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MODELING OF NONLINEAR SYSTEM FOR A HYDRAULIC PROCESS

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This paper deals with the model design of a nonlinear plant which consists of a three open coupled tanks and a collecting reservoir. Through these tanks, water is pumped by means of two pumps and several valves. New contributions are brought by the author to the design of nonlinear model of the plant.

The plant represents a suitable research benchmark for studying hydraulic processes, control and fault tolerant strategies that can be applied. Also this plant has many applications in power plants and petro-chemical industry. The article presents the design and validation of a nonlinear and respectively a linear model of the plant, used for control and optimization purpose.

Keywords: nonlinear modeling, linear modeling, hydraulic process

1. Introduction

Systems with three open tanks are frequently met in water treatment plant that is an important part of any power plan, in petro-chemical industry, in systems of aircraft fuel tanks, in the case of boilers in paper industry, the filter water and many other areas. Many processes can be modeled as systems with three open reservoirs [1]. In such systems, liquid is pumped, stored in tanks, and then pumped/discharged in other reservoirs by means of pumps and valves.

When the objective is to maintain a precise constant water level in such tanks, the design of appropriate controllers can be accomplished by knowing the mathematical model of the plant. Designing a mathematical model of the controlled plant can be achieved by means of analytical and experimental identification techniques. In case of using analytical models, mathematical models are obtained by applying the laws that describe functionality of the process (energy conservation law, the impulse conservation law, etc.) taking into account the characteristic of each process. In general, mathematical models obtained by means of analytical techniques are complex, nonlinear and often time-varying [2,7].

The complexity of the mathematical model is a result of a trade-off between the desired accuracy and the cost of the implementation. It is clear that a more accurate model will lead to a complex one which is hard to be implemented

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and simulated. Today, based on the powerful computing systems, we are able to simulate with dedicated software products (such as Matlab/Simulink[3], Mathematica[4], etc) very complex mathematical models. The problem still arises when dealing with model based control designs [5] or fault detection and diagnosis method [10]. In this case the available algorithms are based on simpler models (the vast majority on linear models).

On the other hand the linear models and some class of nonlinear models are obtained by simplifying the initial hypotheses. All these simplifications aid the design of control strategies but bring some reduction of the designed performances. This is often the case when a designed controller is accompanied with restricted operating ranges in order to guarantee a level of performance. Often these solutions are cheaper and are based on simpler and classical designs.

As today the control research community is concerned with control strategies of nonlinear models[2] it is of actual interest to insist on nonlinear modeling of plant systems.

The plant discussed here, for which a model is proposed, is a process with two inputs and three outputs (MIMO). The main feature of these systems is that multiple inputs generate multiple outputs with strong interactions between variables [2,7].

2. System modeling and model validation

Mathematical models of dynamic processes are primarily obtained by either theoretical/physical modeling or experimentally by the means of identification methods. For *theoretical modeling*, also called *theoretical analysis* or *modeling* by first principles, the model is set up on the basis of mathematically formulated laws of nature [2,7]. The theoretical modeling always begins with simplifying assumptions about the process, which simplifies the calculations or enables them with a tolerable expenditure.

By summarizing the basic equations of all process elements, one receives a system of ordinary and/or partial differential equations of the process. This leads to a theoretical model with a certain structure and certain parameters, if it can be solved explicitly. Frequently, this model is extensive and complicated, so it must be simplified for further application [11].

The simplifications are made by linearization, reduction of the model order or approximation of systems with distributed parameters by lumped parameters when limiting on fixed locations. The first steps of these

[2] Searching in the IEEEXplore Database after "*control*" in title, abstract and keywords after year 2000 till present one retrieves 37.132 papers. Within these papers 7.037 (≈ 20%) contains references to "*nonlinear*" term.

simplifications can be already made by simplifying assumptions while stating the basic equations.

But also if the set of equations cannot be solved explicitly, the individual equations supply important hints for the model structure. So, balance equations are always linear and some phenomenological equations are linear in wide areas. The constitutive equations often introduce nonlinear relations.

During experimental modeling, which is referred to as *identifications*, one obtains the mathematical model of a process from measurements. Here, one always proceeds from a priori knowledge, which was gained, from the theoretical analysis or from preceding measurements. Then, input and output signals are measured and evaluated by means of identification methods in such a way that the relation between the input and the output signal is expressed in a mathematical model. The input signals can be naturally operating signals (occurring in the system) or artificially introduced test signals. The result of the identification is an experimental model.

