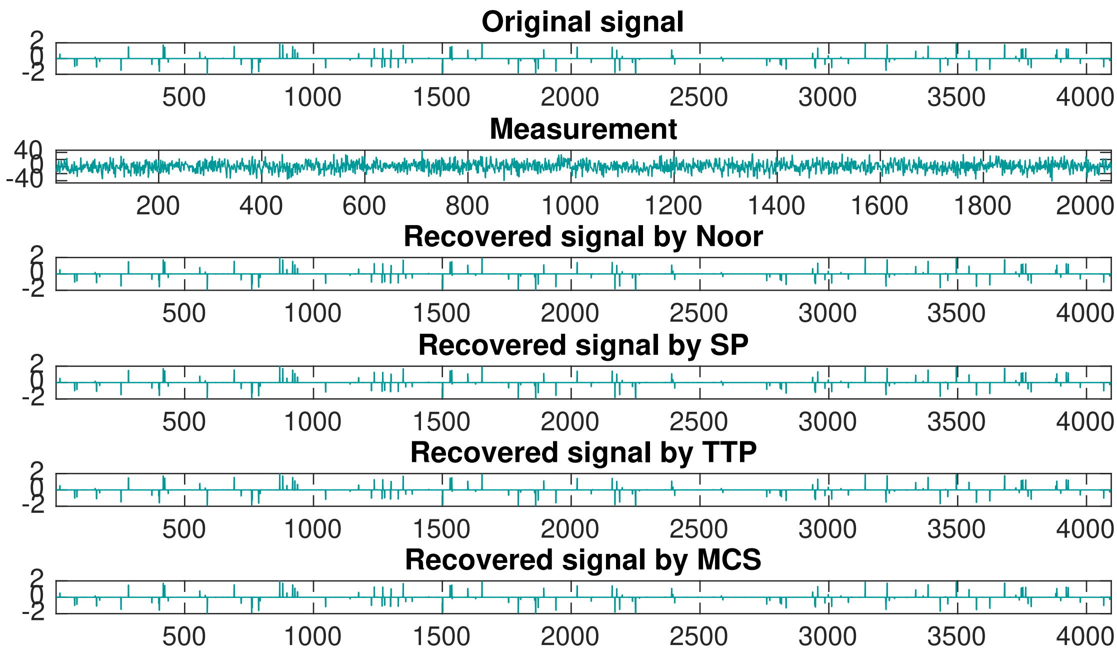

FIGURE 1. From top to bottom: the original signal, the measurement, and the recovery signals by the Noor iteration, the SP-iteration, the TTP-iteration and the MCS-iteration, respectively when $m = 2^7$ and $\sigma = 0.1$.

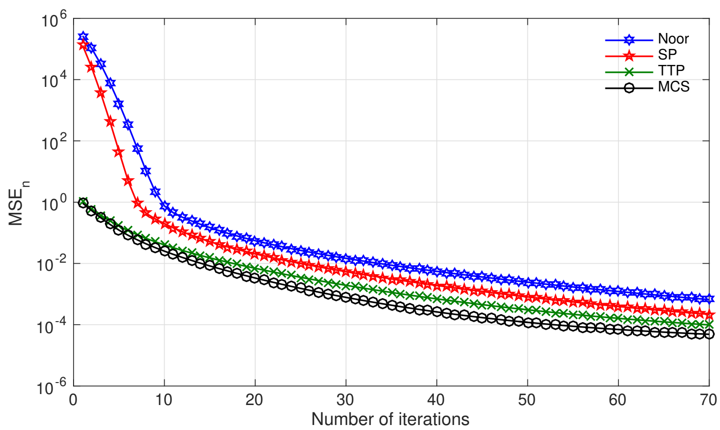

FIGURE 2. Mean-squared error versus number of iterations when $m = 2^7$ and $\sigma = 0.1$.

In conclusion, the MCS-iteration improves the numerical results in these particular cases. To distinguish the difference of these results, we compute the errors of each reconstructed signal are shown in Figure 2.

Additionally, we explore more results in the case when $\sigma = 0.01$ to confirm the previous results. As expected, we gain the similar well-behaved outcome of the MCS-iteration. Table 2 are demonstrated that the MCS-iteration needs less elapsed times and a number of iterations than the other three iterative schemes. Correspondingly, the following figures are provided to compare the reconstructed signals for each iterative scheme when $m = 2^7$ for $\sigma = 0.01$. As can be seen in Figures 3 and 4, the MCS-iteration gives error less than the others as previously.