The theoretical and experimental models can be compared, providing both types of modeling. If the models do not agree, then one can conclude from the type and size of the differences which particular steps of the theoretical or experimental modeling have to be re-evaluated.

Theoretical and experimental modeling mutually completes them. The theoretical model contains the functional description between the physical data of the process and its parameters. Therefore, one will use this model, if the process is to be favorably designed with regard to dynamical behavior or if the process behavior has to be simulated before construction. The experimental model on the other hand, contains parameters as numeric values whose functional relation with the physical basic data of the process remains unknown. In many cases, the real dynamic behavior can be described more exactly or it can be determined at smaller expenditure by experimentally obtained models, which better suited to the adjustment of the feedback controller, the prediction of signals or for fault detection [6].

3. Process plant description

The experimental plant considered in this paper consists of three identical cylindrical tanks with equal cross-sectional area A (Fig.1) and a collectiong reservoir.

These three tanks are interconnected through two cylindrical pipes of the same cross-sectional area, denoted *S*, and have the outflow coefficients az_{13}, az_{32}. The nominal outflow located at tank T_2 has the same cross-sectional area as the coupling pipe between the cylinders with outflow coefficient az_{20}.

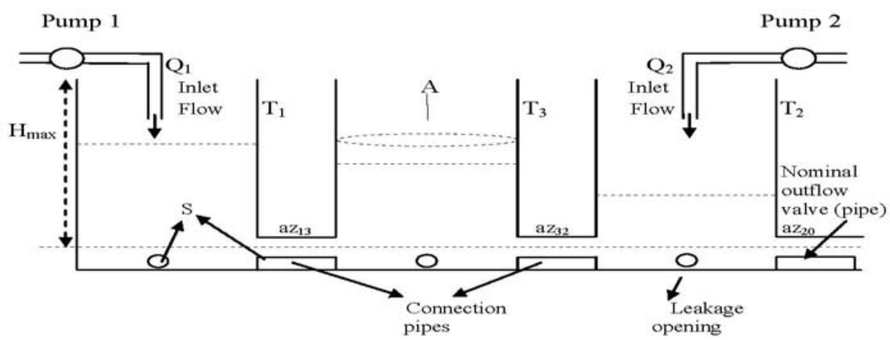

Fig 1. Structure of the plant

Two pumps driven by DC motors supply the first and last tanks. Pumps flow rates (Q_1 and Q_2) are defined by flow per rotation.

A digital/analog converter is used in the control path of each pump. The maximum flow rate for pump *i* is denoted Q_{imax} The out flowing liquid is collected in a reservoir, which supplies the Pumps 1 and 2.

A piezoresistive differential pressure sensor carries out the necessary level measurement. Three transducers provide voltage signal levels. The variable h_j, denotes the level in tank *j* and H_{max} denotes the highest possible liquid level. In case the liquid level of T_1 or T_2 exceeds this value the corresponding pump will be switched off automatically.

For the purpose of simulating leaks each tank additionally has a circular opening with the cross section S and manually adjustable valve. The following pipe ends in the reservoir.

Table 1.

Parameters values of three-tanks system

Variable	Symbol	Value
Section of cylinder	A	0.0154 m^2
Section of pipes	S	5*10-5 m^2
Supplying flow rates	Q_{imax}	100 ml/s
Flow rates between tanks	Q_{13} , Q_{32} , Q_{20}	variable
Outflow coefficients	az_{13}, az_{32} , az_{20}	see Table 2
Maximum level	h_{jmax}	0.63

The connection of the plant with a PC is assured by two acquisition cards : ”Humusoft MF624” for Pump 1 and level transducers and ”National Instruments PCI-6503” which controls the electro valves. The real-time interface is configured by using Matlab/Simulink.

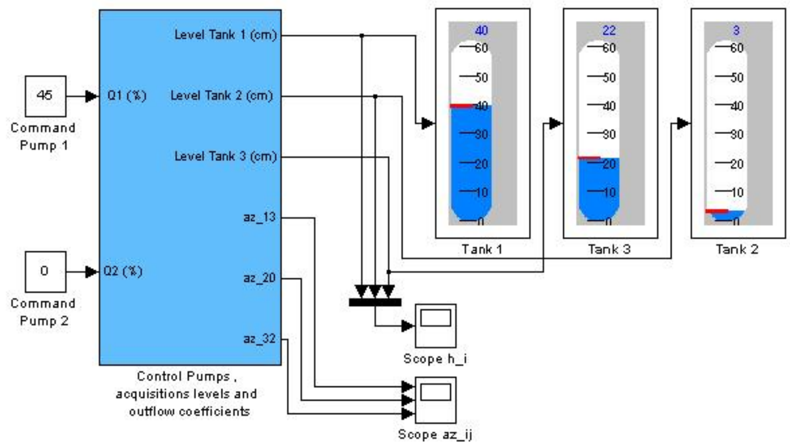

Fig 2. Control Interface in Simulink

By means of this interface, the values of level sensors, which are placed in each one of the three tanks (*Tank 1, Tank 2, Tank 3)* are acquired, in real-time manner. In the Simulink interface these level are represented in an animated way.

4. The model design of the plant

In this section the nonlinear and respective the linear model of the plant is derived. The novelty in these designs is related to the identification of the outflow coefficients, which are often skipped in similar papers which deals with the same problem [1],[8].

4.1 Nonlinear model

The three tank system model from Fig. 1 is written using the mass balance equation. The system can be expressed by next equations.

$$A\frac{dh_1}{dt} = Q_1(t) - Q_{13}(t) \tag{1}$$

$$A\frac{dh_3}{dt} = Q_{13}(t) - Q_{32}(t) \tag{2}$$

$$A\frac{dh_2}{dt} = Q_2(t) + Q_{32}(t) - Q_{20}(t) \tag{3}$$

where Q_{ij} represents the water flow rate from tank *i* to *j, i, j = 1, 2, 3*, which, according to Torricelli's rule is given by

$$Q_{ij}(t) = az_{ij}S\ sgn\left(\Delta h(t)\right)\sqrt{2g\Delta h_{ij}(t)} \tag{4}$$

where $\Delta h_{ij}(t) = h_i(t) - h_j(t)$ and Q_{20} represents the outflow rate described as follows :

$$Q_{20}(t) = az_{20}S\sqrt{2gh_2(t)} \tag{5}$$

The numerical values of the plant parameters are listed in Table 1 and Table 2.

Appling Torricelli's rule the following equations are obtained for all flow rates:

$$Q_{13}(t) = az_{13}S\ sgn\left(h_1(t) - h_3(t)\right)\sqrt{2g\left(h_1(t) - h_3(t)\right)} \tag{6}$$

$$Q_{32}(t) = az_{32}S\ sgn\left(h_3(t) - h_2(t)\right)\sqrt{2g\left(h_3(t) - h_2(t)\right)} \tag{7}$$

$$Q_{20}(t) = az_{20}S\sqrt{2gh_2(t)} \tag{8}$$

For computing the nonlinear model it was considered that Pump 1 is opened and Pump 2 is closed. In this case the flow levels respect the condition $h_1 < h_3 < h_2$.

4.1.1 Computing the outflow coefficients

The outflow coefficients were calculated using at equilibrium the Torricelli's rule. At equilibrium, levels variation is 0 , that is:

$$\frac{dh_{1e}}{dt} = 0; \frac{dh_{2e}}{dt} = 0; \frac{dh_{3e}}{dt} = 0 \tag{9}$$

so, from equations (1), (2) and (3), the next relations for computing *az* coefficients are in use:

$$az_{13} = \frac{Q_1}{S\sqrt{2g(h_{1e} - h_{3e})}} \tag{10}$$

$$az_{32} = \frac{Q_1}{S\sqrt{2g(h_{3e} - h_{2e})}} \tag{11}$$

$$az_{20} = \frac{Q_1}{S\sqrt{2g(h_{2e})}} \tag{12}$$

These relations have been evaluated in the control interface and from the obtained values the next table has been filled depending on the level of Q_1.

Table 2

Outflow coefficients

Outflow cofficients	az_{13}	az_{32}	az_{20}
0 $m^3/s < Q_1 \le 25e\text{-}6\ m^3/s$	0.46	0.44	0.9
$25e\text{-}6\ m^3/s < Q_1 \le 35e\text{-}6\ m^3/s$	0.45	0.43	0.726
$35e\text{-}6\ m^3/s < Q_1 \le 100e\text{-}6\ m^3/s$	0.485	0.485	0.79

Nonlinear model for this process is described by the next differential equations:

$$\frac{dh_1}{dt} = \frac{1}{A}(Q_1 - az_{13}S\sqrt{2g(h_1 - h_3)}) \qquad (13)$$

$$\frac{dh_3}{dt} = \frac{1}{A}(az_{13}S\sqrt{2g(h_1 - h_3)} - az_{32}S\sqrt{2g(h_3 - h_2)}) \qquad (14)$$

$$\frac{dh_2}{dt} = \frac{1}{A}(az_{32}S\sqrt{2g(h_3 - h_2)} - az_{20}S\sqrt{2gh_2}) \qquad (15)$$

The nonlinear model described by relations (13)-(15) was simulated with the model build in Matlab/Simulink.