Iterative schemes		m Nonzero Elements		
		$m = 2^5$	$m = 2^6$	$m = 2^7$
Noor	Elapsed Time (s)	1.2310	1.6270	3.1234
	No. of Iter.	50	67	131
SP	Elapsed Time (s)	0.9023	1.2118	2.3821
	No. of Iter.	37	50	100
TTP	Elapsed Time (s)	0.6273	0.9256	1.8145
	No. of Iter.	26	38	77
MCS	Elapsed Time (s)	0.5022	0.7317	1.3879
	No. of Iter.	21	30	59

TABLE 2. Numerical comparison of four iterative schemes for $\sigma = 0.01$.

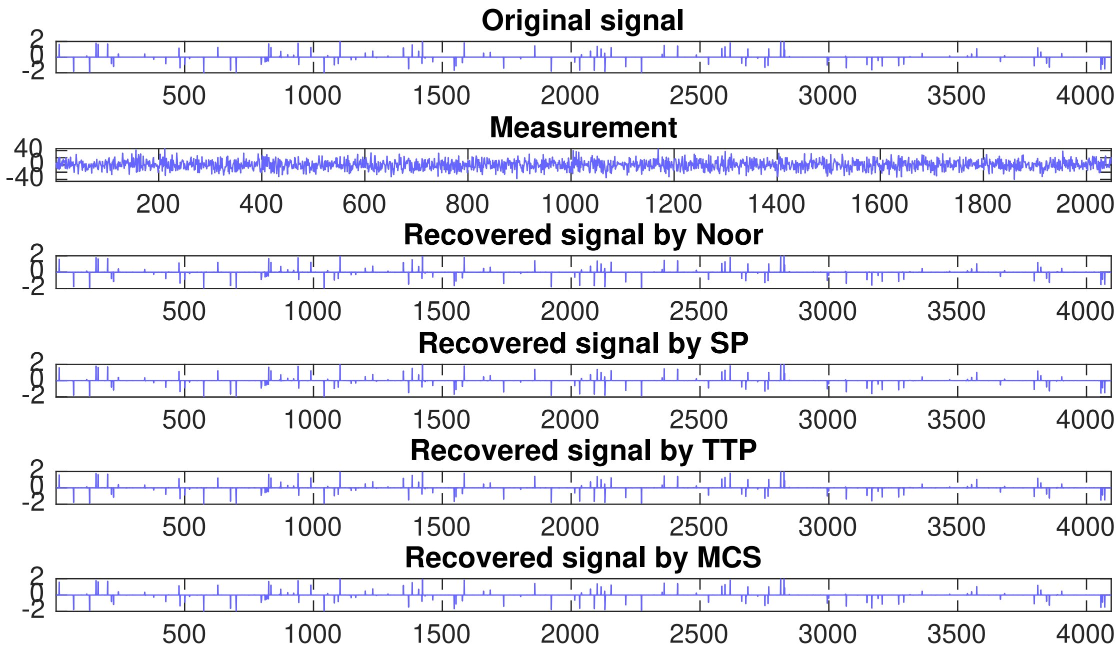

FIGURE 3. From top to bottom: the original signal, the measurement, and the recovery signals by the Noor iteration, the SP-iteration, the TTP-iteration and the MCS-iteration, respectively when $m = 2^7$ and $\sigma = 0.01$.

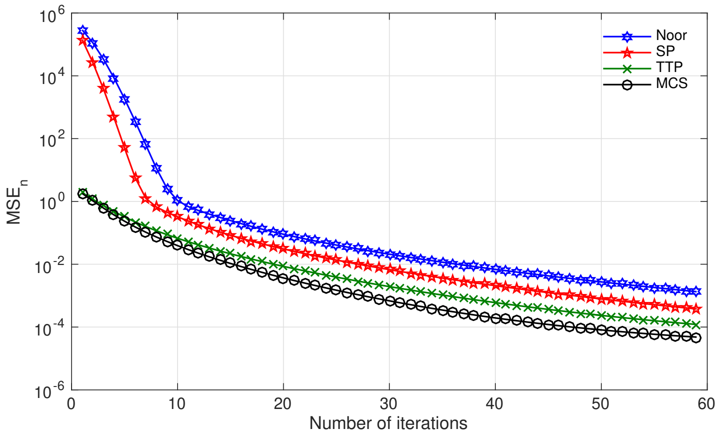

FIGURE 4. Mean-squared error versus number of iterations when $m = 2^7$ and $\sigma = 0.01$.

4. Conclusions

In summary, we present a new iterative method for solving for the fixed points of generalized nonexpansive mappings with property (E). Besides, we validate the weak and strong convergence properties of the iterative scheme under certain conditions. After that applying the iterative scheme to the problem of signal recovery in compressed sensing. The numerical investigations of our iterative scheme presents better solutions than the prior iterative schemes.

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