4.1.2. Validation of the nonlinear model

Nonlinear model validation was done by comparing the evolution of the real system with the evolution of nonlinear model at the same step inputs. These two trajectories were placed in the same scope (Fig. 3), using the same units, and as can be seen the equivalents trajectories are about the same, hence the nonlinear model follows the real one.

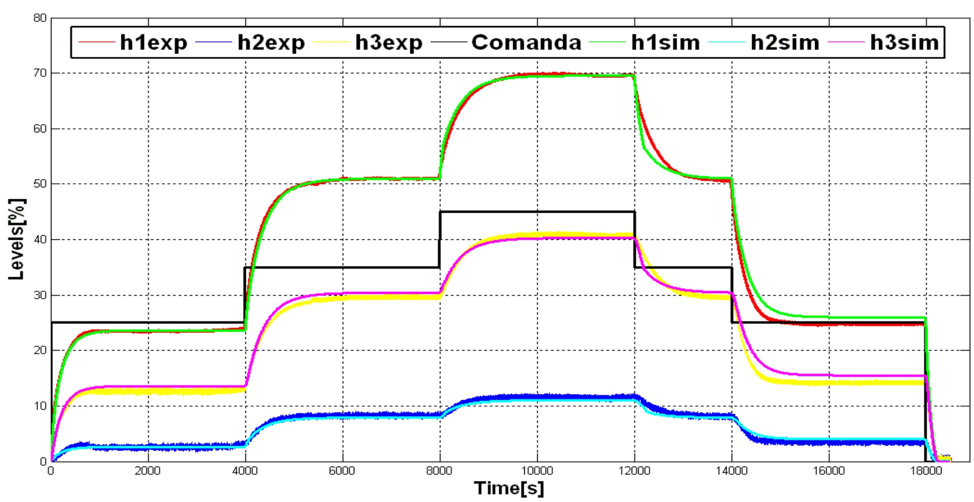

Fig. 3. The command and the responses of the real and simulated process

4.2. Linear model

The linear model is obtained from the nonlinear one, commonly by using the linearization method. The expansion in Taylor's series around the equilibrium point is a very effective approximation of the non-linear model only for some minor deviation of state variables from the equilibrium point [7], [9]. If the previous nonlinear model is valid for all range of the input variables, this model will be restricted.

Let x_{eq}, u_{eq} be the equilibrium points of the system Eq. (13,14,15), i.e.

$$\dot{x}_{eq} = f\left(x_{eq};u_{eq};t\right) \tag{16}$$

where

$$\Delta x = x - x_{eq}, \Delta u = u - u_{eq} \tag{17}$$

are the small differences for the state vector and the input vector, respectively. Assuming that

$$\Delta x = x - x_{eq} = \dot{x} - \dot{f}(x_{eq}, u_{eq}, t) \tag{18}$$

and expanding in Taylor's series the right side of Eq. (13,14,15), respectively neglecting the terms of order higher than first, we obtain the approximation of this equation in the form of the following linear equation

$$\Delta\dot{x} = A\Delta x + B\Delta u \tag{19}$$

Usually the Eq. (19) is written in the linear state space representation:

$$x(t) = A\dot{x}(t) + Bu(t) \tag{20}$$

where,

$$A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_{eq}\\ u=u_{eq}}}, \mathrm{B} = \left.\frac{\partial \mathrm{f}}{\partial \mathrm{u}}\right|_{\substack{\mathrm{x}=\mathrm{x}_{eq}\\ \mathrm{u}=\mathrm{u}_{eq}}} \tag{21}$$

The steady state operating data of the Three-tank system is given in Table 1. The state space model of the three tank system around the operating point is given next:

$$\begin{aligned} \dot{\mathrm{x}} &= \mathrm{Ax} + \mathrm{Bu} \\ y &= \mathrm{Cx} + \mathrm{Du} \end{aligned} \tag{22}$$

$x = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$ - states system $u = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$ - input system $y = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ - output system

Linearization of the nonlinear model is made around the equilibrium point, $u_e = (Q_{1e};Q_{2e})$, which means that after a sufficiently long time, if we applying the command $u_e = (Q_{1e};Q_{2e})$, the system will reach equilibrium.

We note the equilibrium output system with h_{1e}; h_{2e} si h_{3e}. The output is in equilibrium when their variation in time is zero, with other words when the levels derivative in rapport with time is zero:

$$\frac{dh_{1e}}{dt} = 0, \frac{dh_{2e}}{dt} = 0, \frac{dh_{3e}}{dt} = 0 \tag{23}$$

$$\begin{cases} A\dfrac{dh_{1e}}{dt} = Q_{1e} - Q_{13} = 0 \\ A\dfrac{dh_{3e}}{dt} = Q_{13} - Q_{32} = 0 \\ A\dfrac{dh_{2e}}{dt} = Q_{2e} + Q_{32} - Q_{20} = 0 \end{cases} \Leftrightarrow$$

$$\begin{cases} Q_{1e} = az_{13}S\sqrt{2g(h_{1e} - h_{3e})} \\ az_{13}S\sqrt{2g(h_{1e} - h_{3e})} = az_{32}S\sqrt{2g(h_{3e} - h_{2e})} \\ az_{20}S\sqrt{2gh_{2e}} = Q_{2e} + az_{32}S\sqrt{2g(h_{3e} - h_{2e})} \end{cases} \Leftrightarrow$$

$$\begin{cases} h_{1e} - h_{3e} = \left(\dfrac{Q_{1e}}{az_{13}S}\right)^2 \dfrac{1}{2g} \\ h_{3e} - h_{2e} = \left(\dfrac{Q_{1e}}{az_{32}S}\right)^2 \dfrac{1}{2g} \\ h_{2e} = \left(\dfrac{Q_{2e} + Q_{1e}}{az_{20}S}\right)^2 \dfrac{1}{2g} \end{cases} \tag{24}$$

For input flows u_e = *(4.503e-5m*^{3}*/s, 0e-5m*^{3}*/s)* we obtain levels of equilibrium y_e=(0.4177,0.0662,0,2420)

By using partial derivatives for linearization, the next system equations are obtained:

$$\frac{dx_1}{dt} = \frac{1}{A}(-az_{13}S\frac{\sqrt{2g}}{2\sqrt{h_{1e} - h_{3e}}}x_1 + az_{13}S\frac{\sqrt{2g}}{2\sqrt{h_{1e} - h_{3e}}}x_3 + Q_{1e}) \tag{25}$$

$$\frac{dx_2}{dt} = \frac{1}{A}((-az_{32}S\frac{\sqrt{2g}}{2\sqrt{h_{3e} - h_{2e}}} - az_{20}S\frac{\sqrt{2g}}{2\sqrt{2gh_{2e}}})x_2 + az_{32}S\frac{\sqrt{2g}}{2\sqrt{h_{3e} - h_{2e}}}x_3 + Q_{2e}) \tag{26}$$

$$\frac{dx_3}{dt}=\frac{1}{A}(az_{13}S\frac{\sqrt{2g}}{2\sqrt{h_{1e}-h_{3e}}}x_1-az_{32}S_n\frac{\sqrt{2g}}{2\sqrt{h_{3e}-h_{2e}}}x_2+$$
$$+(-az_3S_n\frac{\sqrt{2g}}{2\sqrt{h_{3e}-h_{2e}}}-az_1S_n\frac{\sqrt{2g}}{2\sqrt{h_{1e}-h_{3e}}})x_3) \qquad (27)$$

In the end, the matrices that define relations from equation (22) are:

Matrix A

$$\frac{1}{A}\begin{pmatrix} -az_{13}S\frac{\sqrt{2g}}{2\sqrt{h_{1e}-h_{3e}}} & 0 & az_{13}S\frac{\sqrt{2g}}{2\sqrt{h_{1e}-h_{3e}}} \\ 0 & -az_{32}S\frac{\sqrt{2g}}{2\sqrt{h_{3e}-h_{2e}}}-az_{20}S\frac{\sqrt{2g}}{2\sqrt{h_{2e}}} & az_{32}S\frac{\sqrt{2g}}{2\sqrt{h_{3e}-h_{2e}}} \\ az_{13}S\frac{\sqrt{2g}}{2\sqrt{h_{1e}-h_{3e}}} & az_3S\frac{\sqrt{2g}}{2\sqrt{h_{3e}-h_{2e}}} & -az_{32}S\frac{\sqrt{2g}}{2\sqrt{h_{3e}-h_{2e}}}-az_{13}S\frac{\sqrt{2g}}{2\sqrt{h_{1e}-h_{3e}}} \end{pmatrix}$$

Matrix B Matrix C Matrix D

$$\begin{pmatrix} 1/A & 0 \\ 0 & 1/A \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad (0)$$

4.3. Validation of the linear model

In order to validate the linear model a series of simulations were conducted. In Fig. 5 and 6 the step responses of a linear system are compared with the responses of the nonlinear model. The model is valid if the values of the outputs at equilibrium coincide for linear and nonlinear model.

By applying the command u_e = *(4.2e-5m^3/s, 0e-5m^3/s)* for nonlinear system and for linear system, the results are given in *Fig. 5* :

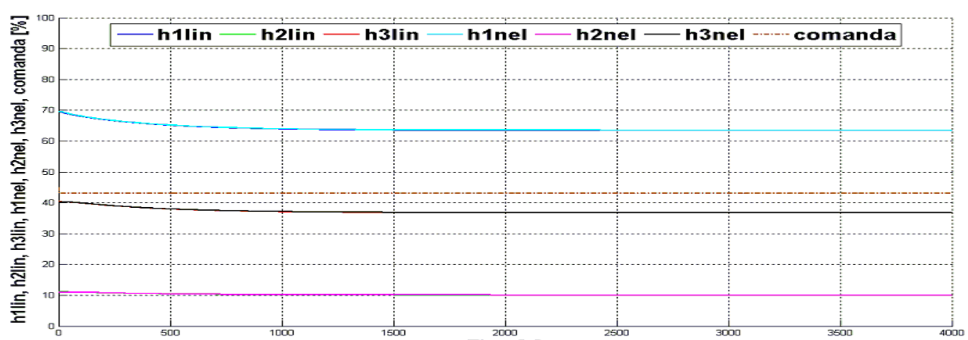

Fig 5. The responses of nonlinear system and linear system when the command is u_e = (4.2e-5m3/s, 0e-5m3/s)

Applying the command u_e = *(4.8e-5m^3/s, 0e-5m^3/s)* for nonlinear system and for linear system we obtained results in Fig. 6.

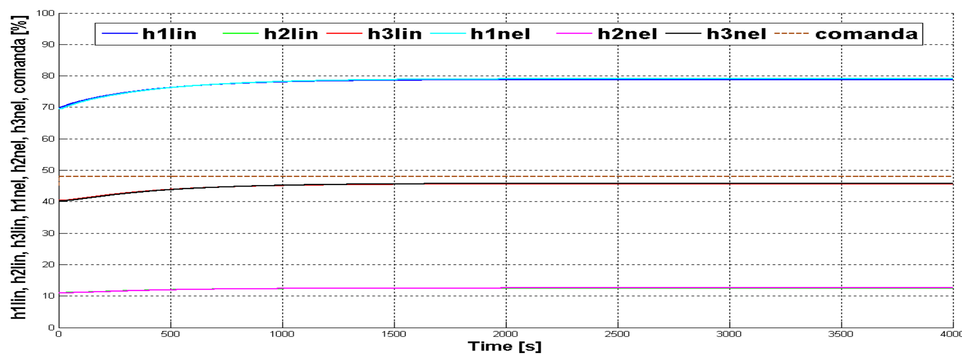

Fig 8. The responses nonlinear system and linear system when the command is u_e = (4.8e-5m^3/s, 0e-5m^3/s)

As can be seen in the graphs in Fig. 5 and 6 answer the two representations (linear and nonlinear) for a simulation time of 4000 sec is about the same.

In conclusion, for a variation of command Δu = 0.3e-5m3/s the linear system is valid.

5. Conclusion

In this paper I have proposed and designed two models (linear and respectively nonlinear) that can be used for the control design of a three tank system. The three tank system can be used as a benchmark for various industrial processes found in power plant and petro-chemical industry. The novelty of this paper is related to the identification of the outflow coefficients, which are often skipped in similar papers.

The necessity of those models rises from the controller design issue. The linear model is simple and suitable to design simple and classical control algorithms. The validity of such model is restricted to control inputs close to the values set for linear model design. The nonlinear model is more complex and gives no restriction on the control input. This model is more accurate and designing model based control solutions represents a more difficult task.

The nonlinear model was developed by using the equilibrium and balance equations for all flow rates. Implementation was realized in Matlab/Simulink tool and set on real process equipment.

The simulation and experiment results of the proposed models show a good description of the real process. The obtained model offers a good basis for future tasks in modeling, optimization, or in process control.

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